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# The core and Shapley function for games on augmenting systems with a coalition structure

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*Abstract*—In this paper, we first introduce the model of games on augmenting systems with a coalition structure, which can be seen as an extension of games on augmenting systems. The core of games on augmenting systems with a coalition structure is defined, and an equivalent form is discussed. Meantime, the Shapley function for this type of games is given, and two axiomatic systems of the given Shapley function are researched. When the given games are quasi convex, the relationship between the core and the Shapley function is discussed, which does coincide as in classical case. Finally, a numerical example is given.

*Keywords*—cooperative game, augmenting system, Shapley function, core

## I. Introduction

IN the model of traditional cooperative games, we always assume each coalition can be formed, which seems to be unrealistic. There are many situations that not all coalitions can be formed for various reasons. One of which is games with a coalition structure. Aumann and Drze [1] first researched this problem, and gave the Shapley function on it. Later, Owen [2,3] further studied games with a coalition structure, where the probability of cooperation among coalitions is considered. Meantime, the author introduced the Owen value and the Banzhaf-Owen value for games with a coalition structure. More researches about games with a coalition structure can be seen in [4-7].

Besides games with a coalition structure, there exists another kind of games. People call it games under precedence constraints, where not all coalitions can be formed and the player payoffs are relevant to their orders in the coalitions. Myerson [8] introduced games with communication situations, and gave the Shapley function for the given model. Later, Faige and Kern [9] introduced a special kind of cooperative games under precedence constraints, and discussed the axiomatic system of the given Shapley value by using *hierarchical strength*. Bilbao [10] defined games on convex geometries, and the Shapley value for this kind of games is studied. Bilbao et al. [11,12] gave another special kind of games under precedence constraints which is named games on matroids, and studied the Shapley values for two cases of games on matroids–the static model and the dynamic model. Recently, Bilbao [13] introduced the model of games on augmenting systems. Furthermore, Algaba et al. [14] presented the model of games on antimatroids, and researched the Shapley value for this class of games.

Different to the above introduced models, we shall research games on augmenting systems with a coalition structure. Namely, there is a coalition structure for the player set where the players can participate in different unions, and the players in one union form an augmenting system.

In section 2, some basic concepts of cooperative games with a coalition structure and games on augmenting systems are introduced, which will be used in the following. In section 3, we first give the model of games on augmenting systems with a coalition structure. Meantime, we research the core and the Shapley function for games on augmenting systems with a coalition structure. Some properties of the given Shapley function are discussed. In section 4, a numerical example is given.

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## II. Preliminaries

### A. Some concepts of games with a coalition structure

Let $N=\{1,2,...,n\}$ be a finite set, and $P(N)$ be the set of all subsets in $N$. The coalitions $P(N)$ in are denoted by $S,T,...$. For any $S\in P(N)$, the cardinality of $S$ is denoted by the corresponding lower case $s$.

A coalition structure $\Gamma=\{B_1,B_2,\dots,B_m\}$ on $N$ is a partition of $N$, i.e., $\cup_{1\le h\le m}B_h=N$ and $B_h\cap B_l=\emptyset$ for all $h,l\in M=\{1,2,\dots,m\}$ such that $h\neq l$, denoted by $(N,\Gamma)$. For any $S\in L(N,\Gamma)$, $S$ is called a feasible coalition, where $L(N,\Gamma)$ denotes the set of all feasible coalitions in $(N,\Gamma)$. A function $v: L(N,\Gamma)\to\Re_+$ satisfying $v(\emptyset)=0$ is called a set function. The set of all set functions in $(N,\Gamma)$ is denoted by $G(N,\Gamma)$.

Aumann and Drze [1] gave the Shapley value on $G(N,\Gamma)$ as follows:

$$\delta_i(N,v,\Gamma)=\sum_{i\in S\subseteq B_k}\frac{(s-1)!(b_k-s)!}{b_k!}(v(S)-v(S\backslash i))\quad \forall i\in N,\quad (1)$$

where$B_k\in\Gamma$.

### B. Cooperative games on augmenting systems

A set system on $N$ is a pair $(N,\boldsymbol{F})$, where $\boldsymbol{F}\subseteq 2^N$ is a family of subsets. The sets belong to $\boldsymbol{F}$ are called feasible.

**Definition 1.** [13] An augmenting system is a set system $(N,\boldsymbol{F})$ with the following properties:

A1: $\emptyset\in\boldsymbol{F}$;

A2: If $S,T\in\boldsymbol{F}$ with $S\cap T\neq\emptyset$, then $S\cup T\in\boldsymbol{F}$;

A3: If $S,T\in\boldsymbol{F}$ with $S\subseteq T$, then there exists $i\in T\backslash S$ such that $S\cup i\in\boldsymbol{F}$.

An augmenting system $(N,\boldsymbol{F})$ is said to be normal, if we have $N=\cup_{S\in\boldsymbol{F}}S$. A compatible ordering of an augmenting system $(N, \boldsymbol{F})$ is given by $i_1 < i_2 < \cdots < i_n$ with $\{i_1, i_2, \ldots, i_j\} \in \boldsymbol{F}$ for all $j = 1, 2, ..., n$. A compatible ordering corresponds to a maximal chain, the set of all maximal chains in $\boldsymbol{F}$ is denoted by $\mathrm{Ch}(\boldsymbol{F})$. The cardinality of $\mathrm{Ch}(\boldsymbol{F})$ is denoted by $c(N)$. For any $S \in \boldsymbol{F}$, $c(S)$ denotes the number of maximal chains from $\emptyset$ to $S$, and $c(S \cup i, N)$ is the number of maximal chains from $S \cup i$ to $N$, where $S \cup i \in \boldsymbol{F}$. For any $S \in \boldsymbol{F}$, let $S^* = \{i \in N \backslash S : S \cup i \in \boldsymbol{F}\}$. A game on an augmenting system is a set function $v : \boldsymbol{F} \to \Re_+$, such that $v(\emptyset) = 0$.

Bilbao and Ordonez [15] defined the Shapley function for games on augmenting systems as follows:

$$\phi_i(N, v, \boldsymbol{F}) = \sum_{\{S \in \boldsymbol{F} : i \in S^*\}} \frac{c(S)c(S \cup i, N)}{c(N)} (v(S \cup i) - v(S) \qquad \forall i \in N. \quad (2)$$

Two axiomatic systems of the given Shapley function are studied by using *hierarchical strength* and *chain axiom*.

## III. Games on augmenting systems with a coalition structure

In this section we shall research games on augmenting systems with a coalition structure. Different to the coalition structure given by Aumann and Drze [1] and Owen [2, 3], the coalition structure given in this paper does not require the intersection of different unions is empty set. Similar to above analysis, we give the following discussion.

A coalition structure $\Gamma' = \{B_1, B_2, \ldots, B_m\}$ on player set $N$ is a set of unions on $N$, where $\cup_{1 \le h \le m} B_h = N$. Let $\boldsymbol{P}(\Gamma')$ be the set of all probability distributions in $\Gamma'$. For any $P \in \boldsymbol{P}(\Gamma')$ and any $B_k \in \Gamma'$, we have $P(B_k) \ge 0$ and $\sum_{B_k \in \Gamma'} P(B_k) = 1$.

Let $B_k \in \Gamma'$, an augmenting system on $B_k$ is a set system $(B_k, \boldsymbol{F}_{B_k})$ as given in Definition 1, where the following conditions hold:

A1: $\emptyset \in \boldsymbol{F}_{B_k}$;

A2: If $S, T \in \boldsymbol{F}_{B_k}$ with $S \cap T \ne \emptyset$, then $S \cup T \in \boldsymbol{F}_{B_k}$;

A3: If $S, T \in \boldsymbol{F}_{B_k}$ with $S \subseteq T$, then there exists $i \in T \backslash S$ such that $S \cup i \in \boldsymbol{F}_{B_k}$.

By $L_M(B_k, \boldsymbol{F}_{B_k})$, we denote the set of all feasible coalitions in with respect to $\boldsymbol{F}_{B_k}$, where $k \in M = \{1, 2, ..., m\}$. A function $v : L_M(B_k, \boldsymbol{F}_{B_k}) \to \Re_+$, such that $v(\emptyset) = 0$, is called a set function. The set of all set functions in $L_M(B_k, \boldsymbol{F}_{B_k})$ is denoted by $G_M(B_k, \boldsymbol{F}_{B_k})$.

**Remark 1.** *In this paper, without special explanation, we always have $B_k \in \boldsymbol{F}_{B_k}$ for each $k \in M$.*

### *A. The core of games on augmenting systems with a coalition structure*

**Definition 2.** Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$, and $P \in \boldsymbol{P}(\Gamma')$ be a probability distribution in $\Gamma'$. The core $C_M(B_k, v, \boldsymbol{F}_{B_k})$ of $v$ in $L_M(B_k, \boldsymbol{F}_{B_k})$ is given as:

$$C_M(B_k, v, \boldsymbol{F}_{B_k}) = \{x \in \Re^n_+ | \sum_{i \in N} x_i = \sum_{k \in M} P(B_k) v(B_k),$$
$$\sum_{i \in S} x_i \ge \sum_{k \in M} P(B_k) v(S \cap B_k), \forall S \in L_M(B_k, \boldsymbol{F}_{B_k})\}.$$

When there is only one union in $\Gamma'$, then Definition 2 degenerates to be the core of games on augmenting systems.

**Definition 3.** Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$, $v$ is said to be quasi convex if we have

$$v(S \cup T) + v(S \cap T) \ge v(S) + v(T)$$

for any$S, T \in L_M(B_k, \boldsymbol{F}_{B_k})$.

**Theorem 1.** *Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$ be quasi convex, and $P \in \mathbf{P}(\Gamma')$ be a probability distribution in $\Gamma'$. Then $C_M(B_k, v, \boldsymbol{F}_{B_k}) \ne \emptyset$, and can be expressed by*

$$C_M(B_k, v, \boldsymbol{F}_{B_k}) = \{z \in \Re^n_+ | \sum_{i \in N} z_i = \sum_{k \in M} P(B_k) x^{B_k},$$
$$\forall x^{B_k} \in C(B_k, v, \boldsymbol{F}_{B_k}), \forall k \in M\},$$

*where $C(B_k, v, \boldsymbol{F}_{B_k})$ denotes the core of $v$ in $\boldsymbol{F}_{B_k}$.*

**Proof:** From the quasi convexity of $v$, we have $C(B_k, v, \boldsymbol{F}_{B_k}) \ne \emptyset$ for any $k \in M$. Let

$$z_i = \sum_{k \in M} P(B_k) x_i^{B_k} \qquad \forall i \in N, \quad (3)$$

where $(x_i^{B_k})_{i \in B_k} \in C(B_k, v, \boldsymbol{F}_{B_k})$.

We first show $z = (z_i)_{i \in N} \in C_M(B_k, v, \boldsymbol{F}_{B_k})$. Since

$$\begin{aligned}\sum_{i \in N} z_i &= \sum_{i \in N} \sum_{k \in M} P(B_k) x_i^{B_k} \\ &= \sum_{k \in M} P(B_k) \sum_{i \in N} x_i^{B_k} \\ &= \sum_{k \in M} P(B_k) \sum_{i \in B_k} x_i^{B_k} \\ &= \sum_{k \in M} P(B_k) v(B_k)\end{aligned}$$

and

$$\begin{aligned}\sum_{i \in S} z_i &= \sum_{i \in S} \sum_{k \in M} P(B_k) x_i^{B_k} \\ &= \sum_{k \in M} P(B_k) \sum_{i \in S} x_i^{B_k} \\ &= \sum_{k \in M} P(B_k) \sum_{i \in S \cap B_k} x_i^{B_k} \\ &\ge \sum_{k \in M} P(B_k) v(S \cap B_k)\end{aligned}$$

for any $S \in L_M(B_k, \boldsymbol{F}_{B_k})$.

From Definition 2, we get $z = (z_i)_{i \in N} \in C_M(B_k, v, \boldsymbol{F}_{B_k}) \ne \emptyset$.

In the following, we shall show $z$ can be expressed by Eq.(3) for any $z \in C_M(B_k, v, \boldsymbol{F}_{B_k})$.

For any $k \in M$ and any $x^{B_k} = (x_i^{B_k})_{i \in B_k} \in C(B_k, v, \boldsymbol{F}_{B_k})$, let

$$\underline{x}_p^{B_k} = \min\left\{x_p^{B_k} \left| x^{B_k} \in C(B_k, v, \boldsymbol{F}_{B_k}), p \in B_k \right.\right\}$$

and

$$\bar{x}_r^{B_k} = \max\left\{x_r^{B_k} \left| x^{B_k} \in C(B_k, v, F_{B_k}), r \in B_k \right.\right\}.$$

It is apparent that $\underline{x}_p^{B_k} = \begin{cases} v(p) & p \in F_{B_k} \\ 0 & \text{otherwise} \end{cases}$.

If there exists $z \in C_M(B_k, v, \boldsymbol{F}_{B_k})$, which can not be expressed by Eq.(3), then there only exist two cases:

(i) $z_p < \sum_{k \in M} P(B_k)\, \underline{x}_p^{B_k}$;

(ii) $z_r > \sum_{k \in M} P(B_k) \overline{x}_r^{B_k}$.

For case (i): When $p \in \boldsymbol{F}_{B_k}$ for some $k \in M$, we have

$$\begin{aligned} z_p &< \sum_{k \in M} P(B_k)\, \underline{x}_p^{B_k} \\ &= \sum_{k \in M, p \in B_k} P(B_k) v(p); \end{aligned}$$

otherwise, $z_p < 0$, which contradict with the quasi convexity of $v$.

For case (ii): Let $R = \left\{ r \,\middle|\, z_r > \sum_{k \in M} P(B_k) \overline{x}_r^{B_k}, r \in N \right\}$, we have

$$\begin{aligned} \sum_{i \in N} z_i &= \sum_{i \in R} z_i + \sum_{i \in N \setminus R} z_i \\ &> \sum_{i \in R} \sum_{k \in M} P(B_k) \overline{x}_r^{B_k} + \sum_{i \in N \setminus R} \sum_{k \in M} P(B_k) x_p^{B_k} \\ &\geq \sum_{i \in N} \sum_{k \in M} P(B_k) x_p^{B_k} \\ &= \sum_{k \in M} P(B_k) \sum_{i \in N} x_p^{B_k} \\ &= \sum_{k \in M} P(B_k) v(B_k), \end{aligned}$$

which contradicts with $z \in C_M(B_k, v, \boldsymbol{F}_{B_k})$. Hence, $R = \emptyset$, and the proof is completed. ■

*B. The Shapley function for games on augmenting systems with a coalition structure*

Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$ and any given $P \in \boldsymbol{P}(\Gamma')$, following the work of Bilbao and Ordonez [15], we define the Shapley function on $G_M(B_k, \boldsymbol{F}_{B_k})$ as follows:

$$\varphi_i^M(B_k, v, \boldsymbol{F}_{B_k}) = \sum_{k \in M} \sum_{\substack{S \in \boldsymbol{F}_{B_k}, \\ i \in S^*}} \frac{P(B_k) c(S) c(S \cup i, B_k)}{c(B_k)} (v(S \cup i) - v(S)) \quad \forall i \in N, \tag{4}$$

which can be equivalently expressed by

$$\varphi_i^M(B_k, v, \boldsymbol{F}_{B_k}) = \sum_{k \in M} P(B_k) \varphi_i(B_k, v, \boldsymbol{F}_{B_k}) \;\; i \in N, \tag{5}$$

where $\varphi_i(B_k, v, \boldsymbol{F}_{B_k}) = \sum_{\{S \in \boldsymbol{F}_{B_k}, i \in S^*\}} \frac{c(S) c(S \cup i, B_k)}{c(B_k)} (v(S \cup i) - v(S))$. $c(B_k)$ is the cardinality of $\mathrm{Ch}(\boldsymbol{F}_{B_k})$.

From Eq.(4), we know when there is only one union in $\Gamma'$, then Eq.(4) degenerates to be Eq.(2). When each union has the same cardinality and all subsets of each union are feasible, then Eq.(4) degenerates to be the Shapley value for games on matroids (see [11]).

Given $i \in B_k (k \in M)$ and a compatible ordering $C \in \boldsymbol{F}_{B_k}$. Let $C(i) = \{i$ is the last element in $C\}$. Similar to Bilbao and Ordonez [15], we define $h_S^{B_k}(i)$ for $i$ in $S \in \boldsymbol{F}_{B_k}$ as follows:

$$h_S^{B_k}(i) = \frac{|\{C \in \boldsymbol{F}_{B_k} : S \subseteq C(i)\}|}{c(B_k)}.$$

**Definition 4.** Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$, $T \in L_M(B_k, \boldsymbol{F}_{B_k})$ is said to be a carrier in $L_M(B_k, \boldsymbol{F}_{B_k})$ of $v$ if $v(S \cap T) = v(S)$ for any $S \in L_M(B_k, \boldsymbol{F}_{B_k})$.

Let $f : G_M(B_k, \boldsymbol{F}_{B_k}) \rightarrow \Re_+^n$ be a solution on $G_M(B_k, \boldsymbol{F}_{B_k})$. Similar to Faige and Kern [9], Bilbao et al, [11] and Bilbao and Ordonez [15], we give the following properties.

*Linearity:* Let $v, w \in G_M(B_k, \boldsymbol{F}_{B_k})$ and all $\alpha, \beta \in \Re$. $P \in \boldsymbol{P}(\Gamma')$ is a probability distribution in $\Gamma'$. Then we have

$$\begin{aligned} & f^M(B_k, \alpha v + \beta w, \boldsymbol{F}_{B_k}) \\ &= \alpha f^M(B_k, v, \boldsymbol{F}_{B_k}) + \beta f^M(B_k, w, \boldsymbol{F}_{B_k}) \end{aligned}$$

*Probabilistic efficiency on unions:* Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$, and $P \in \boldsymbol{P}(\Gamma')$ be a probability distribution in $\Gamma'$. If $T \in L_M(B_k, \boldsymbol{F}_{B_k})$ is a carrier of $v$, then

$$\sum_{i \in T} f_i^M(B_k, v, \boldsymbol{F}_{B_k}) = \sum_{k \in M} P(B_k) v(T \cap B_k)$$

*Hierarchical strength in unions:* Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$ and any $k \in M$, we have

$$h_S^{B_k}(j) f_i(B_k, u_S, \boldsymbol{F}_{B_k}) = h_S^{B_k}(i) f_j(B_k, u_S, \boldsymbol{F}_{B_k})$$

for any $S \in \boldsymbol{F}_{B_k}$ with $i, j \in S$, where $u_S$ is the unanimity game on $S \in \boldsymbol{F}_{B_k}$ such that $u_S(T) = \begin{cases} 1 & S \subseteq T \\ 0 & \text{otherwise} \end{cases}$, and $f(B_k, u_S, \boldsymbol{F}_{B_k})$ is the restriction of $f$ in $\boldsymbol{F}_{B_k}$.

**Theorem 2.** *Let $v \in G_M(B_k, \boldsymbol{F}_{B_k})$, and $P \in \mathbf{P}(\Gamma')$ be a probability distribution in $\Gamma'$. The function $f : G_M(B_k, \boldsymbol{F}_{B_k}) \rightarrow \Re_+^n$ satisfies Linearity, Probabilistic efficiency on unions and Hierarchical strength in unions if and only if $f = \varphi$.*

**Proof:** From Eq.(4), we know *Linearity* holds;

*Probabilistic efficiency on unions:* From Eq.(4) and Definition 4, we get

$$\begin{aligned} & \sum_{i \in T} \varphi_i^M(B_k, v, \boldsymbol{F}_{B_k}) \\ &= \sum_{i \in N} \varphi_i^M(B_k, v, \boldsymbol{F}_{B_k}) \\ &= \sum_{i \in N} \sum_{k \in M} \sum_{\{S \in \boldsymbol{F}_{B_k} : i \in S^*\}} \frac{P(B_k) c(S) c(S \cup i, B_k)}{c(B_k)} \\ & \quad \times (v(S \cup i) - v(S)) \\ &= \sum_{k \in M} \sum_{i \in B_k} \sum_{\{S \in \boldsymbol{F}_{B_k} : i \in S^*\}} \frac{P(B_k) c(S) c(S \cup i, B_k)}{c(B_k)} \\ & \quad \times (v(S \cup i) - v(S)) \\ &= \sum_{k \in M} P(B_k) v(B_k) \\ &= \sum_{k \in M} P(B_k) v(T \cap B_k). \end{aligned}$$

*Hierarchical strength in unions:* From Eq.(5), we have

$$\begin{aligned}
&\varphi_i(B_k, u_S, \boldsymbol{F}_{B_k}) \\
&= \sum_{\{T \in \boldsymbol{F}_{B_k} : i \in T^*\}} \frac{c(T)c(T \cup i, B_k)}{c(B_k)} \left(u_S(T \cup i) - u_S(T)\right) \\
&= \sum_{\{S \subseteq T \in \boldsymbol{F}_{B_k} : i \in T^*\}} \frac{c(T)c(T \cup i, B_k)}{c(B_k)} \left(u_S(T \cup i) - u_S(T)\right) \\
&= \sum_{\{S \subseteq T \in \boldsymbol{F}_{B_k} : i \in T^*\}} \frac{c(T)c(T \cup i, B_k)}{c(B_k)} \\
&= \frac{1}{c(B_k)} |\{C \in \boldsymbol{F}_{B_k} : S \subseteq C(i)\}| \\
&= h_S^{B_k}(i).
\end{aligned}$$

Similarly, we obtain $\varphi_j(B_k, u_S, \boldsymbol{F}_{B_k}) = h_S^{B_k}(j)$. Thus, we get *Hierarchical strength in unions*.

Uniqueness. For any $v \in G_M(B_k, \boldsymbol{F}_{B_k})$, we first show $v$ can be expressed by

$$v = \sum_{k \in M} \sum_{\emptyset \neq S \in \boldsymbol{F}_{B_k}} c_S u_S, \tag{6}$$

where $c_S = \sum_{T \subseteq S, T \in \boldsymbol{F}_{B_k}} (-1)^{s-t} v(T)$.

For any $W \in L_M(B_k, \boldsymbol{F}_{B_k}) \backslash \emptyset$, without loss of generality, suppose $W \in \boldsymbol{F}_{B_k}$, we have

$$\begin{aligned}
&\left( \sum_{k \in M} \sum_{\emptyset \neq S \in \boldsymbol{F}_{B_k}} c_S u_S \right)(W) \\
&= \sum_{k \in M} \sum_{\emptyset \neq S \in \boldsymbol{F}_{B_k}} c_S u_S(W) \\
&= \sum_{\{S \subseteq W,\, S \in \boldsymbol{F}_{B_k}\}} c_S u_S(W) \\
&= \sum_{\{S \subseteq W,\, S \in \boldsymbol{F}_{B_k}\}} c_S \\
&= \sum_{\{S \subseteq W,\, S \in \boldsymbol{F}_{B_k}\}} \sum_{\{T \subseteq S, T \in \boldsymbol{F}_{B_k}\}} (-1)^{s-t} v(T) \\
&= \sum_{\{T \subseteq W, T \in \boldsymbol{F}_{B_k}\}} \sum_{\{S \subseteq W,\, S \in \boldsymbol{F}_{B_k}\}} (-1)^{s-t} v(T).
\end{aligned}$$

The Möbius inversion formula for the lattice $\boldsymbol{F}_{B_k}$ implies

$$\left( \sum_{k \in M} \sum_{\emptyset \neq S \in \boldsymbol{F}_{B_k}} c_S u_S \right)(W) = v(W).$$

From *Linearity*, we only need to show $f = \varphi$ on unanimity games. For any $S \in L_M(B_k, \boldsymbol{F}_{B_k}) \backslash \emptyset$, without loss of generality, suppose $S \in F_{B_k}$, define the unanimity game $u_S$ on $S$ as given above.

From *Hierarchical strength in unions*, we get

$$f_j(B_k, u_S, \boldsymbol{F}_{B_k}) = \frac{h_S^{B_k}(j)}{h_S^{B_k}(i)} f_i(B_k, u_S, \boldsymbol{F}_{B_k}).$$

Fix $i \in S$, we obtain

$$\begin{aligned}
&\sum_{j \in S} f_j(B_k, u_S, \boldsymbol{F}_{B_k}) \\
&= \sum_{j \in S \backslash i} \frac{h_S^{B_k}(j)}{h_S^{B_k}(i)} f_i(B_k, u_S, \boldsymbol{F}_{B_k}) + f_i(B_k, u_S, \boldsymbol{F}_{B_k}) \\
&= \sum_{j \in S} \frac{h_S^{B_k}(j)}{h_S^{B_k}(i)} f_i(B_k, u_S, \boldsymbol{F}_{B_k}) \\
&= \frac{f_i(B_k, u_S, \boldsymbol{F}_{B_k})}{h_S^{B_k}(i)}
\end{aligned}$$

From *Probabilistic efficiency on unions*, we get

$$\sum_{j \in S} f_j^M(B_k, u_S, \boldsymbol{F}_{B_k}) = P(B_k).$$

Thus, we have

$$f_i^M(B_k, u_S, \boldsymbol{F}_{B_k}) = f_i(B_k, u_S, \boldsymbol{F}_{B_k}) = P(B_k) h_S^{B_k}(i).$$

On the other hand, from Eq.(4), we get

$$\varphi_i^M(B_k, u_S, \boldsymbol{F}_{B_k}) = \begin{cases} P(B_k) h_S^{B_k}(i) & i \in S \\ 0 & \text{otherwise} \end{cases}.$$

Namely, $f = \varphi$ on unanimity games. ■

Similar to the property of *strong monotonicity* for the Shapley function on traditional case (see [16]), we give *strong monotonicity* for the Shapley value on $G_M(B_k, \boldsymbol{F}_{B_k})$as follows:

*Strong monotonicity on unions:* Let $v, w \in G_M(B_k, \boldsymbol{F}_{B_k})$, and $P \in \boldsymbol{P}(\Gamma')$ be a probability distribution in $\Gamma'$. If we have $v(S \cup i) - v(S) \geq w(S \cup i) - w(S)$ for any $S \in L_M(B_k, \boldsymbol{F}_{B_k})$ with $i \in S^*$, then

$$f_i^M(B_k, v, \boldsymbol{F}_{B_k}) \geq f_i^M(B_k, w, \boldsymbol{F}_{B_k}).$$

**Theorem 3.** *There is a unique solution* $f : G_M(B_k, \boldsymbol{F}_{B_k}) \rightarrow \Re_+^n$ *that satisfies Probabilistic efficiency on unions, Hierarchical strength in unions and Strong monotonicity on unions.*

**Proof:** From Eq.(4) and Theorem 2, we know existence holds. Next, we shall show uniqueness. Define the index $I$ of $v$ to be the minimum number of non-zero terms in some expression for $v$ of form (6).

When $I = 0$, from *Strong monotonicity*, we have

$$v(S \cup i) - v(S) = w(S \cup i) - w(S) = 0$$

for any $S \in L_M(B_k, \boldsymbol{F}_{B_k})$ with $i \in S^*$.

From *Probabilistic efficiency on unions* and *Hierarchical strength in unions*, we have

$$f_i^M(B_k, w, \boldsymbol{F}_{B_k}) = 0 \qquad \forall i \in N.$$

On the other hand, from Eq.(5), we have

$$\varphi_i^M(B_k, w, \boldsymbol{F}_{B_k}) = 0 \qquad \forall i \in N.$$

When $I = 1$. Without loss of generality, suppose $v = c_S u_S$, where $S \in \boldsymbol{F}_{B_k} \backslash \emptyset$.

From *Probabilistic efficiency on unions* and *Hierarchical strength in unions*, we have

$$f_i^M(B_k,w,\boldsymbol{F}_{B_k})=f_i^M(B_k,v,\boldsymbol{F}_{B_k})\\ =\begin{cases} P(B_k)c_S h_S^{B_k}(i) & i\in S\\ 0 & \text{otherwise}\end{cases}.$$

From Eq.(5), we get $f=\varphi$.
Therefore $f=\varphi$, whenever the index of $v$ is 0 or 1.
Assume now that $f=\varphi$, whenever the index of $v$ is at most $I$, and let $v$ have index $I+1$ with expression

$$v=\sum_{r=1}^{I+1} c_{S_r}u_{S_r},$$

where $S\in L_M(B_k,\boldsymbol{F}_{B_k})\backslash\emptyset$ for all $r=1,2,\ldots,I+1$.
Let $T=\cap_{r=1}^{I+1}S_r$, for any $i\in N\backslash T$, construct the game

$$w=\sum_{r:i\in S_r} c_{S_r}u_{S_r}.$$

The index of $w$ is at most $I$, since $v(S\cup i)-v(S)=w(S\cup i)-w(S)$ for any $S\in L_M(B_k,\boldsymbol{F}_{B_k})$ with $i\in S^*$. By induction and *Hierarchical strength in unions*, we have

$$f_i^M(B_k,v,\boldsymbol{F}_{B_k})=f_i^M(B_k,w,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M:i\in S_r\subseteq \boldsymbol{F}_{B_k},} P(B_k)c_{S_r}h_{S_r}^{B_k}(i).$$

From Eq.(5), we get $f=\varphi$.
When $i\in T$. From *Probabilistic efficiency on unions* and *Hierarchical strength in unions*, we have

$$f_i^M(B_k,v,\boldsymbol{F}_{B_k})=f_i^M(B_k,w,\boldsymbol{F}_{B_k})\\ =\sum_{\substack{k\in M:S_r\subseteq \boldsymbol{F}_{B_k},\\ r=1,2,..,I+1}} P(B_k)c_{S_r}h_{S_r}^{B_k}(i).$$

From Eq.(5), we have $f=\varphi$, and the result is obtained. ■

*C. Some properties*

**Property 1.** *Let $v\in G_M(B_k,\boldsymbol{F}_{B_k})$ be quasi convex, and $P\in\mathbf{P}(\Gamma')$ be a probability distribution in $\Gamma'$, then we have $\left(\varphi_i^M(B_k,v,\boldsymbol{F}_{B_k})\right)_{i\in N}\in C_M(B_k,v,\boldsymbol{F}_{B_k})$.*

**Proof:** From Theorem 2, we only need to show $\sum_{i\in S}x_i\geq\sum_{k\in M}P(B_k)v(S\cap B_k)$ for any $S\in L_M(B_k,\boldsymbol{F}_{B_k})$.
From the quasi convexity of $v$, we have

$$v(S\cup i)-v(S)\geq v(T\cup i)-v(T)$$

for any $S,T\in L_M(B_k,\boldsymbol{F}_{B_k})$ with $T\subseteq S$ and $i\in S^*,\ i\in T^*$. From Eq.(4), we have

$$\sum_{i\in S}\varphi_i^M(B_k,v,\boldsymbol{F}_{B_k})\\ =\sum_{i\in S}\sum_{k\in M}P(B_k)\varphi_i(B_k,v,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M}P(B_k)\sum_{i\in S}\varphi_i(B_k,v,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M}P(B_k)\sum_{i\in S\cap B_k}\varphi_i(B_k,v,\boldsymbol{F}_{B_k})\\ \geq\sum_{k\in M}P(B_k)\sum_{i\in S\cap B_k}\varphi_i(S\cap B_k,v,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M}P(B_k)v(S\cap B_k).$$

■

**Definition 5.** Let $v\in G_M(B_k,\boldsymbol{F}_{B_k})$, and $P\in\boldsymbol{P}(\Gamma')$ be a probability distribution in $\Gamma'$, the vector $y=(y_i)_{i\in N}$ is said to be a population monotonic allocation scheme (PMAS) for $v$ in $L_M(B_k,\boldsymbol{F}_{B_k})$, if
(1) $\sum_{i\in S}y_i(S)=\sum_{k\in M}P(B_k)v(S\cap B_k)\quad\forall S\in L_M(B_k,\boldsymbol{F}_{B_k})$;
(2) $y_i(S)\leq y_i(T)\ \forall i\in S,\ \forall S,T\in L_M(B_k,\boldsymbol{F}_{B_k})$ *s.t.* $S\subseteq T$.

**Property 2.** *Let $v\in G_M(B_k,\boldsymbol{F}_{B_k})$ be quasi convex, and $P\in\mathbf{P}(\Gamma')$ be a probability distribution in $\Gamma'$, then is a PMAS for $v$ in $L_M(B_k,\boldsymbol{F}_{B_k})$.*

**Proof:** From Eq.(5), we have

$$\sum_{i\in S}\varphi_i^M(S,v,\boldsymbol{F}_{B_k})\\ =\sum_{i\in S}\sum_{k\in M}P(B_k)\varphi_i(S,v,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M}P(B_k)\sum_{i\in S}\varphi_i(S,v,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M}P(B_k)\sum_{i\in S}\varphi_i(S\cap B_k,v,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M}P(B_k)v(S\cap B_k);$$

From Property 1, we get the second condition given in Definition 5. ■

**Property 3.** *Let $v\in G_M(B_k,\boldsymbol{F}_{B_k})$ be quasi convex, and $P\in\mathbf{P}(\Gamma')$ be a probability distribution in $\Gamma'$, then we have $\varphi_i^M(B_k,v,\boldsymbol{F}_{B_k})\geq\sum_{k\in M}P(B_k)v(i)$ for any $i\in L_M(B_k,\boldsymbol{F}_{B_k})$.*

**Proof:** From the quasi convexity of $v$, we have

$$v(S\cup i)-v(S)\geq v(i)$$

for any $S\in L_M(B_k,\boldsymbol{F}_{B_k})$ with $i\in S^*$. From Eq.(4), we get

$$\varphi_i^M(B_k,v,\boldsymbol{F}_{B_k})\\ =\sum_{k\in M}\sum_{\{S\in\boldsymbol{F}_{B_k}:i\in S^*\}}\frac{P(B_k)c(S)c(S\cup i,B_k)}{c(B_k)}(v(S\cup i)\\ -v(S))\\ \geq\sum_{k\in M}\sum_{\{S\in\boldsymbol{F}_{B_k}:i\in S^*\}}\frac{P(B_k)c(S)c(S\cup i,B_k)}{c(B_k)}v(i)\\ =\sum_{k\in M}P(B_k)v(i)\sum_{\{S\in\boldsymbol{F}_{B_k}:i\in S^*\}}\frac{c(S)c(S\cup i,B_k)}{c(B_k)}\\ =\sum_{k\in M}P(B_k)v(i).$$

■

TABLE 1: THE COALITION VALUES

| $S$ | $v(S)$ | $S$ | $v(S)$ | $S$ | $v(S)$ |
|---|---|---|---|---|---|
| {1} | 2 | {1,2} | 5 | {4,5} | 7 |
| {2} | 1 | {1,3} | 6 | {1,2,3} | 15 |
| {3} | 2 | {2,3} | 8 | {1,3,4} | 10 |
| {5} | 1 | {2,4} | 5 | {2,4,5} | 13 |

## IV. A NUMERICAL EXAMPLE

Let the player set $N = \{1, 2, 3, 4, 5\}$. The coalition structure is given by $\Gamma' = \{B_1, B_2, B_3\}$, where $B_1 = \{1, 2, 3\}$, $B_2 = \{1, 3, 4\}$ and $B_3 = \{2, 4, 5\}$. The augmenting systems on $B_1, B_2$ and $B_3$ are given as: $\boldsymbol{F}_{B_1} = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{2, 3\}, B_1\}$, $\boldsymbol{F}_{B_2} = \{\emptyset, \{1\}, \{1, 3\}, B_2\}$ and $\boldsymbol{F}_{B_3} = \{\emptyset, \{2\}, \{5\}, \{2, 4\}, \{4, 5\}, B_3\}$. The corresponding coalition values are given by table 1. If we use the values of the unions as the probability distribution in $\Gamma'$, from table 1, we get the probability distribution is

$P(B_1) = 15/38$, $P(B_2) = 10/38$ and $P(B_3) = 13/38$.

From Eq.(4), we obtain the player Shapley values are

$$\varphi_1^M(B_k, v, \boldsymbol{F}_{B_k}) = 2.3,$$
$$\varphi_2^M(B_k, v, \boldsymbol{F}_{B_k}) = 3,$$
$$\varphi_3^M(B_k, v, \boldsymbol{F}_{B_k}) = 3.4,$$
$$\varphi_4^M(B_k, v, \boldsymbol{F}_{B_k}) = 2.8,$$
$$\varphi_5^M(B_k, v, \boldsymbol{F}_{B_k}) = 1.5.$$

From table 1, we know this is a quasi convex game, and the Shapley value given above is an element in its core.

## V. CONCLUSION

We have researched a special kind of games under precedence constraints with a coalition structure, which is named games on augmenting systems with a coalition structure. The core and the Shapley function for the given model is researched. An equivalent form of the core is studied, and two axiomtizations of the given Shapley are discussed. Some properties are also researched, which are the same as classical case.

However, we only study the core and Shapley function for games on augmenting systems with a coalition structure, and it will be interesting to research other payoff indices.

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