Hopf Bifurcation Analysis for a Delayed Predator–prey System with Stage Structure

Kejun Zhuang

Abstract—In this paper, a delayed predator–prey system with stage structure is investigated. Sufficient conditions for the system to have multiple periodic solutions are obtained when the delay is sufficiently large by applying Bendixson’s criterion. Further, some numerical examples are given.

Keywords—Predator-prey system, Stage structure, Hopf bifurcation, Periodic solutions.

I. INTRODUCTION

Since Aiello and Freedman proposed and studied the well-known single species model with delay and stage structure in [1], people have paid great attention to stage-structured population dynamics and obtained significant results, see [2]–[4] and the references therein. This is not only because they are simpler than the models governed by partial differential equations, but also they can exhibit phenomena similar to those of partial differential models.

In 1997, Wang and Chen constructed a predator-prey system with stage structure for predator as follows,

\[
\begin{align*}
\dot{x}(t) &= x(t)(r - ax(t - \tau_1) - by_1(t)), \\
y_1(t) &= kbx(t - \tau_2)y_2(t - \tau_2) - (D + \nu_1)y_1(t), \\
y_2(t) &= Dy_1(t) - \nu_2y_2(t),
\end{align*}
\]

where \(x(t)\) denotes the density of prey at time \(t\), \(y_1(t)\) denotes the density of immature predator at time \(t\), \(y_2(t)\) denotes the density of mature predator at time \(t\), constant \(\tau_1 \geq 0\) corresponds to the time delay in the feedback of prey’s density and constant \(\tau_2 \geq 0\) denotes the time delay due to gestation of mature predator. All coefficients are positive constants and the detailed ecological meanings can be found in [5].

For (1), Wang and Chen studied the permanence and global stability of positive equilibrium, and obtained the existence of orbitally asymptotically stable periodic solutions without time delays. And it shows that stage structure can be a cause of periodic oscillation and can make the population behavior more complex. Further, existence and global stability for the corresponding nonautonomous systems were derived in [6] by using Mawhin’s continuation theorem and constructing a suitable Lyapunov functional respectively. However, there are few results about the properties of positive equilibrium as time delay varies. To reduce the complexity of the analysis, we mainly consider the following system,

\[
\begin{align*}
\dot{x}(t) &= x(t)(r - ax(t) - by_2(t)), \\
y_1(t) &= kbx(t - \tau)y_2(t - \tau) - (D + \nu_1)y_1(t), \\
y_2(t) &= Dy_1(t) - \nu_2y_2(t),
\end{align*}
\]

II. STABILITY OF POSITIVE EQUILIBRIUM AND EXISTENCE OF LOCAL HOPF BIFURCATION

It is known that time delay does not change the location and number of positive equilibrium. According to the results in [5], we have the following lemma.

Lemma 2.1. Let

\[
\frac{r}{a} > \nu_2 \frac{D + \nu_1}{kbD}.
\]

Then (2) has the unique positive equilibrium \(E^* = (x^*, y_1^*, y_2^*)\), where \(x^* = \frac{(D + \nu_1)y_2}{kbD}, y_1^* = \frac{y_2^*}{D} \left(1 - \frac{D + \nu_1}{kbD} \right)\).

The linear part of (2) at \(E^*\) is

\[
\begin{align*}
\dot{x}(t) &= -ax^*x(t) - bx^*y_2(t), \\
y_1(t) &= kbx^*x(t - \tau) - (D + \nu_1)y_1(t) + kbx^*y_2(t - \tau), \\
y_2(t) &= Dy_1(t) - \nu_2y_2(t),
\end{align*}
\]

and the corresponding characteristic equation is

\[
\begin{align*}
\lambda^3 + (ax^* + D + \nu_1 + \nu_2)\lambda^2 + ax^*(D + \nu_1 + \nu_2) \\
+ (D + \nu_1)\nu_2 \lambda + ax^*\nu_2(D + \nu_1) \\
+ [kbDx^*(by_2^* - ax^*) - kbDx^*\lambda] e^{-\lambda\tau} = 0.
\end{align*}
\]

Next, we shall investigate the distribution of roots of (5). When \(\tau = 0\), (5) can be reduced to

\[
\lambda^3 + (ax^* + D + \nu_1 + \nu_2)\lambda^2 + ax^*(D + \nu_1 + \nu_2) + kbDx^*y_2^* = 0.
\]

By Routh–Hurwitz criteria, if

\[
a(ax^* + D + \nu_1 + \nu_2)(D + \nu_1 + \nu_2) > kbDy_2^*
\]

holds, then all roots of (6) have strictly negative real part. For simplicity, we denote (5) as follows,

\[
\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = (b_1\lambda + b_0)e^{-\lambda\tau} = 0,
\]

where \(a_2 = ax^* + D + \nu_1 + \nu_2, a_1 = ax^*(D + \nu_1 + \nu_2) + (D + \nu_1)\nu_2, a_0 = ax^*\nu_2(D + \nu_1), b_1 = -kbDx^*, b_0 = \ldots
\]
Since $kbDx^*(by^2 - ax^*)$. Obviously, $\lambda = i\omega(\omega > 0)$ is a root of (8) if and only if

$$\omega^3 + ai\omega - a_0 - (b_1\omega + b_0)(\cos \omega \tau - i \sin \omega \tau) = 0.$$ \hspace{1cm} (9)

Separating the real part and imaginary part, we can obtain

$$a_2\omega^2 - a_0 b_0 = b_1 \cos \omega \tau + b_0 \sin \omega \tau,$$

and

$$a_1 \omega - a_0 = b_0 \sin \omega \tau - b_1 \cos \omega \tau.$$ \hspace{1cm} (10)

where $p = a_2^2 - 2a_2, \quad q = a_2^2 - 2a_0 a_2 - b_1^2, \quad s = a_0^2 - b_0^2$. Set $z = \omega^2$, then (11) takes the form

$$z^3 + pz^2 + qz + s = 0.$$ \hspace{1cm} (12)

Thus, by Lemma 2.2 and Corollary 2.4 in [8], we can easily

Thus, the following transversality condition holds.

Further, $d\text{Re}(\tau_{(j)}^{(p)})$ and $h'(z_k) \neq 0$ have same sign.

Proof: By direct computation , we obtain

$$\{3\lambda^2 + 2a_2\lambda + a_1 + [b_1 - \tau(b_1 \lambda + b_0)]e^{-\lambda \tau}\} \frac{d\lambda}{d\tau} = \frac{\lambda(b_1 \lambda + b_0)}{\lambda(b_1 \lambda + b_0) - \tau}.$$ \hspace{1cm} (15)

By (10),

$$\frac{d\lambda}{d\tau} = \frac{\lambda(b_1 \lambda + b_0) - \tau}{\lambda(b_1 \lambda + b_0)}.$$ \hspace{1cm} (16)

Thus, $\lambda(b_1 \lambda + b_0) - \tau$ has at least one positive root.

Because $\Lambda, \omega_k > 0$, the sign of $d\text{Re}(\tau_{(j)}^{(p)})$ is consistent with that of $h'(z_k) \neq 0$. This proves the lemma.

By above analysis, we can obtain the following theorem about the stability of positive equilibrium and the existence of periodic solutions for (2).

**Theorem 2.5.** If (3) and (7) are satisfied, then the following results hold.

(a) If $s \geq 0$ and $D = p^2 - 3q \leq 0$, then for any $\tau \geq 0$, all roots of (5) have negative real parts. Further, positive equilibrium of (2) is absolutely stable for $\tau \geq 0$.

(b) If either $s < 0$ or $D = p^2 - 3q > 0$, then $h(\tau) \leq 0$, $r \geq 0$ and $h(\tau) > 0$ holds , then $h(z)$ has at least one positive root $z_k$ , and when $\tau \in (0, \tau_{0})$ , all roots of (5) have negative real parts. So the positive equilibrium of (2) is asymptotically stable for $\tau \in [0, \tau_{0})$.

(c) If conditions in (b) hold and $h'(z_k) \neq 0$, then Hopf bifurcation for (2) occurs at positive equilibrium when $\tau = \tau_{0}^{(j)}$ ($j = 0, 1, 2, \cdots$), which means that small amplified periodic solutions will bifurcate from positive equilibrium.

**III. Existence of global Hopf bifurcation**

Next, we shall establish the existence of global periodic solutions of (2) by ODE's Bendixson criterion. Define $X = C([-\tau, 0], R^2), \quad \Sigma = C([-\tau, 0], \tau, p) \times X \times R \times R^2, \quad \omega(t) = p$ is periodic solution of (2). $E^*$ is the connected component in $\Sigma$ of an isolated center $(E^*, \tau_{0}^{(j)}, \omega_{0}^{(j)})$, and $(E^*, \tau_{0}^{(j)}, \omega_{0}^{(j)})$ is nonempty, where $\tau_{0}^{(j)} = \frac{1}{4\pi} \left\{ \cos^{-1} \left( \frac{b_1 \omega_0^2 + (a_2b_0 - a_1b_1) \omega_0^2 - a_0b_0}{b_0^2 + b_1^2 \omega_0^2} \right) \right\}, \quad j = 0, 1, 2, \cdots$.}

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Lemma 3.1. [9] Let $D \subset \mathbb{R}^{n}$ be a simply connected region. Assume that the family of linear systems

$$z'(t) = \frac{\partial f [z]}{\partial x} (x(t, x_0)) z(t), \quad x_0 \in D$$

is equi-uniformly asymptotically stable. Then

(a) $D$ contains no simple invariant curves including periodic orbits, homoclinic orbits, heteroclinic cycles.
(b) Each semi-orbit in $D$ converges to a single equilibrium. In particular, if $D$ is positively invariant and contains an unique equilibrium $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in $D$.

When $\tau = 0, (2)$ is equivalent to

$$\begin{cases} \dot{x}(t) = x(t) (r - ax - by_1), \\ \dot{y}_1(t) = kbx(t)y_2(t) - (D + \nu_1)y_1(t), \\ \dot{y}_2(t) = Dy_1(t) + \nu_2y_2(t). \end{cases}$$

We make the following assumptions.

(H1) There exist $\alpha > 0$, such that

$$\sup_{x,y \in R} (r - D - \nu_1 - b|y_2(t)| - 2\alpha|x(t)| + \frac{2}{3}kb|x(t)|, + ab|x(t)|z_1^2D + r - \nu_1 - b|y_2(t)| - 2a|x(t)|, \frac{1}{3}kb|y_2(t)| - (D + \nu_1 + \nu_2)) < 0.$$  

Lemma 3.2. If (H1) is satisfied, then (18) has no non-constant periodic solution.

Proof: Denote $u = (x, y_1, y_2)^T$ and $f(x, y_1, y_2) = (x(r - ax - by_2), kbx(y_2 - (D + \nu_1)y_1, Dy_1 - \nu_2y_2))^T$, then we have

$$\frac{\partial f}{\partial x} = \begin{pmatrix} r - by_2 & -2ax & 0 \\ kbx & 0 & -bx \\ 0 & D & -\nu_1 \end{pmatrix} ,$$

and by [10],

$$\frac{\partial f [z]}{\partial x} = \begin{pmatrix} a_{11} & kbx & bx \\ a_{22} & 0 & -D \\ a_{33} & 0 & -\nu_2 \end{pmatrix} ,$$

where $a_{11} = r - D - \nu_1 - by_2 - 2ax$, $a_{22} = r - \nu_1 - by_2 - 2ax$ and $a_{33} = D - \nu_1 - \nu_2$. For the following second compound system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \frac{\partial f [z]}{\partial x} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} ,$$

and

$$\begin{cases} \dot{z}_1 = [r - D - \nu_1 - by_2(t) - 2ax(t)]z_1 + kbx(t)z_2 + bx(t)z_3, \\ \dot{z}_2 = Dz_1 + [r - \nu_1 - by_2(t) - 2ax(t)]z_2, \\ \dot{z}_3 = kby_2(t)z_2 - (D + \nu_1 + \nu_2)z_3, \end{cases}$$

where $u(t) = (x(t), y_1(t), y_2(t))^T$ is a solution of (18) when $u(0) = u_0 \in \mathbb{R}$. Set

$$W(z) = \max \{\alpha |z_1|, \beta |z_2|, |z_3|\} ,$$

where $\alpha$ and $\beta$ are both positive constants. Then we can get

$$\frac{d^+}{dt} |z_1| \leq [r - D - \nu_1 - by_2(t) - 2ax(t)]|z_1| + \frac{2}{3}kb|x(t)||z_2| + ab|x(t)||z_1|, $$

$$\frac{d^+}{dt} |z_2| \leq \frac{1}{3}kby_2(t)|\beta||z_2| - (D + \nu_1 + \nu_2)|z_3|, $$

$$\frac{d^+}{dt} |z_3| \leq \frac{1}{3}kby_2(t)|\beta||z_2| - (D + \nu_1 + \nu_2)|z_3|.$$
When results can be established.

The periods of periodic solutions of (2) are uniformly bounded.

**Lemma 3.4.** The periods of periodic solutions of (2) are uniformly bounded.

**Proof:** Note that if \( u(t) = (x(t), y_1(t), y_2(t))^T \) is a \( \tau \)-periodic solution of (2), then \( u(t) \) is a periodic solution of (18) and this contradicts Lemma 3.2. So (2) has no nontrivial periodic solutions. By the definition of \( \tau_0^{(j)} \), when \( j \geq 1 \), we have \( \frac{2\pi}{\nu} \leq \tau_0^{(j)} \). For \( \tau > \tau_0^{(j)} \), there exists an integer \( m \), such that \( \frac{2\pi}{\nu} \leq \frac{2\pi}{\nu} < \tau \). As system (1.2) has no nontrivial \( \tau \)-periodic solution, for any integer \( n \), (2) has no \( \frac{2\pi}{n} \)-periodic solution. This implies that the period \( p \) of a periodic solution on the connected component \( \ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0}) \) satisfies \( \frac{2\pi}{\nu} < p < \tau \). So we can know that the periods of the periodic solutions of (2) on \( \ell(E^*, \tau_0^{(j)}, \frac{2\pi}{\omega_0}) \) are uniformly bounded.

**Theorem 3.5.** Assume that (H1) and hypothesis (c) in Theorem 2.5 are satisfied. Then (2) still has periodic solutions when \( \tau > \tau_0^{(j)}(j \geq 1) \).

**Proof:** The characteristic equation of (2) at positive equilibrium \( E^* \) is

\[
\Delta(E^*, \tau, p)(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda \tau},
\]

and the characteristic equation of (2) at zero is

\[
\lambda[\lambda^2 + (D + \nu_1 + \nu_2)\lambda + \nu_2(D + \nu_1)] = 0,
\]

equation has no pure imaginary root. By the definition of isolated center in [11], we can easily verify that \((E^*, \tau_0^{(j)}, \nu)\) is the unique isolated center. There exist \( \varepsilon > 0 \), \( \delta > 0 \) and a smooth curve \( \lambda(\tau) : (\tau_0^{(j)} - \delta, \tau_0^{(j)} + \delta) \rightarrow C \), such that for any \( \tau \in [\tau_0^{(j)} - \delta, \tau_0^{(j)} + \delta] \), \( \Delta(\lambda(\tau)) = 0 \), \( |\lambda(\tau) - \omega_0\tau| < \varepsilon \), and \( \lambda(\tau_0^{(j)}) = \omega_0\). The corresponding characteristic equation has a pair of purely imaginary roots \( \lambda = \pm 0.486661i \), and \( \omega_0^{(j)} = 7.93605, \tau_0^{(1)} = 20.8476, \tau_0^{(2)} = 33.7584 \cdots \). The following figures explicit the solutions of (32) with the initial value (1, 1, 1).

IV. NUMERICAL SIMULATION

Finally, we shall give a numerical example:

\[
\begin{align*}
\dot{x}(t) &= x(t)(2 - x(t) - 0.3y_2(t)), \\
y_1(t) &= 0.24x(t - \tau)y_2(t - \tau) - 0.95y_1(t), \\
\dot{y}_2(t) &= 0.8y_1(t) - 0.1y_2(t),
\end{align*}
\]

then (3) and (7) hold and (32) has the unique positive equilibrium \( E^* = (0.494792, 0.62717, 5.01736) \). The corresponding characteristic equation has a pair of purely imaginary roots \( \lambda = \pm 0.486661i \), and \( \omega_0^{(j)} = 7.93605, \tau_0^{(1)} = 20.8476, \tau_0^{(2)} = 33.7584 \cdots \). The following figures explicit the solutions of (32) with the initial value (1, 1, 1).
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