Abstract—Dengue, a disease found in most tropical and subtropical areas of the world. It has become the most common arboviral disease of humans. This disease is caused by any of four serotypes of dengue virus (DEN1-DEN4). In many endemic countries, the average age of getting dengue infection is shifting upwards, dengue in pregnancy and infancy are likely to be encountered more frequently. The dynamics of the disease is studied by a compartmental model involving ordinary differential equations for the pregnant, infant human and the vector populations. The stability of each equilibrium point is given. The epidemic dynamic is discussed. Moreover, the numerical results are shown for difference values of dengue antibody.

Keywords—Dengue antibody, infant, pregnant human, mathematical model.

I. INTRODUCTION

MATHEMATICAL model can provide an alternative quality assessment of dynamic of the infectious disease. The results of the interested model can help us to gain insight into the factor controlling the persistence and stability of transmitted viral infections within large human communities and lead to a better understanding the viral and host interaction proteomics. In 1999 and 2003, Esteva and Vargas [1]-[2] propose two mathematical models for the transmission of dengue fever. They established the global stability of the endemic equilibrium state. They discussed the vector population in term of the threshold condition, which governs the existence and stability of the endemic equilibrium state.

Dengue fever is caused by four closely related but distinct serotypes known as DEN-1, DEN-2, DEN-3, and DEN-4, in the genus Flavivirus of family Flaviviridae (single-strand, nonsegmented RNA virus), which can be distinguished by serological methods [3]. Infection with one serotype confers immunity to the infected serotype for a long period, but not to other serotypes. Humans may therefore be infected with the dengue virus up to four times. The most important vector is the Aedes aegypti mosquitoes, a day-biting time mosquitoes which is usually found in dark place inside human housing. The incubation period of dengue disease is normally 3-8 days. The virus is detected in human subjects 6-18 hours before the onset of symptoms and viremia ends as the fever abates [4]. Infection with dengue virus may be asymptomatic or may cause undifferentiated febrile illness, dengue fever (DF). DF is the most common in older children and adults and occasionally with unusual hemorrhagic, while dengue hemorrhagic fever (DHF) is the most common in school age children having secondary response. DHF is characterized by the onset of an acute fever and associated non-specific constitutional signs and symptoms. In addition, the potentially fatal complication of dengue called Dengue hemorrhagic fever (DHF) can occur.

The global epidemiology and the dynamics of transmission of dengue virus have changed dramatically in South-east Asia since World War II [5]. The disease is now highly endemic in more than 100 tropical countries, and the number of cases has increased dramatically during the past three decades[5]-[7]. Worldwide, it is estimated that up to 100 million cases of DF and 250,000 cases of DHF occur annually [8]. Dengue infection is endemic in Thailand and many other tropical countries. Fig. 1 shows the annual number of dengue cases reported to the Division of Epidemiology, Ministry of Public Health, Thailand during the period 1997-2007. As we see, there were many severe cases with the peak increased in 2001.

Fig. 1 The distribution of DHF patients according to the disease severity between 1997 and 2007 [9]

In Thailand, DHF is common in children (less than 15 years) and cause a significant number of deaths From 1997 to 2007, 9,412 infants were diagnosed with dengue infection,
this was approximated 1.3% of all reported cases of dengue infection [9]. DHF is less common during infancy but when it does occur, the mortality is higher than in older children [10]. In Bangkok, Thailand, infants less than 12 months of age are infected with dengue virus were at high risk for DHF if maternal antibodies to dengue virus were present at subneutralizing levels. This led to the theory that DHF is caused by antibody enhancement of viral infection [11]-[13]. As it is mentioned above, infant is an interesting group for this study.

For maternal antibody to dengue virus in infants are disappeared in 3% by two months of age, in 19% by four months of age, in 72% by six months of age, in 92% by nine months of age, and in 100% by 12 months of age [14]. So that, the maternal antibodies to dengue virus disappeared in the first year of life. In our model, infant is defined as the baby who age not more than 12 months. Fig. 2 shows the number of infant cases in Thailand.

In this paper, we are interested in the transmission model between pregnant woman and infant, maternal antibodies to dengue is incorporated into the model. The aims of this paper are to construct and analysis the transmission of dengue disease between pregnant woman and infant with dengue antibody.

II. MATHEMATICAL MODEL

To study the transmission of dengue virus infection, we divide the human population into two classes, pregnant woman and infant classes. Each class is divided into three subclasses, susceptible, infected and recovered human. The vector population is separated into two classes, susceptible and infected vector populations because it never recovers from infection. We assume the susceptible pregnant woman is never infected with dengue virus and infant is defined as the baby who age not more than 12 months.

In our model, the dynamics of each component of human and vector are given by

\[
\begin{align*}
S_m &\quad \text{number of susceptible pregnant woman}, \\
I_m &\quad \text{number of infected pregnant woman}, \\
R_m &\quad \text{number of recovered pregnant woman}, \\
S_n &\quad \text{number of susceptible infant}, \\
I_n &\quad \text{number of infected infant}, \\
R_n &\quad \text{number of recovered infant}, \\
S_v &\quad \text{number of susceptible vector}, \\
I_v &\quad \text{number of infected vector}, \\
a &\quad \text{percentage of infant who be not die while pregnant}, \\
q &\quad \text{average number of infant which one woman can have in each time of pregnancy}, \\
P &\quad \text{constant recruitment rate of pregnant woman}, \\
N_v &\quad \text{total adult mosquitoes}, \\
\mu_h &\quad \text{average constant death rate of pregnant woman}, \\
\mu_v &\quad \text{average constant death rate of vector}, \\
\gamma_{vm} &\quad \text{transmission rate of dengue virus from vector to mother and the mother is infected}, \\
\gamma_{vn} &\quad \text{transmission rate of dengue virus from vector to infant and infant is infected}, \\
\gamma_{mv} &\quad \text{transmission rate of dengue virus from mother to vector and vector is infected}, \\
\gamma_{nv} &\quad \text{transmission rate of dengue virus from infant to vector and vector is infected}, \\
\gamma_{mn} &\quad \text{transmission rate of dengue virus from mother to infant and infant is infected}, \\
r_m &\quad \text{constant rate at which human populations}
\end{align*}
\]

where the parameters in the above diagram are defined as
We assumed the total populations remains constant and each group of population also remain constant, we obtain

\[ N_m = \frac{(P + aqP)}{\mu_h}, \quad N_n = \frac{aqP}{\mu_h}, \quad N_v = \frac{A}{\mu_v}. \]

We normalize (1) by letting \( S_m = \frac{S_m}{N_m}, \quad I_m = \frac{I_m}{N_m}, \quad R_m = \frac{R_m}{N_m}, \quad S_n = \frac{S_n}{N_n}, \quad I_n = \frac{I_n}{N_n}, \quad R_n = \frac{R_n}{N_n}, \quad S_v = \frac{S_v}{N_v}, \quad I_v = \frac{I_v}{N_v}. \)

and \( I_v = \frac{I_v}{N_v}, \) then our equations become

\[
\begin{align*}
\frac{dS_m}{dt} &= \mu_h - (\mu_h + \gamma_{vm} I_v)(A/\mu_v)S_m, \\
\frac{dI_m}{dt} &= \gamma_{vm} I_v(A/\mu_v)S_m - (\mu_h + r_m)I_m, \\
\frac{dR_m}{dt} &= r_m I_m - \mu_h R_m, \\
\frac{dS_n}{dt} &= aqP - (\mu_h + k\gamma_{vn} I_v + \gamma_{mn} I_m)S_n, \\
\frac{dI_n}{dt} &= (k\gamma_{vn} I_v + \gamma_{mn} I_m)S_n - (\mu_h + r_m)I_n, \\
\frac{dR_n}{dt} &= r_m I_n - \mu_h R_n, \\
\frac{dS_v}{dt} &= A - (\mu_v + \gamma_{vm} I_m + \gamma_{mn} I_n)S_v, \\
\frac{dI_v}{dt} &= (\gamma_{vm} I_m + \gamma_{mn} I_n)S_v - \mu_v I_v
\end{align*}
\]

with the new three conditions \( S_m + I_m + R_m = N_m, \quad S_n + I_n + R_n = N_n, \quad S_v + I_v = N_v. \)

III. ANALYSIS OF THE MATHEMATICAL MODEL

A. Equilibrium States

The equilibrium points are obtained by setting the right hand side of all equations in (3) equal to zero. Doing this, we get two equilibrium points are

i) disease free equilibrium point:

\[ E_1 = (1, 0, 1, 0, 0) \]

(5)

ii) endemic equilibrium point

\[ E_2 = (S^*_m, I^*_m, S^*_n, I^*_n, I^*_v) \]

(6)

where

\[
\begin{align*}
S^*_m &= \frac{\beta_1}{I^*_v}, \\
I^*_m &= \frac{R_m I^*_v}{M_1 I^*_v}, \\
S^*_n &= \frac{\beta_2 M_1 I^*_v}{\beta_2 M_1 I^*_v + \beta_3 M_1 R_m I^*_v + \beta_2 \theta R_v I^*_v} + \beta_2 \theta R_v I^*_v, \\
I^*_n &= \frac{\beta_2 M_1 (1 + I^*_v)}{\mu_h M_1 (\beta_2 M_1 I^*_v + kM_1 R_v I^*_v) + \beta_2 \theta R_v I^*_v}, \\
I^*_v &= \text{the positive solution of}
\end{align*}
\]

(7)

(8)

(9)

(10)
\[ b_1 I_v^* + b_2 I_v^* + b_3 I_v^* + b_4 = 0. \] (11)

This corresponds to Descartes’ Rule of Signs, there exists one positive solution exactly,

with \[ \beta_1 = \frac{b_2 N_m}{\mu_s (N_f + h)}, \quad \beta_2 = \frac{b_3 N_m}{\mu_s (N_f + h)}, \quad \beta_3 = \frac{b_4 N_m}{\mu_s (N_f + h)}. \]

\[ R_1 = \frac{b_1^2 N_m (A / \mu_s)}{\mu_s N_m (N_f + h)^2}, \quad R_2 = \frac{b_2^2 N_m (A / \mu_s)}{\mu_s N_m (N_f + h)^2}, \]

\[ M_1 = \frac{\mu_h + \gamma_n}{\mu_h}, \quad \theta_1 = \frac{\gamma_n}{N_m N_m}, \quad \theta_2 = \frac{\gamma_n}{N_m N_m}, \]

\[ \theta_3 = \frac{N}{N_m N_n}, \quad \Gamma = \beta_1 + R_1 I_v^*, \]

such that

\[ b_3 = k M R_0^2 (\beta_1^2 \theta_1 + \mu_s + R_2 \mu_h (\theta_2 + M \mu_s)), \]

\[ b_2 = R_2 (-\mu_s (M R_0^2 (\beta_1 - \beta_2) \theta_2 + R_1 \beta_2 \theta_2 (\theta_2 + \theta_3)) + M_1 (2 k R_0^2 \theta_2 + M R_0^2 \mu_h), \]

\[ + R_1 \beta_2 (-\mu_s (\beta_1 \theta_2 + (M_1 + \theta_1) \mu_h)) \theta_1), \]

\[ b_1 = -R_1 (M R_0^2 \beta_1 (R_1 - \beta_2) \theta_2 + \beta_2 \theta_2 (-\beta_1 \theta_2 + R_1 (\theta_2 + \theta_3)) \theta_1 + M_1 \beta_2 (k R_0^2 \theta_2 + M R_0^2 \mu_h), \]

\[ + R_1 \beta_2 (-2 k R_0^2 \theta_2 + (M_1 + \theta_1) \mu_h) \theta_1), \]

\[ b_0 = \beta_1 \beta_2 (-R_1 (M \theta_2 + \theta_1 \theta_3) \mu_s + M_1 \beta_2 (-k \beta_1 \theta_2 + M \mu_s) \mu_s). \]

**B. Disease Free Equilibrium Point**

For the system defined by (3), the Jacobian matrix evaluated at \( E_1 \) is the 5x5 matrix given by

\[ J_{E_1} = \begin{bmatrix}
-\mu_s & 0 & 0 & 0 & \frac{\mu_s R_1}{\beta_1} \\
0 & -\mu_s M_1 & 0 & 0 & \frac{\mu_s R_1}{\beta_1} \\
0 & -\mu_s M_1 & 0 & 0 & \frac{\mu_s k R_0^2}{\beta_1} \\
0 & \mu_s M_1 & 0 & 0 & -\frac{\mu_s k R_0^2}{\beta_1} \\
0 & \theta_2 & 0 & 0 & -\mu_s M_1 \\
\end{bmatrix} \]

The eigenvalues can be found by solving the following characteristic equation

\[ \det(J_{E_1} - \lambda I_5) = 0, \]

which is

\[ (\mu_s + \lambda)^2 (\lambda^3 + \mu_h \lambda^2 + \mu_s \lambda + a_0) = 0, \] (13)

with

\[ a_2 = 2 M \mu_h + \mu_s, \]

\[ a_1 = \mu_h (M_1^2 \mu_s + 2 M \mu_s - \frac{R_1 \theta_2}{\beta_1} - \frac{\mu_s k R_0^2}{\beta_2}), \]

\[ a_0 = \mu_s^2 (M_1^2 \mu_s - \frac{R_1 \mu_s}{\beta_1} - \frac{\mu_s k R_0^2}{\beta_2} - \frac{R_1 \theta_2}{\beta_1}). \]

The eigenvalues are \( \lambda_1 = \lambda_2 = -\mu_h, \lambda_3, \lambda_4 \) and \( \lambda_5 \) are found by solving the equation

\[ \lambda^2 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0. \]

It can be seen that the coefficients \( a_2, a_1 \) and \( a_0 \) satisfy the Routh-Hurwitz criteria for local stability [15]

i) \( a_2 > 0, \)

ii) \( a_0 > 0, \) (14)

iii) \( a_2 a_1 > a_0 \)

when \( R_0 < 1. \) This means that all eigenvalues will be negative, leading to the first equilibrium point being locally stable where \( R_0 = \frac{R_1 N M \beta_2 + k R_0^2 \beta_1 M \beta_1 + R_1 \beta_2 \theta_1 \theta_2}{\beta_1 \beta_2 M_1^2 \mu_s} < 1. \)

**C. The Endemic Disease Equilibrium Point**

The local stability of the endemic disease equilibrium point \( E_2 \) is governed by the matrix

\[ J_{E_2} = \begin{bmatrix}
-\mu_s & 0 & 0 & 0 & \frac{\mu_s R_1}{\beta_1} \\
0 & -\mu_s M_1 & 0 & 0 & \frac{\mu_s R_1}{\beta_1} \\
0 & -\mu_s M_1 & 0 & 0 & -\frac{\mu_s k R_0^2}{\beta_1} \\
0 & \mu_s M_1 & 0 & 0 & \frac{\mu_s k R_0^2}{\beta_1} \\
0 & \theta_2 & 0 & 0 & -\mu_s M_1 \\
\end{bmatrix} \]

where \( S_{m}, I_{m}, S_{n}, I_{n}, \) and \( I_{v}^* \) are given by (7)-(11).

The characteristic equation for the Jacobian matrix is

\[ \lambda^5 + \lambda^4 d_1 \lambda^3 + \lambda^2 d_2 + \lambda d_3 + d_4 = 0. \] (15)

It can be seen that \( \lambda_1 = -\mu_h, \lambda_2, \lambda_3, \lambda_4 \) and \( \lambda_5 \) are found by solving the equation

\[ \lambda^2 + d_1 \lambda^2 + d_2 \lambda + d_3 + d_4 = 0. \]

The stability of the endemic equilibrium point can be determined by using Routh-Hurwitz criteria as follows:

i) \( d_3 > 0, \)

ii) \( d_2 > 0, \) (16)

iii) \( d_0 \geq 0, \)

iv) \( d_1 d_2 d_3 \geq d_1^2 + d_2^2 d_0. \)

The endemic equilibrium point is locally stability when

\[ R_0 = \frac{R_1 N M \beta_2 + k R_0^2 \beta_1 M \beta_1 + R_1 \beta_2 \theta_1 \theta_2}{\beta_1 \beta_2 M_1^2 \mu_s} > 1. \] (17)

We present the above four conditions by using the following figures.
Fig. 4 The parameter space for the endemic equilibrium point which satisfies the Routh-Hurwitz criteria with the value of parameters are $A = 500,000$, $h = 0$, $k = 0.5$, $N_t = 10,000$, $N_m = 5,000$, $N_s = 5,000$, $r_m = 1/3$ day$^{-1}$, $\beta_m = \beta_v = \beta_{uv} = 0.1$, $\beta_{mv} = 0.001$, $b = 1/3$ day$^{-1}$, $\mu_h = 1/(365 \times 70)$ day$^{-1}$, $\mu_v = 1/14$ day$^{-1}$.

The quantity

$$R_0 = \sqrt{R_o} = \sqrt{\frac{R_0M_0\beta_i + kR_0\beta_iM_0\theta_i + R_0\beta_i\theta_i}{\beta_i\beta_iM_0\mu_v}}$$

is the basic reproduction number, defined as the average number of new infections generated by a single infectious individual in a fully susceptible population [16]-[17]. The estimation of $R_0$ can determine if dengue sustain its chain of transmission. $R_0$ is also referred to as the threshold parameter [18]. It is the most common measure for quantifying the strength of epidemics.

D. Numerical Results

Numerical solutions are shown for comparing the transmission of dengue disease on the free and endemic regions. The values of the parameters used in this study are $\mu_h = 1/(365 \times 70)$ per day, corresponding to a life expectancy of 70 years; $\mu_v = 1/14$ per day, corresponding to a mosquito mean life of 14 days; $b = 1/3$, one bite providing enough blood meal for three days and there is no alternative host ($h = 0$). The other parameters are arbitrarily chosen.

The numerical solutions of (3) are shown in following figures.

Fig. 5 Numerical solutions of system (3) demonstrate the solution of $S, I, S, I, S$, respectively, for $R_0 < 1$ with $A = 140$, $h = 0$, $k = 0.5$, $N_t = 10,000$, $N_m = 5,000$, $N_s = 5,000$, $\beta_m = \beta_v = \beta_{uv} = \beta_{mv} = 0.1$, $\beta_{mv} = 0.001$, $\mu_h = 1/(365 \times 70)$ day$^{-1}$, $b = 1/3$ day$^{-1}$, $r_m = 1/3$ day$^{-1}$, $R_0 = 0.274669$, $\mu_v = 1/14$ day$^{-1}$. The fractions of populations approach to the disease free state.
Numerical solutions of system (3) demonstrate solution of $S_m, I_m, S_i, I_i$, respectively, for $R_0 > 1$ with $N_s = 5,000$, $A = 500,000$, $h = 0$, $k = 0.5$, $N_T = 10,000$, $N_m = 5,000$, $\beta_{mm} = 0.001$, $b = 1/3 \text{day}^{-1}$, $\beta_{vm} = \beta_{vn} = \beta_{vm} = \beta_{vn} = 1/(365 \times 70) \text{day}^{-1}$, $r_m = 1/3 \text{day}^{-1}$, $\mu_s = 1/14 \text{day}^{-1}$, $R_0 = 16.4146$. The fractions of populations converge to the endemic disease state.

Next section, we will compare the numerical solution behaviors when maternal antibodies are different.

IV. DISCUSSION AND CONCLUSION

The mathematical model which we analyze in our study, the pregnant infant human and the vector population are assumed to have constant sizes. The number of secondary infections, which can result from one primary infection, is defined from the square root of $R_0$. The quantity $R_0$ is the basic reproductive number of the disease where

$$R_0 = \frac{R_m \mu_s \beta + kR_r \mu_s \beta + R_f \mu_s \beta + R_m \mu_s \beta}{\beta \mu_s \mu_c},$$

$$R_0 = \frac{R_m \mu_s \beta + kR_r \mu_s \beta + R_f \mu_s \beta + R_m \mu_s \beta}{\beta \mu_s \mu_c},$$

$$R_0 = \frac{\beta \mu_s \mu_c}{\beta \mu_s \mu_c}.$$

We consider the third term

$$k\gamma_m \beta_{vm} N_m(A/\mu_c) \mu_c(N_T + h)(\mu_s + r_m) = k\beta_m \gamma_m N_m(A/\mu_c) \mu_c(N_T + h)(\mu_s + r_m) \mu_c(N_T + h)(\mu_s + r_m).$$

(20)

It indicates the number of secondary infant cases with maternal antibodies (k). The infective infancy introduced into the susceptible infancy is bitten by $b(A/\mu_c) / (N_T + h)(\mu_s + r_m)$ of these mosquitoes becomes infectious. One of these infectious mosquitoes, $\beta_{vm} \left( \frac{N_m}{N_T + h} \right)$ will in turn bite. Multiplying this number by $k\beta_{vm}$, we get the third term, this number present the infected infancy.

Moreover, we compare the transmission of this disease in human, vector population and the basic reproductive number when the dengue antibodies are different. We show in Fig. 7.

Fig. 7 Bifurcation diagrams of system (3) demonstrate the Equilibrium solutions of $S_m, I_m, S_i, I_i$ and $R_0$ respectively, for the different values of $D_{mm}$ with $A = 500,000$, $h = 0$, $k = 0.5$, $N_T = 10,000$, $N_m = 5,000$, $N_s = 5,000$, $\beta_{mm} = 0.001$, $\mu_s = 1/14 \text{day}^{-1}$, $\beta_{vm} = \beta_{vn} = \beta_{vn} = 0.1$, $r_m = 1/3 \text{day}^{-1}$, $b = 1/3 \text{day}^{-1}$, $\mu_s = 1/(365 \times 70) \text{day}^{-1}$.

From Fig. 5, all proportions of populations approach to the disease free state $(1,0,1,0,0)$. Fig. 6, the fractions of populations spiral to the endemic disease states $(0.0306372, 0.000113806, 0.031889, 0.000113659, 0.0000530724)$. The imaginary part of complex root of the eigenvalue is approximately 0.0666672. The period of oscillation is 95 days.
The bifurcation diagrams demonstrate the equilibrium solutions of all populations and $R_0$ for the different values of $D_{nm}$. If the percentage of dengue antibody which infant received from mother in the beginning is increase, the normalized susceptible pregnant and infant human proportions are increases. But the normalized infectious pregnant, infectious infant, infectious vector population and the basic reproductive number are decreases.

In conclusion, vertical transmission of dengue virus may lead to a full-blown illness in the infant similar to that seen in children and adult cases. Hence, early diagnosis and management of this potentially lethal condition is necessary to reduce perinatal morbidity and mortality, especially in communities where dengue is endemic.

ACKNOWLEDGMENT

The authors would like to thank Prof. Dr. I Ming Tang at Mahidol University, Thailand.

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