Existence of solution for singular two-point boundary value problem of second-order differential equation

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Abstract—In this paper, by constructing a special set and utilizing fixed point theory in coin, we study the existence of solution of singular two points boundary value problem for second-order differential equation, which improved and generalize the result of related paper.

Keywords—singular differential equation, boundary value problem, coin, fixed point theory.

I. INTRODUCTION

THE theory of singular differential equation is emerging as an important area of investigation since it is much richer than the corresponding theory of concerning equation without singular. Recently, some existence results concerning the general boundary value problem of singular differential equation have been obtained ([1-3]). In thesis [4], Wang proved the existence of solution for the general boundary value problem for the second-order differential equation:

\[ x''(t) = f(t, x(t)), \]
\[ \alpha x(0) - \beta x'(0) = 0, \]
\[ \gamma x(1) + \delta x'(1) = 0, \]

however, the f is a function without the term x', motivated by the work of Wang, in this paper we study the following second-order differential equation

\[
\begin{align*}
\left\{ \begin{array}{l}
x''(t) = f(t, x(t)) \quad \text{in} \quad x(t), \\
\alpha x(t) - \beta x'(0) = 0, \\
\gamma x(1) + \delta x'(1) = 0, \\
\end{array} \right.
\end{align*}
\]

(1)

where f ∈ C[J × R², R], J = [0, 1], f(·) may be singular at t = 0, 1, that is \( \lim_{t \to 0} ||f(t, ·)|| = \infty, \lim_{t \to 1} ||f(t, ·)|| = \infty, \) let C[J, R] = \{ x : J → R | x continuous in J \}, and C¹[J, R] = \{ x ∈ C[J, R] | x' is continuous in J \}. It is easy to prove that C[J, R] is a Banach space with norm \( ||x|| = \max_{t \in J} ||x(t)|| \), x(t), C¹[J, R] is also a Banach space with norm \( ||x|| = \max_{t \in J} ||x(t)||, ||x'(t)|| \). A map x ∈ C¹[J, R] ∩ C²[J, R] is called a solution of (1) if it satisfies all equations of (1). For convenience sake, we list the definition and preliminary lemmas.

Definition 1.1 Let E be a real Banach space, if P is a convex closed set and satisfied the following conditions: \( (1) x ∈ P, λ ≥ 0 \Rightarrow x ∈ P; (2) x ∈ P, -x ∈ P \Rightarrow x = 0, \theta \) is element zero of E, we call P is a coin in E.

Lemma 1.1 Assume that \( \Delta = \alpha \gamma + \alpha \delta + \beta \gamma, \) \( G(t, s) = \begin{cases} \frac{1}{\alpha}((\beta + \alpha t)(\delta + \gamma(1 - s))), 0 ≤ t ≤ s ≤ 1, \\
\frac{1}{\alpha}((\beta + \alpha s)(\delta + \gamma(1 - t))), 0 ≤ s ≤ t ≤ 1. \end{cases} \)

Let \( y ∈ C[J, E] \), then

\[ \begin{cases} x''(t) = y, \\
\alpha x(t) - \beta x'(0) = 0, \\
\gamma x(1) + \delta x'(1) = 0, \end{cases} \]

has a unique solution in \( C^2[J, R] \) given by \( x(t) = \int_0^t G(t, s)y(s)ds \). We also easily obtain \( G(t, s) ≤ G(s, s) = e(s) \) and \( G(t, s) ≤ G(t, t) = e(t), 0 < t, s < 1 \).

Lemma 1.2 ([5]) Let K be a coin in a Banach space \( 0 < r < R, B(0, R) = \{ x ∈ K ||x|| ≤ R \}, K_R = \overline{B(0, R)} \backslash K \). Suppose that operator \( A : K_R \to K \) is a completely continuous such that following conditions are satisfied: for \( x ∈ K, ||x|| = R, ||A(x)|| ≤ ||x|| \), and for \( x ∈ K, ||x|| = r \Rightarrow ||A(x)|| ≤ ||x|| \), then A has a fixed point in \( K_R \backslash K \).

II. CONCLUSION

Theorem 2.1 Let \( f : (0, 1) × R^2 → R \) be a continuous function, suppose that the following conditions are satisfied:

\( (H_1) \) For all \( (x, y) ∈ R^2 \) and \( t ∈ (0, 1) \), \( f(t, x, y) = p_1(t)q_1(x) + p_2(t)q_2(y) + r(t) \), where \( p_i, q_i ∈ C[(0, 1), R] \), and \( \int_0^1 e(t)p_i(t)ds < +∞ (i = 1, 2) \), r(t) ∈ C[J, R] and \( \int_0^1 e(t)r(t)ds < +∞ \).

\( (H_2) \) There exist constant \( a > 0 \) such that

\[ \begin{align*}
\int_0^1 e(t)p_1(t)\max_{x \in [0, a]} q_1(x)dt \\
+ \int_0^1 e(t)p_2(t)\max_{y \in [0, a]} q_2(y)dt \\
+ \int_0^1 e(t)r(t)dt < a,
\end{align*} \]

where

\[ G_a(t) = \begin{cases} at, 0 ≤ t ≤ \frac{1}{2}, \\
\frac{1}{3} - \frac{1}{2}t, \frac{1}{2} ≤ t ≤ 1. \end{cases} \]
(H₃) There exist constant b ∈ (0, α) such that
\[ \| \int_{0}^{1} M \varepsilon(t) \min_{x \in [g(t), b]} f(t, x, x') dt \| \geq b, \]
where \( M = \min \{ \beta + \alpha a, \delta + \theta \gamma : \beta + a \delta + \gamma \} \), then the problem (1) has at least one solution \( x \in C^2[0, 1] \).

Proof: Operator \( A : C^1[0, 1] \rightarrow C^1[0, 1] \) is defined as follows: \( Ax(t) = \frac{1}{0} G(t, s) f(s, x(s), x'(s)) ds \). It is easy to see if \( x(t) \) is a solution of (1) if and only if \( x(t) \) is a fixed point of the operator \( Ax(t) = x(t) \). Let \( E = C^1[0, 1] \), \( K = \{ x \in C^1[0, 1], \min_{t \in (0, 1)} x(t) \geq M \| x \| \} \), it is easy to see \( K \) is a coin in \( E \).

First, we’ll prove that operator \( A \) is a completely continuous one mapping \( K \) into \( K \). Let \( D \) be an arbitrary bounded set of \( C^1[0, 1] \), for all \( x \in D \), \( \sup_{t \in [0, 1]} \| x(t) \| \) is defined as follows: \( \sup_{t \in [0, 1]} \| x(t) \| = \sup_{t \in [0, 1]} \{ \| x \| \} \) (i = 1, 2).

Therefore,
\[ Ax(t) = \frac{1}{0} G(t, s) f(s, x(s), x'(s)) ds \]
\[ \leq \frac{1}{0} e(s)p_1(1)q_1(x(s)) ds \]
\[ + \frac{1}{0} e(s)p_2(1)q_2(x'(s)) ds + \frac{1}{0} e(s)r(s) ds, \]

\[ \| Ax \| \leq \frac{1}{0} e(s)f(s, x(s), x'(s)) ds \]
\[ \leq \sup_{t \in [0, 1]} \| x(t) \| \frac{1}{0} e(s)p_1(1) ds \]
\[ + \sup_{t \in [0, 1]} \| x(t) \| \frac{1}{0} e(s)p_2(1) ds + \frac{1}{0} e(s)r(s) ds. \]

In view of (H₁), we easily obtain that \( AD \) are all uniformly bounded on \( J \). On the other hand, according to Lebesque control collect theory, we know that \( A \) is continuous on \( J \).

For \( -\frac{\alpha}{2} < \theta < \frac{\alpha}{2} \), we have
\[ \frac{G(t, s)}{G(s, s)} = \begin{cases} \frac{\varphi(t)}{\varphi(s)} & \text{if } s \leq t, \\ \frac{\varphi(t)}{\varphi(s)} & \text{if } t \leq s, \end{cases} \]
where \( \varphi(t) = (\gamma + \delta - \gamma t), \psi(t) = (\beta + \alpha t), 0 < t \leq 1, \)
so we get
\[ \frac{G(t, s)}{G(s, s)} \geq M, \text{ therefore } G(t, s) \geq M e(s). \]

If \( x \in K \), then
\[ \min \frac{1}{0} \frac{\| Ax(t) \|}{\| x(t) \|} \frac{1}{0} \frac{1}{0} G(t, s) f(s, x(s), x'(s)) ds \]
\[ \geq M \int_{0}^{1} e(s)f(s, x(s), x'(s)) ds \]
\[ \geq M \| Ax \|, \quad (2) \]

so \( Ax \in K \), which imply that \( AK \subset K \). For any \( x \in [0, +\infty) \), we can define \( P_{1,n}(t) = \min \{ p_1(t), p_1(1/n) \}, 0 \leq t \leq 1/n \), \( P_{2,n}(t) = \min \{ p_2(t), p_2(1/n) \}, n - 1/n \leq t \leq 1 \).

Let \( P_{1,n}(t)q_1(x) + P_{2,n}(t)q_2(x') + r(t) = f_n(t, x, x') \), it is easy to see for any \( t \in [0, 1] \),
\[ f_n(t, x, x') = f(t, x'), \quad \min \{ p_1(t), p_1(1/n) \}, 0 \leq t \leq 1/n, \]
\[ f_n(t, x, x') = f(t, x'), \quad \min \{ p_1(t), p_1(1/n) \}, 1/n \leq t \leq 1, \]
correspondingly we can define
\[ A_n(x) = \frac{1}{0} G(t, s) f_n(s, x(s), x'(s)) ds, n \geq 2. \quad (3) \]

Obviously, for any \( n \geq 2 \) we can see that \( f_n(t, x, x') \) is continuous in \( [0, 1] \times [0, +\infty) \times [0, +\infty) \) and \( f_n(t, x, x') \leq f(t, x'), P_{1,n}(t) \leq p_1(t), P_{2,n}(t) \leq p_2(t), n \geq 2 \), \( A_n \) is relatively compact in \( K \). For \( R > 0 \), let \( B_R = \{ x \in K \| x \| \leq R \} \), now we prove that \( A_n \) is approximate to \( A \) in \( B_R \).

By condition \( 0 < \int_{0}^{1} e(s)p_1(1) ds < +\infty \) and \( 0 < p_1(n) \leq p_1(s) \), we can get \( \int_{0}^{1} \frac{1}{0} e(s)p_1(n) ds < +\infty \) which imply that \( A_n \) is approximate to \( A \), so \( A \) is relatively compact in \( K \).

Finally, we show that \( A \) has a fixed point. Let \( \partial B_{\theta} = \partial B(\theta, \alpha) \cap K \), \( \forall x \in \partial B_{\theta}, \| x \| = \alpha \). \( A \) is a convex function, therefore \( x(s) \in [g_a(s), a] \), so \( x(s) \) is continuous on \( J \), and by condition \( H_2 \).
\[ \|Ax\| = \left\| \int_0^1 G(t, s)f(s, x(s), x'(s)) ds \right\| \]
\[ \leq \left\| \int_0^1 e(s)(p_1(s)q_1(x(s))) ds \right\| \]
\[ + \left\| \int_0^1 e(s)(p_2(s)q_2(x'(s))) ds \right\| \]
\[ + \left\| \int_0^1 e(s)\lambda(s) ds \right\| \]
\[ \leq \int_0^1 e(s)p_1(s) \max_{x \in [g_a(s), a]} q_1(x) ds \]
\[ + \int_0^1 e(s)p_2(s) \max_{y \in [g_a(s), a]} q_2(y) ds \]
\[ + \int_0^1 e(s)\lambda(s) ds \]
\[ < a. \]  
(5)

On the other hand, \( \forall x \in \partial B_b, \|x\| = b, x(s) \in [g_b(s), b], \)
\[ \|Ax\| = \left\| \int_0^1 G(t, s)f(s, x(s), x'(s)) ds \right\| \]
\[ \geq \int_0^1 M\kappa(s) \min_{x \in [g_b(s), b]} f(s, x(s), x'(s)) ds \]
\[ \geq b. \]  
(6)

By lemma 2, we can see that (1) has at least one solution.

As an example, we consider the following problem
\[ \begin{cases} 
  x''(t) = t x' + (1 - t) x', & t \in (0, 1) \\
  x(0) = x(1) = 0, 
\end{cases} \]  
(7)
it is easy to see \( (H_1) - (H_3) \) are all satisfied, according to Theorem 2.1, the problem (7) has at least a solution.

REFERENCES