Some results on Interval-valued fuzzy \( BG \)-algebras

Arsham Borumand Saeid

Abstract- In this note the notion of interval-valued fuzzy \( BG \)-algebras (briefly, i-v fuzzy \( BG \)-algebras), the level and strong level \( BG \)-subalgebra is introduced. Then we state and prove some theorems which determine the relationship between these notions and \( BG \)-subalgebras. The images and inverse images of i-v fuzzy \( BG \)-subalgebras are defined, and how the homomorphic images and inverse images of i-v fuzzy \( BG \)-subalgebra becomes i-v fuzzy \( BG \)-algebras are studied.

Keywords- \( BG \)-algebra, fuzzy \( BG \)-subalgebra, interval-valued fuzzy set, interval-valued fuzzy \( BG \)-subalgebra.

I. INTRODUCTION

In 1966, Y. Imai and K. Iseki [5] introduced two classes of abstract algebras: \( BCK \)-algebras and \( BCI \)-algebras. It is known that the class of \( BCK \)-algebras is a proper subclass of the class of \( BCI \)-algebras. In [9] J. Neggers and H. S. Kim introduced the notion of \( d \)-algebras, which is generalization of \( BCK \)-algebras and investigated relation between \( d \)-algebras and \( BCK \)-algebras. Also they introduced the notion of \( BCI \)-algebras.

In [11], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set, also he constructed a method of approximate inference using his i-v fuzzy sets.

Biswas [2], defined interval-valued fuzzy subgroups and S. M. Hong et. al. applied the notion of interval-valued fuzzy to \( BCI \)-algebras.

In the present paper, we using the notion of interval-valued fuzzy set by Zadeh and introduced the concept of interval-valued fuzzy \( BG \)-subalgebras (briefly i-v fuzzy \( BG \)-subalgebras) of a \( BG \)-algebra, and study some of their properties. We prove that every \( BG \)-subalgebra of a \( BG \)-algebra \( X \) can be realized as an i-v level \( BG \)-subalgebra of an i-v fuzzy \( BG \)-subalgebra of \( X \), then we obtain some related results which have been mentioned in the abstract.

II. PRELIMINARY

Definition 2.1. [6] A \( BG \)-algebra is a non-empty set \( X \) with a consonant 0 and a binary operation \( * \) satisfying the following axioms:

I. \( x * x = 0 \),
II. \( x * 0 = x \),
III. \( (x * y) * (0 * y) = x \),
for all \( x, y \in X \).

For brevity we also call \( X \) a \( BG \)-algebra. In \( X \) we can define a binary relation \( \leq \) by \( x \leq y \) if and only if \( x * y = 0 \).

Theorem 2.2. [6] In a \( BG \)-algebra \( X \), we have the following properties:

(i) \( 0 * (0 * x) = x \),
(ii) if \( x * y = 0 \), then \( x = y \),
(iii) if \( 0 * x = 0 * y \), then \( x = y \),
(iv) \( (x * (0 * x)) * x = x \),

For all \( x, y \in X \).

A non-empty subset \( I \) of a \( BG \)-algebra \( X \) is called a subalgebra of \( X \) if \( x * y \in I \) for any \( x, y \in I \).

A mapping \( f : X \rightarrow Y \) of \( BG \)-algebras is called a \( BG \)-homomorphism if \( f(x * y) = f(x) * f(y) \) for all \( x, y \in X \).

We now review some fuzzy logic concept (see [11]). Let \( A \) be a set. A fuzzy set \( A \) in \( X \) is characterized by a membership function \( \mu_A : X \rightarrow [0, 1] \). Let \( f \) be a mapping from the set \( X \) to the set \( Y \) and let \( B \) be a fuzzy set in \( Y \) with membership function \( \mu_B \). The inverse image of \( B \), denoted \( f^{-1}(B) \), is the fuzzy set in \( X \) with membership function \( \mu_{f^{-1}(B)} \) defined by \( \mu_{f^{-1}(B)}(x) = \mu_B(f(x)) \) for all \( x \in X \).

Conversely, let \( A \) be a fuzzy set in \( X \) with membership function \( \mu_A \). Then the image of \( A \), denoted by \( f(A) \), is the fuzzy set in \( Y \) such that:

\[ \mu_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \]

A fuzzy set \( A \) in the \( BG \)-algebra \( X \) with the membership function \( \mu_A \) is said to be have the sup property if for any subset \( T \subseteq X \) there exists \( x_0 \in T \) such that \( \mu_A(x_0) = \sup_{t \in T} \mu_A(t) \).

An interval-valued fuzzy set (briefly, i-v fuzzy set) \( A \) defined on \( X \) is given by \( A = \{ (x, [\mu_A^L(x), \mu_A^H(x)]) \} \), \( \forall x \in X \).

Briefly, denoted by \( A = [\mu_A^L, \mu_A^H] \) where \( \mu_A^L \) and \( \mu_A^H \) are any two fuzzy sets in \( X \) such that \( \mu_A^L(x) \leq \mu_A^H(x) \) for all \( x \in X \).

Let \( \pi_A(x) = [\mu_A^L(x), \mu_A^H(x)] \), for all \( x \in X \) and let \( D[0, 1] \) denotes the family of all closed sub-intervals of \([0, 1]\). It is clear that if \( \mu_A^L(x) = \mu_A^H(x) = c \), where \( 0 \leq c \leq 1 \) then \( \pi_A(x) = [c, c] \) is in \( D[0, 1] \). Thus \( \pi_A(x) \in D[0, 1] \),
for all $x \in X$. Therefore the i-v fuzzy set $A$ is given by $A = \{ (x, \overline{\mu}_A(x)) \}, \forall x \in X$ where $\overline{\mu}_A : X \longrightarrow [0,1]$.

Now we define refined minimum (briefly, rmin) and order $\Rightarrow$ on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as:

$$rmin(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$$

$D_1 \leq D_2 \iff a_1 \leq a_2 \land b_1 \leq b_2$

Similarly we can define $\geq$ and $=.$

**Definition 2.3.** [1] Let $\mu$ be a fuzzy set in a $BG$-algebra. Then $\mu$ is called a fuzzy $BG$-subalgebra ($BG$-algebra) of $X$ if $\mu(x \ast y) \geq \min(\mu(x), \mu(y))$ for all $x, y \in X$.

**Proposition 2.4.** [3] Let $f$ be a $BG$-homomorphism from $X$ into $Y$ and $G$ be a fuzzy $BG$-subalgebra of $Y$ with the membership function $\mu_G$. Then the inverse image $f^{-1}(G)$ of $G$ is a fuzzy $BG$-subalgebra of $X$.

**Proposition 2.5.** [3] Let $f$ be a $BG$-homomorphism from $X$ onto $Y$ and $D$ be a fuzzy $BG$-subalgebra of $X$ with the sup property. Then the image $f(D)$ of $D$ is a fuzzy $BG$-subalgebra of $Y$.

### III. INTERVAL-VALUED FUZZY BG-ALGEBRA

From now on $X$ is a $BG$-algebra, unless otherwise is stated.

**Definition 3.1.** An i-v fuzzy set $A$ in $X$ is called an interval-valued fuzzy $BG$-subalgebra (briefly i-v fuzzy $BG$-subalgebra) of $X$ if:

$$\overline{\mu}_A(x \ast y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$$

for all $x, y \in X$.

**Example 3.2.** Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $X$ is a $BG$-algebra. Define $\overline{\mu}_A$ as:

$$\overline{\mu}_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x \in \{0, 2\} \\ [0.1, 0.6] & \text{Otherwise} \end{cases}$$

It is easy to check that $A$ is an i-v fuzzy $BG$-subalgebra of $X$.

**Lemma 3.3.** If $A$ is an i-v fuzzy $BG$-subalgebra of $X$, then for all $x \in X$

$$\overline{\mu}_A(x) \geq \overline{\mu}_A(x).$$

**Proof.** For all $x \in X$, we have

$$\overline{\mu}_A(0) \geq \overline{\mu}_A(x).$$

**Theorem 3.4.** Let $A$ be an i-v fuzzy $BG$-subalgebra of $X$. If there exists a sequence $\{\overline{x}_n\}$ in $X$, such that $\lim_{n \rightarrow \infty} \overline{\mu}_A(\overline{x}_n) = [1, 1]$. Then $\overline{\mu}_A(0) = [1, 1]$.

**Proof.** By Lemma 3.3, we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$, for all $x \in X$, thus $\overline{\mu}_A(0) \geq \overline{\mu}_A(x_n)$, for every positive integer $n$. Consider

$$[1, 1] \geq \overline{\mu}_A(0) \geq \lim_{n \rightarrow \infty} \overline{\mu}_A(x_n) = [1, 1].$$

Hence $\overline{\mu}_A(0) = [1, 1]$.

**Theorem 3.5.** An i-v fuzzy set $A = [\mu^L_A, \mu^U_A]$ in $X$ is an i-v fuzzy $BG$-subalgebra of $X$ if and only if $\mu^L_A$ and $\mu^U_A$ are fuzzy $BG$-subalgebra of $X$.

**Proof.** Let $\mu^L_A$ and $\mu^U_A$ are fuzzy $BG$-subalgebra of $X$ and $x, y \in X$, consider

$$\overline{\mu}_A(x \ast y) = \overline{\mu}_A(x \ast y) \geq \min\{\mu^L_A(x), \mu^L_A(y)\}, \min\{\mu^U_A(x), \mu^U_A(y)\}$$

$$= \min\{\mu^L_A(x), \mu^L_A(y)\}, \min\{\mu^U_A(x), \mu^U_A(y)\}$$

This completes the proof.

Conversely, suppose that $A$ is an i-v fuzzy $BG$-subalgebra of $X$. For any $x, y \in X$ we have

$$\mu^L_A(x \ast y), \mu^U_A(x \ast y) = \mu_A(x \ast y) \geq \min\{\mu^L_A(x), \mu^L_A(y)\}$$

$$\geq \min\{\mu^L_A(x), \mu^L_A(y)\}, \min\{\mu^U_A(x), \mu^U_A(y)\}$$

$$= [\mu^L_A(x), \mu^L_A(y)], [\mu^U_A(x), \mu^U_A(y)].$$

Therefore $\mu^L_A(x \ast y) \geq \min\{\mu^L_A(x), \mu^L_A(y)\}$ and $\mu^U_A(x \ast y) \geq \min\{\mu^U_A(x), \mu^U_A(y)\}$, hence we get that $\mu^L_A$ and $\mu^U_A$ are fuzzy $BG$-subalgebra of $X$.

**Theorem 3.6.** Let $A_1$ and $A_2$ are i-v fuzzy $BG$-subalgebras of $X$. Then $A_1 \cap A_2$ is an i-v fuzzy $BG$-subalgebra of $X$.

**Corollary 3.7.** Let $\{A_i | i \in I\}$ be a family of i-v fuzzy $BG$-subalgebras of $X$. Then $\bigcap_{i \in I} A_i$ is also an i-v fuzzy $BG$-subalgebra of $X$.

**Definition 3.8.** Let $A$ be an i-v fuzzy set in $X$ and $[\delta_1, \delta_2] \in D[0,1]$. Then the i-v level $BG$-subalgebra $U(A; [\delta_1, \delta_2])$ of $A$ and strong i-v $BG$-subalgebra $U(A; [\delta_1, \delta_2])$ of $X$ are defined as following:

$$U(A; [\delta_1, \delta_2]) := \{x \in X \mid \overline{\mu}_A(x) \geq [\delta_1, \delta_2]\},$$
Theorem 3.9. Let A be an i-v fuzzy BG-subalgebra of X and B be closure of image of μ_A. Then the following condition are equivalent:

(i) A is an i-v fuzzy BG-subalgebra of X.

(ii) For all [δ_1, δ_2] ∈ Im(μ_A), the nonempty level subset \( U(A; [δ_1, δ_2]) \) of A is a BG-subalgebra of X.

(iii) For all \( [δ_1, δ_2] \in Im(μ_A) \setminus B \), the nonempty strong level subset \( U(A; [δ_1, δ_2]) \) of A is a BG-subalgebra of X.

(iv) For all \( [δ_1, δ_2] \in D[0, 1] \), the nonempty strong level subset \( U(A; [δ_1, δ_2]) \) of A is a BG-subalgebra of X.

Proof. (i → iv) Let A be an i-v fuzzy BG-subalgebra of X, \( [δ_1, δ_2] \in D[0, 1] \) and \( x, y \in U(A; [δ_1, δ_2]) \), then we have \( \pi_A(x + y) ≥ \min\{\pi_A(x), \pi_A(y)\} ≥ \min\{\pi_A(δ_1), \pi_A(δ_2)\} = [δ_1, δ_2] \). Thus \( x + y \in U(A; [δ_1, δ_2]) \). Hence \( U(A; [δ_1, δ_2]) \) is a BG-subalgebra of X.

(iv → iii) It is clear.

(iii → ii) Let \([δ_1, δ_2] \in Im(μ_A). Then U(A; [δ_1, δ_2]) \) is a nonempty. Since \( U(A; [δ_1, δ_2]) = \bigcap_{[δ_1, δ_2] > [α_1, α_2]} U(A; [δ_1, δ_2]) \), where \([α_1, α_2] \in Im(μ_A) \setminus B \). Then by (iii) and Corollary 3.8, \( U(A; [δ_1, δ_2]) \) is a BG-subalgebra of X.

(ii → v) Let \([δ_1, δ_2] \in D[0, 1] \) and \( U(A; [δ_1, δ_2]) \) be nonempty. Suppose \( x, y \in U(A; [δ_1, δ_2]) \). Let \( [δ_1, δ_2] = \min\{μ_A(x), μ_A(y)\} \), it is clear that \( [δ_1, δ_2] = \min\{μ_A(x), μ_A(y)\} \). Thus \( x, y \in U(A; [δ_1, δ_2]) \) and \([δ_1, δ_2] \in Im(μ_A). By (ii) \( U(A; [δ_1, δ_2]) \) is a BG-subalgebra of X, hence \( x + y \in U(A; [δ_1, δ_2]) \). Then we have \( \pi_A(x + y) ≥ \min\{μ_A(x), μ_A(y)\} \). Therefore \( x + y \in U(A; [δ_1, δ_2]) \). Then \( U(A; [δ_1, δ_2]) \) is a BG-subalgebra of X.

(v → i) Assume that the nonempty set \( U(A; [δ_1, δ_2]) \) is a BG-subalgebra of X, for every \( [δ_1, δ_2] \in D[0, 1] \). In contrary, let \( x_0, y_0 \in X \) be such that

\[
\pi_A(x_0 + y_0) < \min\{\pi_A(x_0), \pi_A(y_0)\}.
\]

Let \( \pi_A(x_0) = [γ_1, γ_2] \), \( \pi_A(y_0) = [γ_3, γ_4] \) and \( \pi_A(x_0 + y_0) = [δ_1, δ_2] \). Then

\[
[δ_1, δ_2] < \min\{[γ_1, γ_2], [γ_3, γ_4]\} = [min\{γ_1, γ_3\}, min\{γ_2, γ_4\}].
\]

So \( δ_1 < min\{γ_1, γ_3\} \) and \( δ_2 < min\{γ_2, γ_4\} \). Consider

\[
[λ_1, λ_2] = \frac{1}{2}(δ_1, δ_2) + min\{γ_1, γ_3\}, min\{γ_2, γ_4\}) = \frac{1}{2}(δ_1 + min\{γ_1, γ_3\}), \frac{1}{2}(δ_2 + min\{γ_2, γ_4\}]
\]

Therefore

\[
min\{γ_1, γ_3\} > λ_1 = \frac{1}{2}(δ_1 + min\{γ_1, γ_3\}) > δ_1
\]

\[
min\{γ_2, γ_4\} > λ_2 = \frac{1}{2}(δ_2 + min\{γ_2, γ_4\}) > δ_2
\]

Hence

\[
[min\{γ_1, γ_3\}, min\{γ_2, γ_4\}] > [λ_1, λ_2] > [δ_1, δ_2] = \pi_A(x_0 + y_0)
\]

so that \( x_0 + y_0 \notin U(A; [δ_1, δ_2]) \)
which is a contradiction since

\[
\pi_A(x_0) = [γ_1, γ_2] ≥ [\min\{γ_1, γ_3\}, \min\{γ_2, γ_4\}] > [λ_1, λ_2]
\]

\[
\pi_A(y_0) = [γ_3, γ_4] ≥ [\min\{γ_1, γ_3\}, \min\{γ_2, γ_4\}] > [λ_1, λ_2]
\]

imply that \( x_0, y_0 \in U(A; [δ_1, δ_2]) \). Thus \( \pi_A(x + y) ≥ \min\{π_A(x), π_A(y)\} \)
for all \( x, y \in X \). Which completes the proof.

Theorem 3.10. Each BG-subalgebra of X is an i-v level BG-subalgebra of an i-v fuzzy BG-subalgebra of X.

Proof. Let Y be a BG-subalgebra of X, and A be an i-v fuzzy set on X defined by

\[
\pi_A(x) = \begin{cases} [α_1, α_2] & \text{if } x \in Y \\ [0, 0] & \text{Otherwise} \end{cases}
\]

where \( α_1, α_2 \in [0, 1] \) with \( α_1 < α_2 \). It is clear that \( U(A; [α_1, α_2]) = Y \). Let \( x, y \in X \). We consider the following cases:

- **case 1** If \( x, y \in Y \), then \( \pi_A(x + y) = [α_1, α_2] = \min\{[α_1, α_2], [α_1, α_2]\} = \min\{π_A(x), π_A(y)\} \).
- **case 2** If \( x \notin Y \) and \( y \in Y \), then \( \pi_A(x) = [0, 0] = \pi_A(y) \). And so \( \pi_A(x + y) = [0, 0] = \min\{[0, 0], [0, 0]\} = \min\{π_A(x), π_A(y)\} \).
- **case 3** If \( x \in Y \) and \( y \notin Y \), and \( \pi_A(x) = [α_1, α_2] \) and \( \pi_A(y) = [0, 0] \). Thus \( \pi_A(x + y) ≥ [0, 0] = \min\{[α_1, α_2], [0, 0]\} = \min\{π_A(x), π_A(y)\} \).
- **case 4** If \( y \in Y \) and \( x \notin Y \), then by the same argument as in case 3, we can conclude that \( \pi_A(x + y) ≥ \min\{π_A(x), π_A(y)\} \).

Therefore A is an i-v fuzzy BG-subalgebra of X.

**Theorem 3.11.** If A is an i-v fuzzy BG-subalgebra of X, then

\[
X_{π_A} := \{x \in X \mid π_A(x) = π_A(0)\}
\]

is a BG-subalgebra of X.

**Definition 3.12.** [2] Let f be a mapping from the set X into a set Y. Let B be an i-v fuzzy set in Y. Then the inverse image of B, denoted by \( f^{-1}[B] \), is the i-v fuzzy set in X with the
membership function given by \( \overline{\mu}_{f^{-1}[B]}(x) = \overline{\mu}_B(f(x)) \), for all \( x \in X \).

**Lemma 3.13.** [2] Let \( f \) be a mapping from the set \( X \) into a set \( Y \). Let \( m = [m^L, m^U] \) and \( n = [n^L, n^U] \) be \( i-v \) fuzzy sets in \( X \) and \( Y \) respectively. Then
(i) \( f^{-1}(m) = [f^{-1}(n^L), f^{-1}(n^U)] \),
(ii) \( f(m) = [f(m^L), f(m^U)] \).

**Proposition 3.14.** Let \( f \) be a \( BG \)-homomorphism from \( X \) into \( Y \) and \( G \) be an \( i-v \) fuzzy \( BG \)-subalgebra of \( Y \) with the membership function \( \mu_G \). Then the inverse image \( f^{-1}[G] \) of \( G \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( X \).

**Proof.** Since \( B = [\mu_B^L, \mu_B^U] \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( Y \), by Theorem 3.5, we get that \( \mu_B^L \) and \( \mu_B^U \) are fuzzy \( BG \)-subalgebra of \( Y \). By Proposition 2.4, \( f^{-1}[\mu_B^L] \) and \( f^{-1}[\mu_B^U] \) are fuzzy \( BG \)-subalgebra of \( X \), by above lemma and Theorem 3.5, we can conclude that \( f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)] \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( X \).

**Definition 3.15.** [2] Let \( f \) be a mapping from the set \( X \) into a set \( Y \), and \( A \) be an \( i-v \) fuzzy set in \( X \) with membership function \( \mu_A \). Then the image of \( A \), denoted by \( f[A] \), is the \( i-v \) fuzzy set in \( Y \) with membership function defined by:
\[
\overline{\mu}_{f[A]}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \overline{\mu}_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ [0,0] & \text{otherwise} \end{cases}
\]

Where \( f^{-1}(y) = \{ x \mid f(x) = y \} \).

**Theorem 3.16.** Let \( f \) be a \( BG \)-homomorphism from \( X \) onto \( Y \). If \( A \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( X \), then the image \( f[A] \) of \( A \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( Y \).

**Proof.** Assume that \( A \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( X \), then \( A = [\mu_A^L, \mu_A^U] \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( X \) if and only if \( \mu_B^L \) and \( \mu_B^U \) are fuzzy \( BG \)-subalgebra of \( X \). By Proposition 2.5, \( f[\mu_A^L] \) and \( f[\mu_A^U] \) are fuzzy \( BG \)-subalgebra of \( Y \). By Lemma 3.13, and Theorem 3.5, we can conclude that \( f[A] = [f[\mu_A^L], f[\mu_A^U]] \) is an \( i-v \) fuzzy \( BG \)-subalgebra of \( Y \).

**REFERENCES**