Existence of Solution for Four-Point Boundary Value Problems of Second-Order Impulsive Differential Equations (II)

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Abstract—In this paper, we study the existence of solution of the four-point boundary value problem for second-order differential equations with impulses by using Leray-Schauder theory:

\[
\begin{aligned}
&x''(t) = f(t, x(t), x'(t)), \quad t \in [0, 1], t \neq t_k, k = 1, 2, \ldots, m \\
&\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m \\
&\Delta x'(t_k) = T_k(x(t_k), x'(t_k)), \quad k = 1, 2, \ldots, m \\
&x(0) = \alpha x(\xi), \quad x(1) = \beta x(\eta),
\end{aligned}
\]

where \(0 < \xi < \eta < 1, \alpha \geq 0, \beta \geq 0\) and \(\alpha \neq 1, \beta \neq 1\), also \(\alpha \beta = 0\) implies \(\alpha \neq \beta\), \(f \in C^1([0, 1] \times \mathbb{R}^2, \mathbb{R})\), \(I_k \in C(\mathbb{R}, \mathbb{R})\), \(T_k \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})\) and \(x(0)\) is continuous at \(x(0)\). Moreover, for convenience sake, we set \(x(t) = x(t)\). So, for \(x \in PC^1(J, \mathbb{R})\), we have \(x' \in PC^1(J, \mathbb{R})\).

For \(m \in PC^1(J, \mathbb{R})\), by virtue of the mean value theorem ([6]), we know that the left derivation \(x'_-(t_k)\) exists and \(x'_-(t_k) = x'(t_k)\). In (E) and what follows, it is understood that \(x'(t_k) = x'(t_k)\). So, for \(x \in PC^1(J, \mathbb{R})\), we have \(x' \in PC^1(J, \mathbb{R})\).

II. Preliminary Lemmas

Lemma 2.1 ([3]) \(H \subset PC^1(J, \mathbb{R})\) is a relatively compact set if and only if both \(x(t)\) and \(x'(t)\) are uniformly bounded on \(J\) and equicontinuous on every \(J_k(k = 1, 2, \ldots, m)\) for any \(x \in H\).

Lemma 2.2 ([3]) If \(x \in PC^1(J, \mathbb{R}) \cap C^2(J', \mathbb{R})\) satisfies \(x'' = f(t, x(t), x'(t))\), \(t \neq t_k, k = 1, 2, \ldots, m\), then

\[
x(t) = x(0) + \int_0^t f(s, x(s), x'(s))ds + \sum_{0 < t_k < t} (x(t_k^-) - x(t_k)), \quad \forall t \in J, \quad (1)
\]

\[
x(t) = x(0) + x'(0)t + \int_0^t (t-s)f(s, x(s), x'(s))ds + \sum_{0 < t_k < t} (x(t_k^-) - x(t_k)) + \sum_{0 < t_k < t} (x(t_k^-) - x(t_k))(t-t_k), \quad \forall t \in J. \quad (2)
\]
Lemma 2.3  let x ∈ PC[1, R] ∩ C[2, R] be a solution of BVP(E), if and only if x ∈ PC[1, R] ∩ C[2, R] is a solution of the following implication integral-differential equations:

\[ x'(t) = x'(0) + \int_0^t f(s, x(s), x'(s))ds + \sum_{0 < t_k < t} T_k(x(t_k), x'(t_k)). \]  

(3)

\[ x(t) = x(0) + x'(0)t + \int_0^t (t - s)f(s, x(s), x'(s))ds + \sum_{0 < t_k < t} I_k(x(t_k)) \]

\[ x(t) = x(0) + \sum_{0 < t_k < t} I_k(x(t_k)) \]

(4)

where

\[ x(0) = \alpha(\beta - 1) \int_0^x f(s, x(s), x'(s))ds \]

\[ + \sum_{0 < t_k < x} I_k(x(t_k)) \]

\[ + \beta - \sum_{0 < t_k < x} \sum_{0 < t_k < t} I_k(x(t_k)) \]

\[ + \int_0^x (\eta - s)f(s, x(s), x'(s))ds \]

\[ + \sum_{0 < t_k < x} \int_0^x (\eta - t_k)f(s, x(s), x'(s))ds \]

\[ - \frac{1}{1 - \beta} \sum_{k=1}^m I_k(x(t_k)) \]

\[ + \sum_{k=1}^m (1 - t_k)f(s, x(s), x'(s))ds \]

\[ + \sum_{k=1}^m (1 - t_k)f(s, x(s), x'(s))ds \]

\[ + \sum_{k=1}^m (1 - t_k)f(s, x(s), x'(s))ds \]

(5)

Proof If x(t) is a solution of BVP(E), then

\[ x(t) = x(0) + x'(0)t + \int_0^t (t - s)f(s, x(s), x'(s))ds \]

\[ + \sum_{0 < t_k < t} (x(t_k^+) - x(t_k)) \]

\[ + \sum_{0 < t_k < t} (x(t_k^+) - x(t_k))(t - t_k), \quad \forall t \in J. \]

In view of \( x'(0) = \alpha x'(x), x(1) = \beta x(\eta) \), we easily obtain (5) and (6). The combination of (1), (2), (5) and (6), yields (3) and (4).

On the other hand, assume that x ∈ PC[1, R] is a solution of Eqs (3) and (4). It is clear that x'(0) = \( \alpha x'(x), x(1) = \beta x(\eta), \Delta x(t_k) = I_k(x(t_k)). \) By performing differentiation of (4) twice, we get

\[ x'(t) = x'(0) + \int_0^t f(s, x(s), x'(s))ds \]

\[ + \sum_{0 < t_k < t} T_k(x(t_k), x'(t_k)), \quad t \neq t_k, \]

(6)

which implies x ∈ PC[1, R] and \( \Delta x(t_k) = T_k(x(t_k), x'(t_k)) \). Therefore x ∈ PC[1, R] and x is a solution of BVP(E).

Lemma 2.4 Operator A is a completely continuous one mapping PC[1, R] into PC[1, R].

Proof By (3), we get

\[ (Ax)'(t) = x'(0)t + \int_0^t f(s, x(s), x'(s))ds + \sum_{0 < t_k < t} I_k(x(t_k)) \]

\[ + \sum_{0 < t_k < t} T_k(x(t_k), x'(t_k)), \]

(8)

where x'(0) is defined by (6).

From (7) and (8), it is easy to see that A is continuous operator from PC[1, R] into PC[1, R]. Let S be a bounded set of PC[1, R], then A(S) ⊆ PC[1, R] is bounded and the elements of A(S) and their derivatives are all uniformly bounded on J and equicontinuous on each \( J_k = \{ x \in J | x = \lambda \}, \) \( \lambda = 1 \).

Therefore, A(S) is a relatively compact set of PC[1, R] by Lemma 2.1. So, operator A is completely continuous.

Lemma 2.5 (7)

Let X be a real normed linear space and T : X → X be a compact operator. Suppose that \( \Omega = \bigcup_{\varepsilon \in (0, 1)} \Omega_x \) is a bounded set, where \( \Omega_x = \{ x \in X | x = \lambda \varepsilon x \} \), then the equation \( x = \lambda T x \) has at least a solution when \( \lambda = 1 \).

III. Existence results for BVP(E)

In this section, we will prove existence results for BVP(E) in following cases:

(i) \( \alpha > 1, \beta > 1, \beta \eta < 1 \).

(ii) \( 0 \leq \alpha < 1, \beta > 1, \beta \eta > 1 \).

(iii) \( \alpha > 1, 0 \leq \beta < 1 \).

(iv) \( \alpha > 1, \beta > 1, \beta \eta > 1 \).

(v) \( 0 \leq \alpha < 1, \beta > 1, \beta \eta < 1 \).

(vi) \( 0 \leq \alpha < 1, 0 \leq \beta < 1 \).
Let \( f : [0, 1] \times R^2 \rightarrow R \) be a continuous function, \( I_k \in C[R, R], \lambda_k \in C[R \times R, R] \). Assume that 

\((H_1)\) There exist functions \( p, q, r \) in \( L^1[0, 1] \), such that for all \( (x, y) \in R^2, t \in [0, 1] \)

\[ |f(t, x, y)| \leq p(t)|x| + q(t)|y| + r(t). \quad (9) \]

\((H_2)\) There exist constants \( 0 \leq \beta_k < 1, M_k \geq 0 \) satisfying \( |I_k(x)| \leq M_k \) for any \( x \in R \), and

\[ \lim_{|x| + |y| \to \infty} \frac{|I_k(x, y)|}{|x| + |y|} = \beta_k, \quad k = 1, 2, \ldots, m. \quad (10) \]

\((H_3)\) There exist constants \( \alpha, \beta, \eta \) satisfying (i).

Then BVP(E) has at least one solution in \( PC^1[J, R] \cap C^2[J', R] \) provided that

\[ \|p\|_1 + \|q\|_1 + \sum_{k=1}^m 2\beta_k < \frac{\Lambda_2}{\beta(2\alpha - 1 - \eta)}. \quad (11) \]

**Proof**

We will verify the statement of the set of all possible solution of the family of equations:

\[
\begin{align*}
\lambda x'(t) &= \lambda f(t, x(t), x'(t)), \quad t \neq t_k, k = 1, 2, \ldots, m \\
\Delta x_k &= \lambda I_k(x(t_k)), \quad k = 1, 2, \ldots, m \\
\Delta x_k' &= \lambda \lambda_k (x(t_k), x'(t_k)), \quad k = 1, 2, \ldots, m \\
x'(0) &= \alpha x'(\xi), \quad x(1) = \beta x(\eta),
\end{align*}
\]

is prior bounded in \( PC^1[J, R] \cap C^2[J', R] \) by a constant independent of \( \lambda \in (0, 1) \).

If \( x \in PC^1[J, R] \cap C^2[J', R] \), with \( x'(0) = \alpha x'(\xi), x(1) = \beta x(\eta) \), from the previous equation, we have

\[ |x(t)| \leq |x(0)| + \int_0^1 |x'(s)| ds + \sum_{k=1}^m M_k. \quad (12) \]

From (3) and (6), we get

\[ x'(t) = x'(0) + \int_0^t f(s, x(s), x'(s)) ds + \sum_{0 < t_k < t} T_k(x(t_k), x'(t_k)). \]

That is

\[ |x'(t)| \leq \frac{\alpha}{\alpha - 1} \left[ \int_0^t |f(s, x(s), x'(s))| ds + \sum_{0 < t_k < t} |T_k(x(t_k), x'(t_k))| \right] + \int_0^t |x'(s)| ds + \sum_{0 < t_k < t} |T_k(x(t_k), x'(t_k))| + \sum_{k=1}^m |I_k(x(t_k))| \]

\[ \leq \frac{2\alpha - 1}{\alpha - 1} \left[ \int_0^1 |f(s, x(s), x'(s))| ds + \sum_{0 < t_k < t} |T_k(x(t_k), x'(t_k))| \right] + \sum_{k=1}^m |I_k(x(t_k))| \]

The combination of (4), (5), and (12), yields

\[ |x(t)| \leq \frac{\alpha(1 - \beta)}{\alpha - 1} \left[ \int_0^t |f(s, x(s), x'(s))| ds + \sum_{0 < t_k < t} |T_k(x(t_k), x'(t_k))| \right] + \frac{\beta}{\beta - 1} \left[ \int_{t_k}^t |f(s, x(s), x'(s))| ds + \sum_{0 < t_k < t} |T_k(x(t_k), x'(t_k))| \right] + \frac{\beta}{\beta - 1} \sum_{k=1}^m |I_k(x(t_k))| \]

\[ \leq \frac{\beta(2\alpha - 1 - \eta)}{\alpha - 1} \left[ \int_0^1 |f(s, x(s), x'(s))| ds + \sum_{k=1}^m |I_k(x(t_k))| \right] \]
\[
+ \sum_{k=1}^{m} [T_k(x(t_k), x'(t_k))] \\
+ \frac{2\beta}{\beta - 1} \sum_{k=1}^{m} [I_k(x(t_k))].
\]

By (11), set \( ||p||_1 + ||q||_1 + \sum_{k=1}^{m} 2\beta_k = M \), then exists \( \varepsilon_0 > 0 \), such that

\[
m\varepsilon < \frac{1}{4} \frac{\Lambda_2}{\beta(2\alpha - 1 - \eta)} - M, \quad \forall \varepsilon < \varepsilon_0.
\]

From (10), we know exists an \( M(\varepsilon) \) for any \( \varepsilon \) defined above such that \( |x| + |x'| \geq M(\varepsilon), |T_k(x(t_k), x'(t_k))| \leq (\beta_k + \varepsilon)(|x| + |x'|) \). Now, we assume that \( |x'(t)| \) is unbounded, that is there exists some \( \lambda \in (0,1) \) such that \( |x'| > \max \{ M(\varepsilon), \overline{M}_2 \} \), where

\[
\overline{M}_2 = \left[ 1 - \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \right] \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right]^{1 - \lambda}.
\]

where

\[
M_{21} = \frac{\beta(2\alpha - 1)(2\alpha - \eta - 1)}{\Lambda_2(\alpha - 1)} ||r||_1 \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right]^{1 - \lambda}.
\]

\[
M_{22} = \frac{2\beta}{\beta - 1} \sum_{k=1}^{m} M_k + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} ||r||_1 \left[ 1 - \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \right] \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right]^{1 - \lambda}.
\]

Hence

\[
|x'(t)| \leq \frac{2\alpha - 1}{\alpha - 1} \left[ ||p||_1 ||x|| + ||q||_1 ||x'|| + ||r||_1 \right] + \sum_{k=1}^{m} (\beta_k + \varepsilon)(|x| + |x'|)
= \frac{2\alpha - 1}{\alpha - 1} \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] [x'] + \frac{2\alpha - 1}{\alpha - 1} ||r||_1 (15)
\]

and

\[
|x(t)| \leq \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \left[ ||p||_1 ||x|| + ||q||_1 ||x'|| + ||r||_1 \right] + \sum_{k=1}^{m} (\beta_k + \varepsilon)(|x| + |x'|) + \frac{2\beta}{\beta - 1} \sum_{k=1}^{m} M_k
= \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] |x| + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \left[ ||q||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] |x'| + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} ||r||_1 + \frac{2\beta}{\beta - 1} \sum_{k=1}^{m} M_k. (16)
\]

The combination of (15) and (16), yields

\[
|x'(t)| \leq \frac{2\alpha - 1}{\alpha - 1} \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] |x| + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \left[ ||q||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] |x'| + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} ||r||_1 + \frac{2\beta}{\beta - 1} \sum_{k=1}^{m} M_k.
\]

where \( d = \alpha - 1 - (2\alpha - 1) \left[ ||q||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] \), and

\[
|x(t)| \leq 1 - \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \left[ ||p||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] - \frac{2\alpha - 1}{\alpha - 1} \left[ ||q||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] |x| + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} \left[ ||r||_1 + \sum_{k=1}^{m} (\beta_k + \varepsilon) \right] |x'| + \frac{\beta(2\alpha - \eta - 1)}{\Lambda_2} ||r||_1 + \frac{2\beta}{\beta - 1} \sum_{k=1}^{m} M_k. (17)
\]

It lead to \( \|x\|_C \leq \overline{M}_2 \) for any \( \lambda \in (0, 1) \), a contradiction. It is now immediate from (17), that \( \|x'||_C \) is also bounded, so is \( \|x''\|_C \). This completes the proof.

By using the same method as the proof of Theorem 3.1, we can show that the following Theorem 3.2 - Theorem 3.6 hold.

**Theorem 3.2** Let \( f : [0, 1] \times R^2 \rightarrow R \) be a continuous function, \( I_k \in C[R, R], T_k \in C[R \times R, R]. \) Assume that the conditions \( \lambda_1 \) and \( \lambda_2 \) of Theorem 3.1 are satisfied and \( \lambda_4 \) There exist constants \( \alpha, \beta, \eta \) satisfying (ii). Then BVP(E) has at least one solution in \( PC^1[I, R] \) and \( \|C^2[J', R] \) provided that

\[
||p||_1 + ||q||_1 + \sum_{k=1}^{m} 2\beta_k < -\frac{\Lambda_2}{\beta(\alpha - 2\beta)}. \lambda_4
\]

**Theorem 3.3** Let \( f : [0, 1] \times R^2 \rightarrow R \) be a continuous function, \( I_k \in C[R, R], T_k \in C[R \times R, R]. \) Assume that the conditions \( \lambda_1 \) and \( \lambda_2 \) of Theorem 3.1 are satisfied and \( \lambda_4 \). There exist constants \( \alpha, \beta, \eta \) satisfying (ii). Then BVP(E) has at least one solution in \( PC^1[I, R] \) and \( \|C^2[J', R] \) provided that

\[
||p||_1 + ||q||_1 + \sum_{k=1}^{m} 2\beta_k < -\frac{\Lambda_2}{4\alpha \beta - 2\beta}. \lambda_4
\]
Theorem 3.4 Let $f : [0, 1] \times R^2 \to R$ be a continuous function, $I_k \in C[R, R]$, $T_k \in C[R \times R, R]$. Assume that the conditions $(H_1)$ and $(H_2)$ of Theorem 3.1 are satisfied and $(H_6)$ There exist constants $\alpha$, $\beta$, $\eta$ satisfying (iv).

Then BVP(E) has at least one solution in $PC^1[J, R] \cap C^2[J', R]$ provided that

$$
\|p\|_1 + \|q\|_1 + \sum_{k=1}^m 2\beta_k < \frac{\Lambda_2}{2\alpha\beta - 2\alpha - \beta}.
$$

Theorem 3.5 Let $f : [0, 1] \times R^2 \to R$ be a continuous function, $I_k \in C[R, R]$, $T_k \in C[R \times R, R]$. Assume that the conditions $(H_1)$ and $(H_2)$ of Theorem 3.1 are satisfied and $(H_7)$ There exist constants $\alpha$, $\beta$, $\eta$ satisfying (v).

Then BVP(E) has at least one solution in $PC^1[J, R] \cap C^2[J', R]$ provided that

$$
\|p\|_1 + \|q\|_1 + \sum_{k=1}^m 2\beta_k < \frac{\Lambda_2}{\beta(\eta + 1 - 2\alpha\beta/\beta)}.
$$

Theorem 3.6 Let $f : [0, 1] \times R^2 \to R$ be a continuous function, $I_k \in C[R, R]$, $T_k \in C[R \times R, R]$. Assume that the conditions $(H_1)$ and $(H_2)$ of Theorem 3.1 are satisfied and $(H_8)$ There exist constants $\alpha$, $\beta$, $\eta$ satisfying (vi).

Then BVP(E) has at least one solution in $PC^1[J, R] \cap C^2[J', R]$ provided that

$$
\|p\|_1 + \|q\|_1 + \sum_{k=1}^m 2\beta_k < \frac{\Lambda_2}{\beta\eta + 2 - 2\alpha\beta - \beta}.
$$

Example Consider the BVP:

$$
\begin{align*}
  x''(t) &= \frac{1}{30}x(t) + \frac{1}{40}y(t) + 3\ln(1 + t^2), \quad t \neq 1/2, \\
  \Delta x(1/2) &= \cos^2 x(1/2), \\
  \Delta y(1/2) &= \frac{1}{3}(x(1/2) - y(1/2)), \\
  x'(0) &= 2x'(\xi), \quad x(1) = \frac{1}{2}x(1/2), \\
  \end{align*}
$$

(E')

where $f \in C[J \times R^2, R]$, $I_1 \in C[R, R]$, $T_1 \in C[R^2, R]$. Note that $m = 1$, $t_1 = 1/2$, $\alpha = 2$, $\xi \in (0, 1/4)$, $\beta = 1/2$, $\eta = 1/4$.

Furthermore

$$
|f(t, x, y)| \leq \frac{1}{30}|x| + \frac{1}{40}|y| + 3\ln(1 + t^2),
$$

$$
|I_1(x)| = |\cos^2 x| \leq 1, \\
|T_1(x, y)| \leq \frac{1}{40}(|x| + |y|), \quad \forall t \in J, \ x, y \in R.
$$

Therefore

$$
\frac{\Lambda_2}{\beta(2\alpha - 1 - \eta)} = \frac{1}{8}, \quad \beta_1 = \frac{1}{40} \|p\|_1 + \|q\|_1 + 2\beta_1 = \frac{13}{120} < \frac{1}{8}.
$$

Hence from Theorem 3.1, there exists a solution $x \in PC^1[J, R] \cap C^2[J', R]$ to $(E')$, where $(J' = [0, 1] \cup (1/2, 1)])$.

REFERENCES


