The Sizes of Large Hierarchical Long-Range Percolation Clusters

Yilun Shang

Abstract—We study a long-range percolation model in the hierarchical lattice $\Omega_N$ of order $N$ where probability of connection between two nodes separated by distance $k$ is of the form $\min(\alpha \beta^{-k}, 1)$, $\alpha \geq 0$ and $\beta > 0$. The parameter $\alpha$ is the percolation parameter, while $\beta$ describes the long-range nature of the model. The $\Omega_N$ is an example of so called ultrametric space, which has remarkable qualitative difference between Euclidean-type lattices. In this paper, we characterize the sizes of large clusters for this model along the line of some prior work. The proof involves a stationary embedding of $\Omega_N$ into $\mathbb{Z}$. The phase diagram of this long-range percolation is well understood.

Keywords—percolation, component, hierarchical lattice, phase transition.

I. INTRODUCTION

Percollation theory in the Euclidean lattice $\mathbb{Z}^d$ started with the work of Broadbent and Hammersley in 1957. The infinity of the space of vertices and its geometry are principal features of this model; see e.g. [11] and references therein. Some questions of percolation in other non-Euclidean infinite systems is formulated in [4]. The study of long-range percolation on $\mathbb{Z}^d$ traces back to [15] and leads to a range of interesting results in probability theory and statistical physics [1], [5], [6], [8], [18], [21]. On the other hand, hierarchical structures have been used in applications in the physics, genetics and social sciences thanks to the multi-scale organization of many natural objects [3], [13], [19], [20].

Recently, long-range percolation is studied on the hierarchical lattice $\Omega_N$ of order $N$ (to be defined below), where classical methods for the usual lattice break down. The asymptotic long-range percolation on $\Omega_N$ is addressed in [10] for $N \to \infty$. The work [9], [12], [16] and [17] analyze the phase transition of long-range percolation on $\Omega_N$ for finite $N$ using different connection probabilities and methodologies. The contact process on $\Omega_N$ for fixed $N$ has been investigated in [2]. In this paper, we investigate the sizes of large connected components (or clusters) in the resulting percolation graph on $\Omega_N$ for fixed $N$. The form of the connection probabilities used here follow from a prior work [16].

For an integer $N \geq 2$, we define the set

$$\Omega_N := \left\{ x = (x_1, x_2, \cdots) : x_i \in \{0, 1, \cdots, N - 1\}, \sum_{i=1}^{\infty} x_i < \infty \right\}.$$  \hspace{1cm} (1)

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and define a metric $d$ on it:

$$d(x, y) = \left\{ \begin{array}{ll} 0, & x = y, \\ \max \{i : x_i \neq y_i\}, & x \neq y. \end{array} \right.$$ \hspace{1cm} (2)

The pair $(\Omega_N, d)$ is referred to as the hierarchical lattice of order $N$, which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two vertices is the number of levels (generations) from the bottom to their most recent common ancestor. Figure 1 shows the lattice $\Omega_2$ along with its metric generating tree.

Such a distance $d$ satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\},$$ \hspace{1cm} (3)

for any triple $x, y, z \in \Omega_N$. Hence, $(\Omega_N, d)$ is an ultrametric (or non-Archimedean) space [14]. From its ultrametricity, it is clear that for every $x \in \Omega_N$ there are $(N - 1)N^{k-1}$ vertices at distance $k$ from it.

Now consider a long-range percolation on $\Omega_N$. For each $k \geq 1$, the probability of connection between $x$ and $y$ such that $d(x, y) = k$ is given by

$$p_k = \min \left\{ \frac{\alpha}{\beta^k}, 1 \right\},$$ \hspace{1cm} (4)

where $0 \leq \alpha < \beta < 0$, all connections being independent. Two vertices $x, y \in \Omega_N$ are in the same cluster if there exists a finite sequence $x = x_0, x_1, \cdots, x_n = y$ of vertices such that each pair $(x_{i-1}, x_i)$, $i = 1, \cdots, n$, of vertices presents an edge.

The rest of the paper is organized as follows. In Section 2, we provide the main results and Section 3 is devoted to the proofs.

II. MAIN RESULTS

Let $\mathbb{N}$ be the non-negative integers including 0, and denote by $\ell := \min\{k \in \mathbb{N} : \alpha \leq \beta^{k+1}\}$. Let $|S|$ be the size of a set $S$. The connected component containing the node $x \in \Omega_N$ is denoted by $C(x)$. Since, for every node $x$, $|C(x)|$ has the same distribution, it suffices to consider only $|C(0)|$. The percolation probability is defined as

$$\theta(\alpha, \beta) := P(|C(0)| = \infty),$$ \hspace{1cm} (5)

and the critical percolation value is defined as

$$\alpha_c(\beta) := \inf \{\alpha \geq 0 : \theta(\alpha, \beta) > 0\}. $$ \hspace{1cm} (6)

The following theorem characterizes the phase transition for this model.
Lemma 1. For any constant $K > 0$,  
\[ 1\{\|C(0)\| = \infty\} \cap \{\|C_n(0)\| < K(\beta/N)^n\} \rightarrow 0, \]  
almost surely as $n \rightarrow \infty$.

Proof. By multiplication principle, we only need to show that the conditional probability  
\[ P\left(\|C(0)\| = \infty\right|\{n \in \mathbb{N} : \|C_n(0)\| \leq K \left(\frac{\beta}{N}\right)^n\}) = 0. \]  
(9)

First, we assume that $\beta > N$. Let $n_1$ be the smallest $n$ for which $C_n(0) \leq K(\beta/N)^n$. If $C_n(0) \not\leftrightarrow B_n(0)$, then $n_{i+1} = n_i$. If $C_n(0) \leftrightarrow B_n(0)$, then $n_{i+1}$ is the smallest $n > n_i$ such that $C_n(0) \not\leftrightarrow B_n(0)$ and $|C_n(0)| \leq K(\beta/N)^n$. Note that $|C_n(0)| \leq K(\beta/N)^n$, and then we have  
\[ P(C_n(0) \leftrightarrow B_n(0)) \leq P\left(C_n(0) \leftrightarrow B_n(0) : \|C_n(0)\| = K \left(\frac{\beta}{N}\right)^{n_i}\right) \]
\[ = 1 - \prod_{j=n_i+1}^{\infty} \left(1 - \min\{\alpha(\beta - j), 1\}\right)^{K(\beta/N)^{n_i}(N-1)^{N-j-1}} \]  
(10)

If $n_{i+1} = \ell$, then we have a trivial bound, i.e., the above probability less than 1. If $n_{i+1} > \ell$, then  
\[ P(C_n(0) \leftrightarrow B_n(0)) \leq 1 - \prod_{j=n_i+1}^{\infty} (1 - \alpha(\beta - j)^{K(\beta/N)^{n_i}(N-1)^{N-j-1}}) \]
\[ < 1 - \alpha \frac{N - 1}{\beta - N}, \]  
(11)

involving the inequality $\exp\left(-\frac{1}{\beta - N}\right) < 1 - \frac{1}{\beta}$ as in [16]. The right-hand side of (11) is strictly less than 1 and is independent of $n_i$. Recall that $\{C_n(0) \leftrightarrow B_n(0)\}_{i \geq 1}$ are independent.
events. If there are infinitely many different \( n_i \), then there must be some \( n_i \) for which \( \{C_n(0) \neq B_{n+1}(0)\} \) holds. If there are only finitely many different \( n_i \), then by definition the same thing holds. The above comments clearly yield (9) for any \( \beta > N \). By monotonicity, we know that (9) holds for any \( 0 < \beta < N^2 \).

**Lemma 2.** For any constant \( K > 0 \). The fraction of the vertices in \( B_n(0) \) which are in a cluster of size at least \( K(\beta/N)^n \), converges to \( \theta \) almost surely as \( n \to \infty \).

**Proof.** First assume that \( \beta > N \). We will use the random embedding of the hierarchical lattice in \( Z \) [17]. From the ergodic theorem we obtain for any \( k > 0 \),

\[
\frac{1}{2N^n + 1} \sum_{x \in \mathbb{Z}^n} |C_n(x)| > K(\beta/N)^n \rightarrow P(\cap_{x \in \mathbb{Z}^n} |C_n(x)| > K(\beta/N)^n),
\]

almost surely as \( n \to \infty \).

By virtue of Lemma 1, the right-hand side of (12) increases to \( \theta \) as \( k \to \infty \). Hence, we have

\[
A(n) := \frac{1}{2N^n + 1} \sum_{x \in \mathbb{Z}^n} |C_n(x)| > K(\beta/N)^n \rightarrow \theta,
\]

almost surely as \( n \to \infty \). By our construction in [17], the collection vertices \( \{-N^n, -N^n + 1, -N^n + 2, \ldots, N^n\} \) contains the image under the embedding of the ball \( B_n(0) \) and this image contains a fraction \( N^n/(2N^n + 1) \) of those vertices. The events \( \{|C_n(x)| > K(\beta/N)^n\} \) are independent for vertices in different \( n \)-balls, and then

\[
A_1(n) := \frac{1}{2N^n + 1} \sum_{x \in B_n(0)} |C_n(x)| > K(\beta/N)^n
\]

and

\[
A_2(n) := A(n) - A_1(n)
\]

are independent.

It is easy to see that \( A_1(n) \) and \( A_2(n) \) are bounded above by \( 1 \) and have asymptotically the same mean. By (13) we obtain that

\[
\frac{1}{N^n} \sum_{x \in B_n(0)} 1_{\{|C_n(x)| > K(\beta/N)^n\}} \rightarrow \theta,
\]

almost surely as \( n \to \infty \) for \( \beta > N \). When \( \beta \leq N \), we have \( \theta = 1 \) by Theorem 1. It is direct to check that the above derivations still hold. □

**Proof of Theorem 3.** From Lemma 2 we have for every \( K > 0 \) and \( \varepsilon > 0 \)

\[
P\left(\left\{x \in B_n(0) : |C_n(x)| > K\left(\frac{\beta}{N}\right)^n\right\} \right) > (\theta - \varepsilon)N^n
\]

\[
> 1 - \varepsilon,
\]

for \( n \) large enough. A ball \( B_n(y) \) is said to be good if and only if

\[
\left\{x \in B_n(y) : |C_n(x)| > K\left(\frac{\beta}{N}\right)^n\right\} \rightarrow (\theta - \varepsilon)N^n.
\]

In what follows, we condition on the event that all \( n \)-balls in \( B_{n+1}(0) \) are good. The probability of this event is bounded below by \( (1 - \varepsilon)^N \geq 1 - N\varepsilon \).

For each good ball \( B_n(y), y \in \Omega_N \), we make a partition of the set

\[
B'_{n}(y) := \left\{x \in B_n(y) : |C_n(x)| > K\left(\frac{\beta}{N}\right)^n\right\}
\]

into super vertices. For \( x \in B'_{n}(y) \) we make a partition of \( C_n(x) \) into \( |C_n(x)|/|K(\beta/N)^n| \) super vertices, all of which have size at least \( {K(\beta/N)^n} \). Denote by \( V_n \) the collection of super vertices that contain vertices in \( B_{n+1}(0) \). For \( K \) large enough, if \( B_n(y) \) is good, then \( V_n \) contains at least \((\theta - \varepsilon)N^n/(2K(\beta/N)^n) \geq (\theta - \varepsilon)N^n/(3K(\beta/N)^n)) \) super vertices.

As in [12], we construct a new \( N \)-partite graph on \( V_n \) as follows. Let \( V_n \) be the vertex set and let \( E_n \) be the edge sets. Choose \( |K(\beta/N)^n| \) original vertices from every super vertex in \( V_n \). Choosing those vertices may be done in any way that is independent of the presence of edges of length \( \geq n+1 \). Denote these sets by \( A_n \). The super vertices \( x, y \in V_n \) are connected by an edge if there is at least one edge in the original graph which is present between vertices that make up the sets in \( A_n \), corresponding to \( x \) and \( y \), respectively, and if the original vertices that make up \( x \) and \( y \) are at distance \( n+1 \) of each other. Otherwise, there is no edge between the super vertices.

Since \( \beta < N^2 \), \((\theta - \varepsilon)N^n/(3K(\beta/N)^n) \) tends to infinity as \( n \to \infty \). Hence, the expected degree of a vertex in \( V_n \) is larger than

\[
\frac{(N-1)(\theta - \varepsilon)N^n}{3K(\beta/N)^n} \left(1 - \left(1 - \frac{\alpha}{\beta + 1}\right)^{K^2(\beta/N)^n}\right)
\]

\[
> \frac{(N-1)(\theta - \varepsilon)N^n}{3K(\beta/N)^n} \left(1 - \exp\left(-\frac{\alpha}{\beta + 1}K^2\left(\frac{\beta}{N}\right)^n\right)\right),
\]

which exceeds \( \lambda := (N-1)(\theta - \varepsilon)K^2/(6\beta) \) for large \( n \). Clearly, the parameter \( \lambda \) can be made large enough by choosing \( K \) large enough.

The \( N \)-partite graph \( (V_n, E_n) \) is an inhomogeneous random graph; see [7] for backgrounds. The degree of every super vertex is asymptotically Poisson distributed, with mean bounded below by \( \lambda \). The unique largest cluster of such an \( N \)-partite graph contains a fraction \( \eta \) of the super vertices almost surely as \( n \to \infty \), where \( \eta \) is the largest solution of the equation

\[
1 - \eta = e^{-\lambda \eta}.
\]

We can choose \( \lambda \) sufficiently large and \( \eta > 1 - \varepsilon \). Hence, for each \( \varepsilon > 0 \) and large \( n \), the graph \( (V_n, E_n) \) contains a unique giant cluster containing a fraction \( (1 - \varepsilon)N \) of the vertices in \( V_n \) with probability at least \( 1 - \varepsilon \).

Since we have conditioned on the event that all \( n \)-balls in \( B_{n+1}(0) \) are good, the fraction of vertices in \( B_{n+1}(0) \) that are part of vertices in \( V_n \) is larger than \( \theta - 2\varepsilon \). Accordingly, conditional on the same event, the largest cluster in \( B_{n+1}(0) \) is at least of size \( (\eta - \varepsilon)(\theta - 2\varepsilon)N^n > (1 - 2\varepsilon)(\theta - 2\varepsilon)N^n \) with probability at least \( 1 - \varepsilon \). By the multiplication principle, we have the probability that the largest cluster in \( B_{n+1}(0) \) is at least of size \( (1 - 2\varepsilon)(\theta - 2\varepsilon)N^n \) is bounded below by.
Now, choosing $\varepsilon' < \varepsilon / \max\{4, N+1\}$, we finally obtain that
\[
P(|C_n^m(0)| > (\theta - \varepsilon')N^n) \geq 1 - \varepsilon'.
\]
(21)

The proof then readily follows. $\square$

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REFERENCES


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