New exact three-wave solutions for the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov system

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Abstract—New exact three-wave solutions including periodic two-solitary solutions and doubly periodic solitary solutions for the (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov (ANNV) system are obtained using Hirota’s bilinear form and generalized three-wave type of ansatz approach. It is shown that the generalized three-wave method, with the help of symbolic computation, provides an effective and powerful mathematical tool for solving high dimensional nonlinear evolution equations in mathematical physics.

Keywords—Soliton Solution; Hirota Bilinear Method; ANNV System.

I. INTRODUCTION

In last decades, increasing attention has been paid to the study of the solution theory in many natural sciences particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, nonlinear optics and condensed matter physics.

Integrable models play a prominent role in theoretical physics. The reason is not only the direct phenomenological interest of some of them, but also the fact that they often provide some deep insights into the mathematical structure of theory in which they arise. Up to now, the (1+1)-dimensional integrable models are well understood due to many systematic methods such as the inverse scattering transformation [9], the Darboux transformation, Hirota’s bilinear method [3], [10], [11], [12], [13], [15], Bäcklund transformation method [18], Painlevé expansion method [8], tanh function method [16], homogeneous balance method [19], variable separation approach [6] and the three-wave approach [2], [5]. However, studies of the (2+1)-dimensional cases are fewer in number and such systems are being actively investigated from different viewpoints.

In this paper, we consider the following (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov (ANNV) equation:

\[
\begin{align*}
u_t + u_{xxx} + 3 \left( u \left( \int u_x \, dy \right) \right)_x &= 0. 
\end{align*}
\]

(1)

Boiti et al. [4] was first derived Eq. (1) by using the weak Lax pair. It may be considered as a model for an incompressible fluid where \( u \) and \( v \) are the components of the (dimensionless) velocity [8]. The spectral transformation for this system has been investigated in [1], [4]. Moreover, Hu et al. used the Darboux transformations to find the variable separation solutions of this equation in [14]. Wazwaz used the Hirota bilinear method to derive N-soliton solutions of this completely integrable equation [17], Dai and Wang used the exp-function method for a new general variable separation solutions [7] and the Bäcklund transformation for the ANNV equation was derived by using the extended homogeneous balance method by Zhang in [20].

II. ANNV SYSTEM

The ANNV equation can be written as

\[
\begin{align*}
u_t + u_{xxx} + 3 \left[ u^2 \right]_x &= 0, \\
u_x &= v_y.
\end{align*}
\]

(2)

To solve the ANNV system (2), we substitute the dependent variable transformation

\[
\begin{align*}
u &= 2 (\ln f)_{xy}, \\
v &= 2 (\ln f)_{xx},
\end{align*}
\]

(3)

where \( f(x, y, t) \) is an unknown real function. Substituting (3) into system (2), we can get

\[
\begin{align*}
\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial f}{\partial x} \left( \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y \partial x} \right) + 3 \left( \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial x \partial y} \right) &= 0,
\end{align*}
\]

or equivalently in bilinear form

\[
\begin{align*}D_y (D_x + D_y^2) f : f &= 0, \end{align*}
\]

(4)

where the Hirota D-operator is defined by [10]

\[
\begin{align*}D_x = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial \xi_1} \right)^n \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial \xi_2} \right)^m f(x, t)g(\xi_1, \xi_2). \end{align*}
\]

We propose a novel test function of extended three-soliton method

\[
\begin{align*}f(x, y, t) &= e^{\xi_1} + d_1 \cos(\xi_2) + d_2 \cosh(\xi_3) + d_3 e^{-\xi_1}, \end{align*}
\]

(5)

where

\[
\begin{align*}\xi_i = a_i x + b_i y + c_i t, \quad i = 1, 2, 3 \end{align*}
\]

and \( a_i, b_i, c_i, d_i \) are some constants to be determined later.

Substituting (5) into Eq. (4), and equating all the coefficients of \( \sin(a_2 x + b_2 y + c_2 t), \cos(a_2 x + b_2 y + c_2 t), \sinh(a_3 x + b_3 y + c_3 t) \)
Solving the resulting equations simultaneously with the help of symbolic computation, we obtain the following:

**Case 1:**

\[ a_1 = a_2 = a_3 = c_2 = b_3 = 0, \quad c_3 = -a_3^2, \]

(7)

where \(a_3, b_1, b_2, c_3, d_3 \) (\(i = 1, 2, 3\)) are some free constants.

Substituting (5) into Eq. (3) with (7) and (6), we obtain the periodic two-soliton solution as follows

\[
\begin{align*}
  u(x, y, t) &= 2a_3 d_3 
  \sin\left(a_3 x - a_3^2 t\right) 
  \times \\
  & \left(\frac{d_1 d_3}{2} \sinh(b_3 y) - 2b_3 \sqrt{\Delta} \sinh(b_3 y - \theta)\right) 
  \left(\frac{d_2 + d_3 \cos(b_3 y) \cosh(a_3 x + a_3^2 t) + d_3 \cosh(b_3 y)}{2 \sqrt{\Delta} \sinh(b_3 y - \theta) + d_3 \cosh(b_3 y)}\right)^2
  \end{align*}
\]

(8)

where \(\theta = \frac{1}{2} \ln d_3\) and \(d_3 > 0\). Moreover, when \(\theta = \frac{1}{2} \ln (-d_3)\) and \(d_3 < 0\)

\[
\begin{align*}
  u(x, y, t) &= 2a_3 d_3 
  \sinh\left(-a_3 x + a_3^2 t\right) 
  \times \\
  & \left(\frac{d_1 d_3}{2} \sinh(b_3 y) + 2b_3 \sqrt{\Delta} \sinh(b_3 y - \theta)\right) 
  \left(\frac{d_2 + d_3 \cos(b_3 y) \cosh(a_3 x + a_3^2 t) + d_3 \cosh(b_3 y)}{2 \sqrt{\Delta} \sinh(b_3 y - \theta) + d_3 \cosh(b_3 y)}\right)^2
  \end{align*}
\]

(9)

**Case 2:**

\[ b_1 = b_2 = c_3 = a_3 = 0, \quad c_1 = -a_1^2, \quad c_2 = -a_2^2, \]

(12)

where \(a_1, a_2, b_3, d_1 \) (\(i = 1, 2, 3\)) are some free constants.

Substituting (5) into Eq. (3) with (12) and (6), we obtain the periodic two-soliton solution

\[
\begin{align*}
  u(x, y, t) &= 2b_3 d_3 
  \sinh\left(b_3 y\right) 
  \times \\
  & \left(\frac{a_{2d} d_3 \sinh(a_{2d} + a_{2d}^2) - 2a_{2d} \sqrt{\Delta} \sinh(a_{2d} + a_{2d}^2)\cosh(b_3 y)}{2 \sqrt{\Delta} \sinh(a_{2d} + a_{2d}^2) + d_3 \cosh(b_3 y)}\right)^2
  \left(\frac{d_2 + d_3 \cos(b_3 y) \cosh(a_{2d} + a_{2d}^2) + d_3 \cosh(b_3 y)}{2 \sqrt{\Delta} \sinh(a_{2d} + a_{2d}^2) + d_3 \cosh(b_3 y)}\right)^2.
  \end{align*}
\]

(13)

where \(\theta = \frac{1}{2} \ln d_3\) and \(d_3 > 0\). In the case when \(\theta = \frac{1}{2} \ln (-d_3)\) and \(d_3 < 0\), we get

\[
\begin{align*}
  u(x, y, t) &= 2b_3 d_3 
  \sinh\left(-b_3 y\right) 
  \times \\
  & \left(\frac{a_{2d} d_3 \sinh(a_{2d} + a_{2d}^2) + 2a_{2d} \sqrt{\Delta} \sinh(a_{2d} + a_{2d}^2)\cosh(b_3 y)}{2 \sqrt{\Delta} \sinh(a_{2d} + a_{2d}^2) - d_3 \cosh(b_3 y)}\right)^2
  \left(\frac{d_2 + d_3 \cos(b_3 y) \cosh(a_{2d} + a_{2d}^2) - d_3 \cosh(b_3 y)}{2 \sqrt{\Delta} \sinh(a_{2d} + a_{2d}^2) - d_3 \cosh(b_3 y)}\right)^2.
  \end{align*}
\]

(14)

**III. Conclusion**

By choosing different ansatz of extended three-soliton type, we obtain a new type of three-wave solution, periodic type of three-wave solutions, including periodic two-soliton-wave, doubly periodic solitary-wave of (2+1)-dimensional ANN system. Actually, our present short paper investigates different mechanical features of these wave solutions. It is worth noting that this is merely a beginning work, and we can obtain richer exact solutions by a more general ansatz of extended three-soliton type.

**REFERENCES**


