A systematic approach for finding Hamiltonian cycles with a prescribed edge in crossed cubes

Jheng-Cheng Chen, Chia-Jui Lai, and Chang-Hsiung Tsai

Abstract—The crossed cube is one of the most notable variations of hypercube, but some properties of the former are superior to those of the latter. For example, the diameter of the crossed cube is almost the half of that of the hypercube. In this paper, we focus on the problem embedding a Hamiltonian cycle through an arbitrary given edge in the crossed cube. We give necessary and sufficient condition for determining whether a given permutation with \( n \) elements over \( \mathbb{Z}_n \) generates a Hamiltonian cycle pattern of the crossed cube. Moreover, we obtain a lower bound for the number of different Hamiltonian cycles passing through a given edge in an \( n \)-dimensional crossed cube. Our work extends some recently obtained results.

Keywords—Interconnection network, Hamiltonian, Crossed cubes, Prescribed edge.

I. INTRODUCTION

The ring structure is a fundamental network for multi-processor systems and suitable for developing simple algorithms with low communication cost. Many efficient algorithms were designed with respect to rings for solving a variety of algebraic problems, graph problems, and some parallel applications, such as those in image and signal processing [2], [9]. To carry out a ring-structure algorithm on a multiprocessor computer or a distributed system, the processes of the parallel algorithm need to be mapped to the nodes of the interconnection network in the system such that any two adjacent processes in the cycle are mapped to two adjacent node of the network. Due to execute a parallel program efficiently, the targeted interconnection network posses a Hamiltonian cycle, i.e., a cycle that passes every node of the network exactly once if the number of processes in the ring-structure parallel algorithm equals the number of nodes of the interconnection network. On the other hand, each link in a parallel distributed system may be assigned with distinct bandwidth, thus, it is meaningful to study the problem of how to embed a Hamiltonian cycle into a network such that these cycles pass through a special edge.

Hypercubes are the most well known of all interconnection networks for parallel computing, given their basic simplicity, their generally desirable topological and algorithmic properties. Thus, many practical parallel computer systems, such as Intel iPSC, the nCUBE family [6], the SGI’s Origin 2000 [10], and the Connection Machine [11], employ the hypercubes as the interconnection network. The crossed cube proposed by Efe [3] is one of the most notable variations of hypercube, but some properties of the former are superior to those of the latter. For example, the diameter of the crossed cube is almost the half of that of the hypercube. With regard to cycles embedding of crossed cubes, many interesting results have received considerable attention [1], [5], [7], [8], [12], [13], [14]. In particular, Zheng and Latifi [14] introduced to the notion of reflected link label sequences and proposed a kind of codeword, called Generalized Gray Code. Applying these concepts, they showed that \( CQ_n \) can embed cycles of arbitrary length from 4 to \( 2^n \). In this paper, we consider the problem of embedding a Hamiltonian cycle passing through a prescribed edge in the crossed cube. We introduce a new concept, the cycle pattern, and use it to propose a systematic approach for embedding a desired Hamiltonian cycle in the crossed cube. In particular, we give necessary and sufficient condition for determining whether or not a given permutation with \( n \) elements over \( \mathbb{Z}_n \) generates a Hamiltonian cycle pattern of the crossed cube. Our work extends some recently obtained results in [12], [14].

The rest of this paper is organized as follows. Section II introduces definitions and reflected edge label sequence that will be used throughout this paper. In Section III, we propose cycle pattern concept and give necessary and sufficient condition for determining whether or not a given permutation with \( n \) elements over \( \mathbb{Z}_n \) generates a Hamiltonian cycle pattern of the crossed cube. Based on this concept, how many distinct Hamiltonian cycles pass through a given edge in \( CQ_n \) is calculated in Section IV. Conclusions are given in the final section.

II. PRELIMINARIES

A topology of an interconnection network is conveniently represented by an undirected simple graph \( G = (V, E) \), where \( V(G) \) and \( E(G) \) is the vertex set and the edge set of \( G \), respectively. Throughout this paper, vertex and node, edge and link, graph and network are used interchangeably. For graph terminology and notation not defined here we refer the reader to [9]. A walk in a graph is a finite sequence \( \omega : \lambda_0, e_1, \lambda_1, e_2, \lambda_2, \ldots, \lambda_{k-1}, e_k, \lambda_k \) whose terms are alternately vertices and edges such that, for \( 1 \leq i \leq k \), the edge \( e_i \) has ends \( \lambda_{i-1} \) and \( \lambda_i \), thus each edge \( e_i \) is immediately preceded and succeeded by the two vertices with which it is incident. In particular, a walk \( \omega \) is called a path if all internal vertices, \( \lambda_i \) for \( 1 \leq i \leq k-1 \), of the walk \( \omega \) are distinct. The first vertex \( \lambda_0 \) of \( \omega \) is called its start vertex, and the vertex \( \lambda_k \) is called a last vertex. Both of them are called end-vertices of the path \( \omega \). For simplicity, the path \( \omega \) is also denoted by \( \lambda_0, \lambda_1, \ldots, \lambda_k \). If \( \lambda_0 = \lambda_k \), then \( \omega \) is called a cycle. A cycle of

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Fig. 1. Crossed cubes CQ3 and CQ4.

length l is called a l-cycle. A path (respectively, cycle) which traverses each vertex of G exactly once is Hamiltonian path (respectively, Hamiltonian cycle).

An n-dimensional crossed cube, denoted as CQn, was first proposed by Efe [3]. It is derived by "crossing" some edges in Qn. With exactly same hardware cost as hypercube, it has been shown that such a simple variation gains important benefits such as greatly reduced diameter. To define crossed cubes, the notion so called "paired related" relation is introduced. Let R = \{(00, 00), (10, 10), (01, 11), (11, 01)\}. Two binary strings u1v0 and v1u0 are paired related, denoted as u ~ v, if and only if \((u, v) \in R\). Subsequently, a crossed cube of dimension n is an undirected graph consisting of 2^n vertices labeled from 0 to 2^n − 1 and defined recursively as following:

**Definition 1:** [3] The crossed cube CQ1 is a complete graph with two vertices labeled by 0 and 1, respectively. For \(n \geq 2\), an n-dimensional crossed cube CQn consists of two \((n-1)\)-dimensional sub-crossed cubes, CQ_{n-1} and CQ'_{n-1}, and a perfect matching between the vertices of CQ_{n-1} and CQ'_{n-1} according to the following rule:

Let \(V(CQ_{n-1}) = \{0u_{n-2}u_{n-3}\cdots u_0 : u_0 = 0 \text{ or } 1\}\) and \(V(CQ'_{n-1}) = \{1v_{n-2}v_{n-3}\cdots v_0 : v_0 = 0 \text{ or } 1\}\). The vertex \(u = 0u_{n-2}u_{n-3}\cdots u_0 \in V(CQ_{n-1})\) and the vertex \(v = 1v_{n-2}v_{n-3}\cdots v_0 \in V(CQ_{n-1}')\) are adjacent in CQn if and only if

\[
\begin{align*}
(1) &\quad u_{n-2} = v_{n-2} \text{ if } u \text{ is even, and} \\
(2) &\quad 2u_{j+1}v_ju_j \sim v_{j+1}v_ju_j, \quad 0 \leq i < \left\lfloor \frac{n-1}{2} \right\rfloor.
\end{align*}
\]

An edge \((u, v) \in E(CQ_n)\) is labeled by \(j\) if \(u_j \neq v_j\) and \(u_i = v_i\) for \(j + 1 \leq i \leq n - 1\), i.e., \(v\) is the \(j\)-th dimensional neighbor (abbreviated as \(j\)-neighbor) of \(u\), denoted by \(v[j]u\) or \(v[j]v\). It is observed that each vertex \(u\) in CQn has \(n\) neighbors in CQn; \(u\) has exactly one \(j\)-neighbor for \(0 \leq j \leq n - 1\). As a consequence, there are \(2^n-1\) edges labeled by \(j\), \(0 \leq j \leq n - 1\), in CQn. For example, the graphs shown in Figure 1 are CQ3 and CQ4.

A path in CQn might be specified by the source vertex and a sequence of labels detailing the edges to be traversed, for example, the path in CQ3 detailed as having the source vertex 000 and then following the edges labeled 1,2,1 (also denoted \([1,2,1]\)) is actually the path 000, 010, 110, 100, also denoted 000[1,2,1]100. Besides, 000[1,2]100, 000[0,1]100, 000[0,1][2,1]100, and 000[1,2][1,2]100 are represented the identical path 000, 010, 110, 100. Therefore, the sequence \(L = [d_1, d_2, \ldots, d_m]\) is called an Edge Label Sequence in CQn if two adjacent labels are not identical where \(d_i \in Z_n\), \(Z_n = \{0, 1, \ldots, n - 1\}\), for \(1 \leq i \leq m\).

A walk, \(\omega(L, u) = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m\) in CQn can be generated with respect to a given edge label sequence \(L\) and a given vertex \(u\) as follows: \(\lambda_0 = u\), and \(\lambda_j\) is the \(d_j\)-neighbor of \(\lambda_{j-1}\) in CQn where \(1 \leq j \leq m\), i.e., \(\lambda_{j-1}[d_{j-1}]\lambda_j\). This way, this walk \(\omega(L, u)\) is also represented as \(\lambda_0[\lambda_1[\cdots[\lambda_m]]]\). In particular, the edge label sequence \(L\) is interesting in this paper when it generates a loop-free path \(\omega(L, u)\) starting from any vertex \(u\) in CQn.

Hereafter, we are interesting a special edge label sequence called reflected link label sequence generated by a systematic method. A reflected edge label sequence of length \(2^k\) is generated from a permutation with \(k\) elements over \(Z_n\). Let \(\pi_k = (d_1, d_2, \ldots, d_k)\), \(1 \leq k \leq n\), be a permutation over \(Z_n\) with \(k\) elements, and let \(\pi_k(i) = \langle d_1, d_2, \ldots, d_i \rangle\). The Reflected Edge Label Sequence, RL_{\pi_k} defined by \(\pi_k\), be generated recursively as follows:

\[
\begin{align*}
RL_{\pi_k}(1) &\quad = d_1 \\
RL_{\pi_k}(i) &\quad = RL_{\pi_k}(i-1), d_i, RL_{\pi_k}(i-1), 1 \leq i \leq k; \text{ and} \\
RL_{\pi_k} &\quad = RL_{\pi_k}(k) \quad \text{as a result, the } RL_{\pi_k}\text{ defined by arbitrary permutation } \pi_k \text{ over } Z_n \text{ is a symmetry edge label sequence in } CQ_n. \text{ Zheng and Latifi} [14] \text{ observe the following lemma.}
\end{align*}
\]

**Lemma 1:** [14] For any vertex \(u\) in CQn and any \(\pi_k\) permutation over \(Z_n\) with \(n\) elements, the walk \(\omega(RL_{\pi_k}, u)\) corresponds to a Hamiltonian path of CQn, that start from \(u\).

**Proof.** Let \(\pi_0 = (d_1, d_2, \ldots, d_n)\) and \(\pi_1 = (d_1, d_2, \ldots, d_n)\) be two distinct permutations over on \(Z_n\). Since \(\pi_0 \neq \pi_1\), there exist \(k\) and \(h\) such that \(k \neq h\) and \(d_k = d_h\). By Lemma 2 and for any two vertices \(u, v \in V(CQ_n)\), the path \(\omega(RL_{\pi_k}, u)\) and \(\omega(RL_{\pi_1}, v)\) passes through \(2^n-k\) and \(2^n-h\) edges in dimension \(d_1\) and \(d_h\) of CQn, respectively. As a conclusion, \(\omega(RL_{\pi_1}, u)\) and \(\omega(RL_{\pi_k}, v)\) correspond to two distinct Hamiltonian paths of CQn.

\[\square\]

**III. Cycle patterns**

Let \(L\) be an edge label sequence of CQn with \(l\) elements. We called \(L\) an l-cycle pattern, l-CP for short, of CQn if the walk \(\omega(L, u)\) forms an l-cycle for every vertex \(u\). Particularly, appending \(d_k\) to the end of the sequence the \(RL_{\pi_k}\), we obtain an edge label sequence \([RL_{\pi_k}, d_k]\) where \(\pi_k = (d_1, d_2, \ldots, d_k)\). For convenience, we use C_{\pi_k} to denote the special sequence \([RL_{\pi_k}, d_k]\) in the following; thus, C_{\pi_k} contains \(2^k\) edge labels for any permutation, \(\pi_k\), with \(k\) elements. In other word, we are interested the special permutations, \(\pi_k\) for \(1 \leq k \leq n\), with \(k\) elements over \(Z_n\), satisfy that C_{\pi_k} is a 2^k-CP of CQn. Such permutation \(\pi_k\) is called a 2^k-CP generator of CQn. In particular, \(\pi_0\) is called a Hamiltonian cycle pattern generator if C_{\pi_0} is a 2^n-CP.

Herein, fundamental properties of C_{\pi_0} are proposed in order to help for constructing 2^k-CP generator of CQn. Let B_2 = \{00, 01, 10, 11\} and \(\gamma : B_2 \rightarrow B_2\) be a bijection mapping, which is defined as follows: \(\gamma(v_0u_0) = v_1u_0\) if and
only if \( u_1v_0 \) and \( v_1v_0 \) are pair related, i.e., \( u_1u_0 \sim v_1v_0 \). Let \( C_0 : B_2 \rightarrow B_2 \) and \( C_1 : B_2 \rightarrow B_2 \) be two bijection mappings, which are defined as follows \( C_0(u_1u_0) = u_1v_0 \) and \( C_1(u_1u_0) = v_1v_0 \), respectively. Thus, a composite function \( g \circ f : B_2 \rightarrow B_2 \) defined by \( (g \circ f)(u) = g(f(u)) \) for all \( u \in B_2 \) is obtained where \( g, f \in \{ C_0, C_1 \} \). The following lemma is useful in the proof of Lemma 5. It is not difficult to verify by straightforward; hence, the detail proof is omitted.

**Lemma 4:** For \( k \geq 1 \), let \( u \) be any 2-bit binary string. Then,

1. \( f^2(u) = u \) where \( f \in \{ C_0, C_1 \} \).
2. \( C_1 \circ f(u) = C_0 \circ f(u) \).
3. \( C_1 \circ f(u) = f^2(u) \).

Consequently, Lemma 5 gives necessary and sufficient conditions for determining \( 2^k \)-CP generator of \( CQ_n \) for \( 2 \leq k \leq n \).

**Lemma 5:** For \( n \geq 3 \) and \( 2 \leq k \leq n \), let \( \pi_k = \langle d_1, d_2, d_3, \ldots, d_k \rangle \) be a permutation over \( Z_n \) with \( k \) elements where \( 2 \leq k \leq n \). Then, \( \pi_k \) is not a \( 2^k \)-CP generator if and only if \( \min\{ d_1, d_k \} \) is even and \( |d_1 - d_k| \geq 2 \). Moreover, for any vertex \( u \in CQ_n \), two end-vertices of the walk \( \omega(C_{\pi_k}, u) \) are only different from the \( g \)-th bit position where \( g = \min\{ d_1, d_k \} + 1 \) if \( \pi_k \) is not a \( 2^k \)-CP generator in \( CQ_n \).

**Proof.** The lemma is proved by induction on \( k \). We first claim the base case for \( k = 2 \). Without loss of generality, we may assume that \( d_1 < d_2 \). Also let \( u \) be a vertex of \( CQ_n \).

\((\Rightarrow)\) Suppose that \( d_1 = d_2 - 1 \) if \( d_1 \) is even; otherwise, \( d_1 \) is an odd integer. It is claimed that \( u[d_1, d_2, d_1, d_2] \) forms a 4-cycle in \( CQ_n \), i.e., \( u = v \). It is obvious that \( u = v \) if \( d_1 = d_2 - 1 \). So, we just consider only \( d_1 \) is an odd integer. Since \( d_1 < d_2 \) and \( d_1 \) is odd, \( u_1 = v_1 \) for \( j > d_1 \) or \( j < d_1 - 1 \). We check only whether \( u_1d_1u_1d_1 \) is a valid cycle or not. Note that \( u_1d_1u_1d_1 = (\gamma \circ f)^2(u_1d_1u_1d_1) \). By Lemma 4, \( (\gamma \circ f)^2(u_1d_1u_1d_1) = u_1d_1u_1d_1 \). Therefore, \( u = v \).

\((\Leftarrow)\) Suppose that \( d_1 \) is even and \( d_1 < d_2 - 1 \). It is observed that \( u_2 = v_2 \) for \( j > d_1 + 1 \) or \( j < d_1 - 1 \), and \( u_2u_1u_2 = (\gamma \circ f)^2(u_2u_1u_2) \). By Lemma 4, \( (\gamma \circ f)^2(u_2u_1u_2) = C_1(u_2u_1u_2) = u_2u_1u_2 \). Therefore, \( u_1d_1u_2u_1d_2 \) is not a 4-cycle and \( u, v \) are only different from the \( d_1 \)-th bit position.

As a consequence, we conclude that the lemma is true for \( k \leq m - 1 \). Let \( \pi_m \) be a permutation, \( (d_1, d_2, \ldots, d_m, d_m) \), with \( m \) elements over \( Z_n \). The \( \pi_m \) will be represented by \( [RL_{\pi_m(m-1)}]_{d_m}[RL_{\pi_m(m-1)}]_{d_m} \) where \( RL_{\pi_m(m-1)} \) is a reflected edge label sequence generated by the permutation \( (d_1, d_2, \ldots, d_m, d_m) \). Let \( u = \{ u_{m-1}u_{m-2} \ldots u_0 \} \) be an arbitrary vertex of \( CQ_n \). The walk \( \omega(C_{\pi_m}, u) \) is written as \( [RL_{\pi_m(m-1)}]_{d_m}v_{\{RL_{\pi_m(m-1)}\}_{d_m}} \). Thus, \( u[RL_{\pi_m(m-1)}]_{d_m}v_{\{RL_{\pi_m(m-1)}\}_{d_m}} \) is a \( 2^{m-1} \)-CP generator, the proof is divided into two parts as follows.

**Case 1:** \( \pi_m(m-1) \) is a \( 2^{m-1} \)-CP generator.

Hence \( u \) and \( y \) is a \( d_m \)-neighbor of \( u \) and \( y \), respectively, i.e., \( u[d_m]u[y[d_m]] \). Therefore, \( u[d_m]x[d_m]y[d_m]z[d_m]v_{\{RL_{\pi_m(m-1)}\}_{d_m}} \) is a path having the source vertex \( u \) and then following the edge label sequence \( [d_m-1, d_m, d_m-1, d_m] \).
Since $d_{n-1} > d_n$ and by induction hypothesis, $d_m$ is an even integer. Hence the vertex $y = y_{n-1}y_{n-2} \ldots y_0$ satisfies that

$$y_i = u_i \text{ for } i > h,$$

$$y_{h}y_{h-1} = \langle \gamma(u_{h-1}u_{h}) \rangle \text{ for } \frac{k-3}{2} \geq j \geq \frac{d_m+2}{2},$$

$$y_{d_m+1}y_{d_m} = C_0 \circ \gamma(u_{d_m+1}u_{d_m}),$$

and

$$y_j = u_j \text{ for } (d_m - 1) \geq j \geq 0.$$

Note that $y_iRL_{\pi_{m-1}}(n-1)\gamma$. Thus, the vertex $z = z_{n-1}z_{n-2} \ldots z_0$ satisfies that

$z_i = u_i \text{ for } i > h,$

$z_{h}z_{h-1} = u_{h}u_{h-1},$

$z_{d_m+1}z_{d_m} = \gamma(u_{d_m+1}u_{d_m})$, and

$z_{2j+1}z_{2j} = \gamma(u_{2j+1}u_{2j})$ for $\frac{d_m+1}{2} \geq j \geq 0.$

Since $C_0 \circ C_0 = C_0 \circ C_0 \circ C_0$, $\gamma \circ \gamma(u_{d_m}, u_{d_m-1}) = \gamma \circ C_0(u_{d_m+1}u_{d_m-1})$. Consequently, the vertex $z$ can be represented by

$$z_i = u_i \text{ for } i \neq d_m,$$

$$z_{d_m+1}z_{d_m} = u_{d_m+1}u_{d_m},$$

and

$$z_{2j+1}z_{2j} = \gamma(u_{2j+1}u_{2j}) \text{ for } \frac{d_m+1}{2} \geq j \geq 0.$$

Thus, it is observed that vertex $z$ and $u$ are not adjacent in $CQ_n$. Since $z[d_m]u$, the vertex $v = v_{n-1}v_{n-2} \ldots v_0$ satisfies that

$$v_i = u_i \text{ for } i \neq d_m,$$

and

$$v_{d_m+1} = u_{d_m+1}.$$

Therefore, vertex $u$ and $v$ are different from the $g$-th bit position where $g = d_m + 1$, that is, $g = \min\{d_m, d_m\} + 1$.

Given a permutation $\pi_n$ over $Z_n$ with $n$ elements, one can determine whether or not the permutation $\pi_n$ can generate a Hamiltonian cycle pattern, $C_{\pi_n}$, of $CQ_n$. If so, then the last two numbers $d_{n-1}$ and $d_n$ of $\pi_n$. Thus, we have the following corollary.

**Corollary 1:** For $n \geq 3$, let $\pi_n = \langle d_1, \ldots, d_{n-1}, d_n \rangle$ be a permutation over $Z_n$ with $n$ elements. Then, $C_{\pi_n}$ is a Hamiltonian cycle pattern of $CQ_n$ if and only if

1. $\min\{d_{n-1}, d_n\}$ is odd, or
2. $|d_n - d_{n-1}| = 1$.

### IV. DISTINCT HAMILTONIAN CYCLES PASSING A GIVEN EDGE

Given a Hamiltonian cycle pattern generator $\pi_n$ and any vertex $u$ in $CQ_n$, the walk $\omega(C_{\pi_n}, u)$ corresponds to a Hamiltonian cycle in $CQ_n$. In this section, we will consider several distinct Hamiltonian cycles with respect to Hamiltonian cycle pattern in use that they pass through the same prescribed edge.

**Lemma 6:** For $n \geq 3$, let $\pi_n$ be arbitrary permutation, $\langle d_1, d_2, \ldots, d_n \rangle$, with $n$ elements over on $Z_n$ and $(u, v)$ be any edge of $CQ_n$. Then, there exists a vertex $z$ such that the Hamiltonian path $\omega(RL_{\pi_n}, z)$ of $CQ_n$ passes through the edge $(u, v)$.

**Proof:** Let $\pi_n$ be a permutation, $\langle d_1, \ldots, d_{k-1}, d_k, \ldots, d_n \rangle$, with $n$ elements over on $Z_n$ and $(u, v)$ be an edge in dimension $d$. Without loss of generality, $d_k = d$. Let $RL_{d_1, \ldots, d_{k-1}}$ be a reflected edge label sequence defined by the permutation $(d_1, \ldots, d_{k-1})$. Obviously, $RL_{d_1, \ldots, d_{k-1}}$ is a substring of $RL_{\pi_n}$. The proof is trivial if $k = 1, 2$; thus, we consider only the case of $k \geq 3$.

Let $z$ be the $d_k$-th neighbor of $u$ if $(d_1, \ldots, d_{k-1})$ is a $2k-1$-CP generator; otherwise, $z$ is a vertex satisfying that

$$z_i = u_i \text{ for } i > h,$$

$$z_{d_k-1}z_{d_k} = u_{d_k-1}u_{d_k},$$

$$z_{d_k+1}z_{d_k} = \gamma((u_{d_k+1}u_{d_k})) \text{ for } \frac{k-3}{2} \geq j \geq \frac{d_k+2}{2},$$

and

$$z_{2j+1}z_{2j} = \gamma(u_{2j+1}u_{2j}) \text{ for } \frac{d_k+1}{2} \geq j \geq 0.$$

Thus, it is observed that vertex $z$ and $u$ are not adjacent in $CQ_n$. Since $z[d_k]u$, the vertex $v = v_{n-1}v_{n-2} \ldots v_0$ satisfies that

$$v_i = u_i \text{ for } i \neq d_k,$$

and

$$v_{d_k+1} = u_{d_k+1}.$$

Therefore, vertex $u$ and $v$ are different from the $g$-th bit position where $g = d_k + 1$, that is, $g = \min\{d_k, d_k\} + 1$.

**Theorem 1:** For $n \geq 3$, let $\pi_n$ be arbitrary Hamiltonian cycle pattern generator and $(u, v)$ be any edge in $CQ_n$. Then, there exists a vertex $z$ such that the Hamiltonian cycle $\omega(C_{\pi_n}, z)$ of $CQ_n$ passes through the edge $(u, v)$.

**Theorem 2:** For $n \geq 3$, let $(u, v)$ be any edge in $CQ_n$. Then, there are at least $m$ different Hamiltonian cycles in $CQ_n$ passing through the edge $(u, v)$ where $m = \min\{d_k, d_k\} \times (n-2)!$ if $n$ is even; otherwise, $m = \frac{n^2-1}{4} \times (n-2)!$.

### V. CONCLUSION

The crossed cube is one of most prominent variants of hypercube. Because crossed cubes are neither edge- nor vertex-symmetric, producing Hamiltonian cycles to pass any prescribed edge in a crossed cube is more intricate of a process than in a regular hypercube. In this paper, we apply the characterization of Hamiltonian cycles pattern extended from [14] to build a simple systematic approach to generate a Hamiltonian cycle passing through arbitrary prescribed edge.

Numerous variants of hypercube, for example, Möbius cubes, Twisted cubes, and Locally Twisted cubes, have been proposed and proved that there exists a Hamiltonian cycle passing through any given edge in them. Finding an algorithm to generate a desired Hamiltonian cycle passing through arbitrary given edge in these variants of hypercube is still open. We conjecture that such approach may be constructed by applying the concept of Hamiltonian cycle pattern to these networks.

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