Simplex Method for Solving Linear Programming Problems with Fuzzy Numbers

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Abstract—The fuzzy set theory has been applied in many fields, such as operations research, control theory, and management sciences, etc. In particular, an application of this theory in decision making problems is linear programming problems with fuzzy numbers. In this study, we present a new method for solving fuzzy number linear programming problems, by use of linear ranking function. In fact, our method is similar to simplex method that was used for solving linear programming problems in crisp environment before.

Keywords—Fuzzy number linear programming, ranking function, simplex method.

I. INTRODUCTION

Fuzzy linear programming first formulated by Zimmermann [10]. Recently, these problems are considered in several kinds, that is, it is possible that some coefficients of the problem in the objective function, technical coefficients, the right-hand side coefficients or decision making variables be fuzzy number [3], [4], [5], [6], [7], [8], [9]. In this work, we focus on the linear programming problems with fuzzy numbers in the objective function. Verdegay and et al [4], [9] proposed the equivalent parametric linear programming problems for these problems by use of a certain membership function and proposed a dual method for fuzzy number linear programming problems. Here, we first explain the concept of the comparison of fuzzy numbers by introducing a linear ranking function. Moreover, we describe basic feasible solution for the FNLP problems and state optimality conditions for these problems. Finally, we provide some important results for FNLP problems and we propose simplex algorithm for solving these problems.

II. DEFINITIONS AND NOTATIONS

We first review necessary backgrounds of fuzzy sets theory.

A. Fuzzy Sets

Let $X$ be a classical set of objects, called the universe, whose generic elements are denoted by $x$. The membership in a crisp subset of $X$ is often viewed as characteristic function $\mu_A(x)$ from $X$ to $\{0,1\}$ such that:

$\mu_A(x) = 1$, if $x \in A$

$= 0$, otherwise

where $\{0,1\}$ is called a valuation set.

If the valuation set is allowed to be the real interval $[0,1]$, $A$ is called a fuzzy set proposed by Zadeh [2]. $\mu_A(x)$ is the degree of membership of $x$ in $A$. The closer the value of $\mu_A(x)$ is to 1, the more $x$ belong to $A$. Therefore, $A$ is completely characterized by the set of ordered pairs:

$A = \{(x, \mu_A(x)) | x \in X\}$.

The support of a fuzzy set $A$ is the crisp subset of $X$ and is presented as:

$Supp A = \{x \in X | \mu_A(x) > 0\}$.

The $\alpha$-level ($\alpha$-cut) set of a fuzzy set $A$ is a crisp subset of $X$ and is denoted by

$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$.

A fuzzy set $A$ in $X$ is convex if

$\mu_A(\lambda x + (1-\lambda) y) \geq \min\{\mu_A(x), \mu_A(y)\}$

$x, y \in X$, and $\lambda \in [0,1]$. Alternatively, a fuzzy set is convex if all $\alpha$-level sets are convex. Note that in this paper we suppose that $X = R$.

A fuzzy number $A$ is a convex normalized fuzzy set on the real line $R$ such that

1) It exists at least one $x_0 \in R$ with $\mu_A(x_0) = 1$.

2) $\mu_A(x)$ is piecewise continuous.

Among the various types of fuzzy numbers, triangular and trapezoidal fuzzy numbers are of the most important. Note that, in this study we only consider trapezoidal fuzzy numbers.

A fuzzy number is a trapezoidal fuzzy number if the membership function of it be in the following form:
We show any trapezoidal fuzzy number by 
\[ \tilde{a} = (a^L, a^U, \alpha, \beta), \]
where the support of \( \tilde{a} \) is 
\[ (a^L - \alpha, a^U + \beta), \]
and the modal set of \( \tilde{a} \) is \[ [a^L, a^U]. \]
Let \( F(R) \) be the set of trapezoidal fuzzy numbers. In the next subsection we describe arithmetic on \( F(R) \).

**B. Arithmetic on Fuzzy Numbers**

Let \( \tilde{a} = (a^L, a^U, \alpha, \beta) \) and \( \tilde{b} = (b^L, b^U, \gamma, \theta) \) be two trapezoidal fuzzy numbers and \( x \in R \). Then, the results of applying fuzzy arithmetic on the trapezoidal fuzzy numbers as shown in the following:

**Image of \( \tilde{a} \):**
\[ -\tilde{a} = (-a^U, -a^L, \beta, \alpha). \]

**Addition:**
\[ \tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta). \]

**Scalar Multiplication:**
\[ x \geq 0, x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta). \]
\[ x < 0, x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha). \]

**III. RANKING FUNCTIONS**

A convenient method for comparing of the fuzzy numbers is by use of ranking functions. A ranking function is a map from \( F(R) \) into the real line. Now, we define orders on \( F(R) \) as following:

\[ \tilde{a} \succeq \tilde{b} \quad \text{if and only if} \quad \Re(\tilde{a}) \geq \Re(\tilde{b}) \]  
\[ \tilde{a} \succ \tilde{b} \quad \text{if and only if} \quad \Re(\tilde{a}) > \Re(\tilde{b}) \]  
\[ \tilde{a} = \tilde{b} \quad \text{if and only if} \quad \Re(\tilde{a}) = \Re(\tilde{b}) \]

where \( \tilde{a} \) and \( \tilde{b} \) are in \( F(R) \). It is obvious that we may write \( \tilde{a} \succeq \tilde{b} \) if and only if \( \tilde{b} \succeq \tilde{a} \). Since there are many ranking function for comparing fuzzy numbers we only apply linear ranking functions. So, it is obvious that if we suppose that \( \Re \) be any linear ranking function, then

i) \( \tilde{a} \succeq \tilde{b} \) if and only if \( \tilde{a} - \tilde{b} \geq 0 \) if and only if \( -\tilde{b} \geq -\tilde{a} \)

ii) If \( \tilde{a} \succeq \tilde{b} \) and \( \tilde{c} \succeq \tilde{d} \), then \( \tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{d} \).

One suggestion for a linear ranking function as following:

\[ \Re(\tilde{a}) = a^L + a^U + \frac{1}{2}(\beta - \alpha) \]  

where \( \tilde{a} = (a^L, a^U, \alpha, \beta) \in F(R) \).

**IV. FUZZY LINEAR PROGRAMMING**

In this section, we introduce fuzzy linear programming (FLP) problems. So, we first define linear programming problems.

**A. Linear Programming**

A linear programming (LP) problem is defined as:
\[
\begin{align*}
\text{Max} & \quad z = c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

where \( c = (c_1, \ldots, c_n), b = (b_1, \ldots, b_m)^T \), and \( A = [a_{ij}]_{m \times n} \).

In the above problem, all of the parameters are crisp [1]. Now, if some of the parameters be fuzzy numbers we obtain a fuzzy linear programming which is defined in the next subsection.

**B. Fuzzy Linear Programming**

Suppose that in the linear programming problem some parameters be fuzzy numbers. Hence, it is possible that some coefficients of the problem in the objective function, technical coefficients, the right-hand side coefficients or decision making variables be fuzzy number [3], [4], [5], [6], [7], [8], [9]. Here, we consider the linear programming problems with fuzzy numbers in the objective function.

**V. FUZZY NUMBER LINEAR PROGRAMMING**

A fuzzy number linear programming (FNLP) problem is defined as follows:
\[
\begin{align*}
\text{Max} & \quad \tilde{z} = \tilde{c}^T \tilde{x} \\
\text{s.t.} & \quad \tilde{A} \tilde{x} = \tilde{b} \\
& \quad \tilde{x} \geq 0
\end{align*}
\]

where \( \tilde{b} \in R^n, \tilde{x} \in R^n, \tilde{A} \in R^{m \times n}, \tilde{c}^T \in \Re(F(R))^n \), and \( \Re \) is a linear ranking function.

**Definition 5.1.** We say that vector \( \tilde{x} \in R^n \) is a feasible solution to (6) if and only if \( \tilde{x} \) satisfies the constraints of the problem.

**Definition 5.2.** A feasible solution \( \tilde{x} \) is an optimal solution for (6), if for all feasible solution \( \tilde{x} \) for (6), we have \( \tilde{c} \tilde{x} \geq \tilde{c} \tilde{x} \).

**A. Fuzzy Basic Feasible Solution**

Here, we introduce basic feasible solutions for FNLP problems. Consider the system \( \tilde{A} \tilde{x} = \tilde{b} \) and \( \tilde{x} \geq 0 \), where \( \tilde{A} \) is an \( m \times n \) matrix and \( \tilde{b} \) is an \( m \) vector. Now, suppose that \( \text{rank}(\tilde{A}, \tilde{b}) = \text{rank}(\tilde{A}) = m \). Partition after possibly rearranging the columns of \( \tilde{A} \) as \([ B, N ]\) where \( B \),
m \times m$, is nonsingular. It is obvious that $\text{rank}(B) = m$.

The point $x = (x_B^T, x_N^T)^T$ where $x_B = B^{-1}b$, $x_N = 0$ is called a basic solution of the system. If $x_B \geq 0$, then $x$ is called a basic feasible solution (BFS) of the system. Here $B$ is called the basic matrix and $N$ is called the nonbasic matrix. The components of $x_B$ are called basic variables, and the components of $x_N$ are called nonbasic variables. If $x_B > 0$, then is $x$ a nondegenerate basic feasible solution, and at least one component of $x_B$ is zero, then $x$ is called a degenerate basic feasible solution.

The following theorem characterizes optimal solutions. The result corresponds to the so-called nondegenerate problems, where all fuzzy basic variables corresponding to every basis $B$ are nonzero (and hence positive) [5].

**Theorem 5.1.** Assume the FNLP problem is nondegenerate. A basic feasible solution $x_B = B^{-1}b, x_N = 0$ is optimal to (6) if and only if $\bar{z}_j \geq \bar{c}_j$ for all $1 \leq j \leq n$.

**Proof.** Suppose that $x_* = (x_B^*, x_N^*)^T$ is a basic feasible solution to (6), where $x_B = B^{-1}b, x_N = 0$. Then, $\bar{z}_* = \bar{c}_Bx_B = \bar{c}_B^{-1}b$. On the other hand, for all feasible solution $x$ , we have $b = Ax = Bx_B + Nx_N$. Hence, we may obtain

$$\bar{z} = \bar{c}_Bx_B + \bar{c}_N x_N = \bar{c}_B^{-1}b - \sum_{j \in B} (\bar{c}_B^{-1}a_j - \bar{c}_j)x_j$$

Then,

$$\bar{z} = \bar{z}_* - \sum_{j \in B}(\bar{z}_j - \bar{c}_j)x_j$$  \hspace{1cm} (7)

Therefore, the result follows immediately from (7) and the assumptions of theorem.

In the next section, we propose simplex method for solving FNLP problems.

**VI. SIMPLEX METHOD FOR THE FNLP PROBLEMS**

Consider the FNLP problem as defined in (6).

$$\max \quad \bar{z} = \bar{c}^T_Bx_B + \bar{c}_N x_N$$

s.t. $Bx_B + Nx_N = b$

$x_B, x_N \geq 0$

Hence, we may write $x_B + B^{-1}N x_N = B^{-1}b$. Therefore, $\bar{z} + (\bar{c}_B^{-1}N - \bar{c}_N)x_N = \bar{c}_B^{-1}b$. Currently $x_N = 0$, and then $x_B = B^{-1}b$, and $\bar{z} = \bar{c}_B^{-1}b$. Then we rewrite the above FNLP problem in the following tableau format:

<table>
<thead>
<tr>
<th>$\bar{z}$</th>
<th>$x_B$</th>
<th>$x_N$</th>
<th>R.H.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$\bar{c}_B^{-1}N - \bar{c}_N$</td>
<td>$\bar{c}_B^{-1}b$</td>
</tr>
</tbody>
</table>

The above tableau gives us all the information we need to proceed with the simplex method. The fuzzy cost row in the above tableau is $\bar{y}_j = (\bar{c}_B^{-1}a_j - \bar{c}_j)$ for the nonbasic variables. According to the optimality condition for these problems we are at the optimal solution if $\bar{y}_j \geq 0$ for all $j \neq B_i$. On the other hand, if $\bar{y}_k < 0$, for a $k \neq B_i$, then we may exchange $x_B$ with $x_k$. Then we compute the vector $y_k = B^{-1}a_k$. If $y_k \leq 0$, then $x_k$ can be increase indefinitely, and then the optimal objective is unbounded. On the other hand, if $y_k$ has at least one positive component, then the increase in will be blocked by one of the current basic variables, which drops to zero.

**Theorem 6.1.** If in a simplex tableau, an $l$ exists such that $\bar{z}_l - \bar{c}_l < 0$ and there exists a basic index $i$ such that $y_{il} > 0$, then a pivoting row $r$ can be found so that pivoting on $y_{ir}$ will yield a feasible tableau with a corresponding nondecreasing fuzzy objective value.

**Proof.** We need a criterion for choosing a basic variable to leave the basis so that the new simplex tableau will remain feasible and the new objective value is nondecreasing. Assume column $l$ is the pivot column. Also, suppose that is $x = (x_B^T, x_N^T)^T$ a basic feasible solution to the FNLP problem, where $x_B = B^{-1}b$, and $x_N = 0$. Then, the corresponding fuzzy objective value is $\bar{z} = \bar{c}_B^{-1}b = \bar{c}_Bx_N$.

On the other hand, for any basic feasible solution to the FNLP problem, we have

$$x_B + \sum_{j \in B} y_{ij}x_j = y_0$$  \hspace{1cm} (8)

where $y_{ij} = B^{-1}a_j$. So, if $x_l$ enters into the basis we may write

$$x_B = y_0 - y_{il}x_l$$  \hspace{1cm} (9)

Since, we want $x_B$ be feasible, hence $x_{Bl} \geq 0$, or

$$y_{il}x_l \geq 0, \text{ for all } i = 1, \ldots, m.$$

If $y_{il} \leq 0$, then it is obvious that the above condition is hold. Hence, for all $y_{il} > 0$, we need to have
To satisfy (10) it is sufficient to let
\[
y_{i0} = \min \{ y_{i0} : y_{il} > 0 \} \quad (11)
\]
Also, for any basic feasible solution to the FNLP problem, we have
\[
z = \sum_{j=1}^{m} (\bar{c}_j - \tilde{c}_j)x_j \quad (12)
\]
So, if we enter \(x_i\) into the basis we have
\[
z = \sum_{j=1}^{m} (\bar{c}_j - \tilde{c}_j)x_j \quad (13)
\]
We note that the new objective value is nondecreasing, since
\[
z = \sum_{j=1}^{m} (\bar{c}_j - \tilde{c}_j)x_j \geq \bar{c}_y y_0 \quad (14)
\]
We have
\[
x_i \leq \frac{y_{i0}}{y_{il}} \quad (10)
\]
For an illustration of the above method we solve a FNLP problem by use of simplex method.

**Example 8.1.**
Max \(\tilde{z} = (5, 8, 2, 5)x_1 + (6, 10, 2, 6)x_2\)
s.t. \(2x_1 + 3x_2 \leq 6\)
\(5x_1 + 4x_2 \leq 10\)
\(x_1, x_2 \geq 0\)

We may rewrite
\[2x_1 + 3x_2 + x_3 = 6\]
\[5x_1 + 4x_2 + x_4 = 10\]
\(x_1, x_2, x_3, x_4 \geq 0\)

We may write the first feasible simplex tableau as follows:

<table>
<thead>
<tr>
<th>basis</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>R.H.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{z})</td>
<td>((-8, -5, 5, 2))</td>
<td>((-10, -6, 6, 2))</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x_3)</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>(x_4)</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Now, from \(\tilde{z} = (\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = ((-8, -5, 5, 2), (-10, -6, 6, 2))\),
and \((\gamma_1, \gamma_2) = (\Re(\tilde{\gamma}_1), \Re(\tilde{\gamma}_2)) = (-14.5, -18)\), then \(x_2\) enters the basis and the leaving variable is \(x_3\). The new tableau is:

<table>
<thead>
<tr>
<th>basis</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>R.H.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{z})</td>
<td>((-4, \frac{5}{3}, \frac{10}{3}, 0))</td>
<td>0</td>
<td>((2, \frac{10}{3}, \frac{2}{3}, 2))</td>
<td>0</td>
<td>((12, 20, 4, 12))</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(\frac{2}{3})</td>
<td>1</td>
<td>(\frac{1}{3})</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(x_4)</td>
<td>(\frac{7}{3})</td>
<td>0</td>
<td>(\frac{4}{3})</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Now, from \((\tilde{\gamma}_1, \tilde{\gamma}_2) = ((-4, \frac{5}{3}, \frac{10}{3}, 6), (2, \frac{10}{3}, \frac{2}{3}, 2))\) and 
\((\Re(\tilde{\gamma}_1), \Re(\tilde{\gamma}_2)) = ((\gamma_1, \gamma_2) = (\frac{-5}{3}, 6))\), it follows that \(x_1\) is an entering variable and \(x_4\) is a leaving variable.

The last tableau is shown in the below.
Now, using optimality condition there is not any variable that enters the basis. Therefore, this basis is optimal.

VIII. CONCLUSION

We considered fuzzy number linear programming problems and introduced the basic feasible solution for these problems. Finally, we obtained some important results and we proposed a new algorithm for solving these problems directly, by use of linear ranking function.

REFERENCES