The Slant Helices According to Bishop Frame

Bahaddin Bukcu, Murat Kemal Karacan

Abstract—In this study, we have defined slant helix according to Bishop frame in Euclidean 3-Space. Furthermore, we have found some necessary and sufficient conditions for the slant helix.

Keywords—Slant helix, Bishop frame, Parallel transport frame

I. INTRODUCTION

Let \( M \) be an \( n \)-dimensional smooth manifold equipped with a metric \( \langle \cdot, \cdot \rangle \). A tangent space \( T_p(M) \) at a point \( p \in M \) is furnished with the canonical inner product. If \( \langle \cdot, \cdot \rangle \) is positive definite, then \( M \) is a Riemannian manifold. A curve on an Riemannian manifold \( M \) is a smooth mapping \( \alpha : I \to M \), where \( I \) is an open interval in the real line \( \mathbb{R} \). As an open submanifold of \( \mathbb{R} \), \( I \) has a coordinate system consisting of the identity map \( u \) of \( I \). The velocity vector of \( \alpha \) at \( s \in I \)

\[
\alpha'(s) = \frac{d\alpha(u)}{du} |_{u=\alpha(s)} \in T_{\alpha(s)}M.
\]

A curve \( \alpha(s) \) is said to be regular if \( \alpha'(s) \) is not equal to zero for any \( s \). Let \( \alpha(s) \) be a curve on \( M \); denote by \( \{T, N, B\} \) the moving Frenet frame along the curve \( \alpha \). Then \( T, N \) and \( B \) are the tangent, the principal normal and binormal vectors of the curve \( \alpha \) respectively. If \( \alpha \) is a space curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

\[
\alpha'(s) = T, \\
D_T T = \kappa N, \\
D_T N = -\kappa T + \tau B, \\
D_T B = -\tau N,
\]

where \( D \) denotes the covariant differentiation in \( M \).

In a Riemann manifold \( M \), a curve is described by the Frenet formula. For example, if all curvatures of a curve are identically zero, then the curve is a geodesic. If only the curvature \( \kappa \) is a non-zero constant and the torsion \( \tau \) is all identically zero, then the curve is called a circle. If the curvature \( \kappa \) and the torsion \( \tau \) are non-zero constants, then the curve is called a helix. If the curvature \( \kappa \) and the torsion \( \tau \) are not constant but \( \frac{\kappa}{\tau} \) is constant, then the curve is called a general helix [4, 7].

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while \( T(s) \) for a given curve model is unique, we may choose any convenient arbitrary basis \( (N_1(s), N_2(s)) \) for the remainder of the frame, so long as it is in the plane perpendicular to \( T(s) \) at each point. If the derivatives of \( (N_1(s), N_2(s)) \) depend only on \( T(s) \) and not each other we can make \( N_1(s) \) and \( N_2(s) \) vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations

\[
\begin{bmatrix}
T' \\
N_1' \\
N_2'
\end{bmatrix} = 
\begin{bmatrix}
0 & k_1 & k_2 \\
-k_1 & 0 & 0 \\
-k_2 & 0 & 0
\end{bmatrix} 
\begin{bmatrix}
T \\
N_1 \\
N_2
\end{bmatrix}.
\]

One can show (see, Bishop [3]) that

\[
\kappa(s) = \sqrt{k_1^2 + k_2^2}, \\
\theta(s) = \arctan\left(\frac{k_2}{k_1}\right), k_1 \neq 0 \\
\tau(s) = -\frac{d\theta(s)}{ds}
\]

so that \( k_1 \) and \( k_2 \) effectively correspond to a Cartesian coordinate system for the polar coordinates \( \kappa, \theta \) with \( \theta = -\int \tau(s)ds + \theta_0 \). The orientation of the parallel transport frame includes the arbitrary choice of integration constant \( \theta_0 \), which disappears from \( \tau \) (and hence from the Frenet frame) due to the differentiation [1, 2].

Bahaddin Bukcu is with the Gazi Osmanpasa University, Faculty of Sciences and Arts, Department of Mathematics, Tokat-Turkey (e-mail: bbukcu@yahoo.com).

Murat Kemal Karacan is with Usak University, Faculty of Sciences and Arts, Department of Mathematics 64200, Usak-Turkey (corresponding author; e-mail: murat.karacan@usak.edu.tr).
II. THE SLANT HELICES ACCORDING TO BISHOP FRAME

Definition 2.1. A regular curve \( \alpha : I \to E^3 \) is called a slant helix provided the unit vector \( N_i(s) \) of \( \alpha \) has constant angle \( \theta \) with some fixed unit vector \( u \); that is, \( \langle N_i(s), u \rangle = \cos \theta \) for all \( s \in I \).

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. Slant helices can be identified by a simple condition on natural curvatures.

Theorem 2.1. Let \( \alpha : I \to E^3 \) be a unit speed curve with nonzero natural curvatures. Then \( \alpha \) is a slant helix if and only if \( \frac{k_1}{k_2} \) is constant.

Proof. Let \( \alpha \) be slant helix in \( E^3 \) and \( \langle N_1, u \rangle = \text{const} \). Then \( \alpha \) is slant helix; from the definition, we have

\[
\langle N_1, u \rangle = \text{const.}
\]

where \( u \) is a unit vector, called the axis of slant helix. By differentiation we get

\[
\langle N'_1, u \rangle = \langle -k_1 T, u \rangle = -k_1 \langle T, u \rangle = 0.
\]

Hence

\[
\langle T, u \rangle = 0.
\]

Again differentiating from the last equality, we can write as follows

\[
\langle T', u \rangle = \langle k_1 N_1 + k_2 N_2, u \rangle = k_1 \langle N_1, u \rangle + k_2 \langle N_2, u \rangle = k_1 \cos \theta + k_2 \sin \theta = 0.
\]

Therefore we obtain

\[
\frac{k_1}{k_2} = -\tan \theta
\]

as desired.

Suppose that \( \frac{k_1}{k_2} = -\tan \theta \). Then we can write

\[
u \in Sp\{N_1, N_2\}, i.e.,
\]

\[
u = N_1 \cos \theta + N_2 \sin \theta.
\]

Differentiating the last equality,

\[
u' = (k_1 \cos \theta + k_2 \sin \theta)T = 0.
\]

So \( u \) is a constant vector. Thus, the proof is done.

Theorem 2.2. Let \( \alpha = \alpha(s) \) be a unit speed curve in \( E^3 \). Then \( \alpha \) is a slant helix iff

\[
\det(N'_1, N''_1, N''''_1) = 0.
\]

Proof. \((\Rightarrow)\) Suppose that \( \frac{k_1}{k_2} \) be constant. We have equalities as

\[
-N'_1 = k_1 T
\]

\[
-N''_1 = k_1'T + k_1k_2 N_1 + k_1k_2 N_2
\]

\[
-N''''_1 = \left( k_1'' - k_1^3 - k_1k_2^2 \right) T
\]

\[
+ \left( 3k_1 k_1' \right) N_1 + \left( 2k_1 k_2 + k_1 k_2' \right) N_2
\]

So we get

\[
\det(N'_1, N''_1, N''''_1) = k_1^2 \left[ \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right] \star \begin{array}{cc}
1 & k_1 \\
k_2 & 2k_1k_2 + k_1k_2'
\end{array}
\]

\[
= k_1 \left( \frac{k_1}{k_2} \right)^2 \left( \frac{k_1}{k_2} \right)'.
\]

Since \( \alpha \) is a slant helix, \( \frac{k_1}{k_2} \) is constant. Hence, we have

\[
det(N'_1, N''_1, N''''_1) = 0, k_2 \neq 0.
\]

\((\Leftarrow)\) Suppose that \( det(N'_1, N''_1, N''''_1) = 0 \). Then it is clear that the \( \frac{k_1}{k_2} = \text{const.} \) for being

\[
\left( \frac{k_1}{k_2} \right)' = 0.
\]

Theorem 2.3. Let \( \alpha = \alpha(s) \) be a unit speed curve in \( E^3 \). Then \( \alpha \) is a slant helix iff

\[
\det(N'_2, N''_2, N''''_2) = 0.
\]

Proof. \((\Rightarrow)\) Suppose that \( \frac{k_1}{k_2} \) be constant. From Eq. (1) one can find

\[
-N'_2 = -k_2 T
\]

and

\[
-N''_2 = (k_2')T + (k_2k_2)N_1 + (k_2k_2')N_2,
\]

\[
-N''''_2 = (k''_2 - k_1k_2 - k_1k_2') T
\]

\[
+ (2k_1k_2 + k_1k_2') N_1 + (3k_1k_2') N_2
\]

Moreover, we have
det\left(N'_2, N''_2, N'''_2\right) = -k_2^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & k_1 & k_2 \\ 2k_1 k'_2 + k'_1 k_2 & 3k_1 k'_2 \end{bmatrix} = k_2^2 \left(\frac{k_1}{k_2}\right)'.

Since $\alpha$ is a slant helix curve $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N'_2, N''_2, N'''_2) = 0$$

($\Leftarrow$) Suppose that $\det(N'_2, N''_2, N'''_2) = 0$. Then it is clear that the $\frac{k_1}{k_2} = \text{const}$ for being

$$\left(\frac{k_1}{k_2}\right)' = 0.$$ 

Next we consider general slant helices in a Euclidean manifold $M$. Then we have equalities

$$\alpha'(s) = T,$$

$$D_T T = k_1 N_1 + k_2 N_2,$$

$$D_T N_1 = -k_1 T,$$

$$D_T N_2 = -k_2 T,$$

for any $s \in I$, where $N_1(s)$ and $N_2(s)$ are vector fields and $k_1$ and $k_2$ are functions of parameter $s$.

**Theorem 2.4.** A unit speed curve $\alpha$ on $M$ is a general slant helix iff

$$D_T \left(D_T D_T N_1\right) = -AD_T N_1 - 3k_1' D_T T$$

where

$$A = k^2 - \frac{k''_1}{k_1}, k_1^2 + k_2^2 = k^2.$$ 

**Proof.** Suppose that $\alpha$ is general slant helix. Then, from Eq. (2), we have

$$D_T \left(D_T N_1\right) = D_T \left(-k_1 T\right) = -k_1' T - k_1 D_T T = -k_1' T - k_1^2 T - k_1 k_2 N_2$$

and

$$D_T \left(D_T D_T N_1\right) = \left(-k_1'' + k_1 k_2^2\right) T - k_1^2 D_T N_1 - 2k_1 k_2' N_1 - 3k_1 k_2 D_T T$$

$$- \left(k_1 k'_2 - k_1 k_2'\right) N_2 - k_1 D_T T.$$ 

Now, since $\alpha$ is a general slant helix, we have

$$\frac{k_1}{k_2} = \text{const}.$$ 

and this upon the derivation give rise to

$$k_1' k_2 = k_1 k'_2.$$ 

If we substitute the values

$$T = -\frac{1}{k_1} D_T N_1$$

and

$$\left(\frac{k_1}{k_2}\right)' = 2k_1' k_2,$$

in Eq.(2.4) we obtain

$$D_T \left(D_T D_T N_1\right) = \left(\frac{k_1''}{k_1} - k^2\right) D_T N_1 - 3k_1' D_T T.$$

$$D_T \left(D_T D_T N_1\right) = \left(\frac{k_1''}{k_1} - k^2\right) D_T N_1 - 3k_1' D_T T.$$ 

So we get as desired.

Conversely let us assume that Eq. (2) holds. We show that the curve $\alpha$ is general slant helix. Differentiating covariantly Eq. (6) we obtain

$$D_T T = D_T \left(-\frac{1}{k_1} D_T N_1\right)$$

$$= \frac{k_1''}{k_1} D_T N_1 - \frac{1}{k_1} D_T D_T N_1$$

and so,

$$D_T D_T T = \left(\frac{k_1''}{k_1} + \frac{k_1'}{k_1^2} \frac{k_1'}{k_1^2} D_T T\right) D_T N_1 + \frac{k_1'}{k_1^2} D_T D_T N_1$$

$$+ \frac{k_1'}{k_1^2} D_T D_T N_1 - \frac{1}{k_1} D_T D_T D_T N_1$$

If we use Eq. (2) in Eq. (7), we get

$$D_T D_T T = \left[\left(\frac{k_1'}{k_1^2}\right) + A\right] D_T N_1 + \frac{2k_1'}{k_1^2} D_T D_T N_1$$

$$+ \frac{3k_1'}{k_1^2} D_T T_1.$$ 

Substituting Eq. (4) and Eq. (5) in this last equality we obtain

$$D_T D_T T = \left[\left(\frac{k_1'}{k_1^2}\right) + A\right] D_T N_1 - \frac{2k_1' k_1'}{k_1^2} T$$

$$- 2k_1' N_1 - \frac{2k_1' k_1'}{k_1^2} N_2 + 3k_1' N_1 + \frac{3k_1' k_1'}{k_1^2} N_2.$$ 

From the last equality we have
\[ D_t D_T T = \left[ \left( \frac{k_1'}{k_1^2} \right) + \frac{A}{k_1} \right] D_T N_1 - \frac{2k_1'^2}{k_1^2} T + k_1' N_1 + k_1' k_2 N_2. \] (8)

On the other hand we can write \( D_t D_T T \) as follows

\[ D_T D_T T = k_1 D_T N_1 - k_2^2 T + k_1' N_1 + k_2' N_2. \] (9)

From comparison the Eq. (8) and Eq. (9) we obtain equalities below

\[ \frac{k_1' k_2^2}{k_1} = k_2' \]

and so

\[ \frac{k_1'}{k_1} = \frac{k_2'}{k_2}. \] (10)

Integrating Eq. (10), we get

\[ \frac{k_1}{k_2} = \text{const}. \]

Thus \( \alpha \) is a general slant helix. Hence, the proof is done.

REFERENCES