On $\lambda$– Summable of Orlicz Space of Gai Sequences of Fuzzy Numbers

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Abstract—In this paper the concept of strongly $(\lambda_M)$– Cesàro summability of a sequence of fuzzy numbers and strongly $\lambda_M$– statistically convergent sequences of fuzzy numbers is introduced.

Keywords—Fuzzy numbers, statistical convergence, Orlicz space, gai sequence.

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced Zadeh[18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka[10] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Matloka[10] also has shown that every convergent sequence of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed by Nanda[12], Fang and Huang[5], and many others. The notion of statistical convergence was introduced by Fast[6] and Schoenfeld[16] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory. Later on it was further investigated from the sequence point of view and linked with summability theory by Fridy[7], Kwon[9], Nuray[14] and Savas[15] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of Stone–Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the natural numbers. The natural density of a set $A$ of positive integers is defined by

$$\delta (A) = \lim_{n \to \infty} \frac{|\{k \leq n : k \in A\}|}{n},$$

where $|\{k \leq n : k \in A\}|$ denotes the number of elements of $A \subseteq N$ not exceeding $n$ [13]. It is clear that any finite subset of $N$ have zero natural density and $\delta (A^c) = 1 - \delta (A)$. If a property $P(k)$ holds for all $k \in A$ with $\delta (A) = 1$, we say that $P$ holds for almost all $k$, we abbreviate this by "a.a.k". A sequence $(x_k)$ is said to be statistically convergent to $L$ if for every $\epsilon > 0, \delta (\{k \in N : |x_k - L| \geq \epsilon\}) = 0$. In this case we write $S - \lim x_k = L$. The existing literature on statistical convergence appears to have restricted to real or complex sequences, but in [9] Kwon, Nuray[14] and Savas[15] extended the idea to apply to sequences of fuzzy numbers.

Let $C (R^n) = \{A \subseteq R^n : A$ compact and convex $\}$. The space $C (R^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \subseteq C (R^n)$ and $\lambda \in R$. The Hausdorff distance between $A$ and $B$ of $C (R^n)$ is defined as

$$\delta_{\infty} (A, B) = \max \{\sup_{a \in A} \|f_a - b\|, \sup_{b \in B} \|f_a - A\| \},$$

It is well known that $(C (R^n), \delta_{\infty})$ is a complete metric space.

The fuzzy number is a function $X$ from $R^n$ to $[0,1]$ which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the $\alpha$–level set $[X]^\alpha = \{x \in R^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of $R^n$, with support $X^0 = \{x \in R^n : X(x) > 0\}$. Let $L (R^n)$ denote the set of all fuzzy numbers. The linear structure of $L (R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$, in terms of $\alpha$–level sets, by $[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha, [\lambda X]^\alpha = \lambda [X]^\alpha$ for each $0 \leq \alpha \leq 1$. Define, for each $1 \leq q < \infty$,

$$d_q (X, Y) = \left(\int \delta_{\infty} (X^\alpha, Y^\alpha)^q \, dx\right)^{1/q},$$

$$d_{\infty} = \sup_{0 \leq \alpha \leq 1} \delta_{\infty} (X^\alpha, Y^\alpha),$$

where $\delta_{\infty}$ is the Hausdorff metric. Clearly $d_{\infty} (X, Y) = lim_{q \to \infty} d_q (X, Y)$ with $d_q \leq d_r$, if $q \leq r$ [4]. Throughout the paper, $d$ will denote $d_q$ with $1 \leq q \leq \infty$. Let $w$ be set of all sequences of fuzzy numbers. The generalized de la Vallée-Pousin mean is defined by

$$t_n (x) = \frac{1}{\lambda^n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a nondecreasing sequence of positive numbers such that $\lambda_n + 1 \leq \lambda_n + 1 = 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. A sequence $x (x_k)$ is said to be $(V, \lambda)$– summable to a number $L$ [8] if
all values of \( M \) summability reduces to \((C,1)\) summability when \( \lambda_n = n \) for all \( n \). A complex sequence, whose \( k^{th} \) terms is \( x_k \) is denoted by \( \{ x_k \} \) or simply \( x \). Let \( \phi \) be the set of all finite sequences. Let \( \ell_{\infty}, c, c_0 \) be the sequence spaces of bounded, convergent and null sequences \( x = \{ x_k \} \), respectively. In respect of \( \ell_{\infty}, c, c_0 \) we have
\[
\| x \| = \sup k | x_k |, \quad \text{where} \quad x = \{ x_k \} \in c_0 \subset c \subset \ell_{\infty}. \]
A sequence \( x = \{ x_k \} \) is said to be analytic if \( \sup k | x_k |^{1/k} < \infty \).

The vector space of all analytic sequences will be denoted by \( A \). A sequence \( x \) is called gai sequence if \( \lim k \rightarrow \infty (k!)^{| x_k |^{1/k}} = 0 \). The vector space of all gai sequences will be denoted by \( \chi \).

Orlicz [26] used the idea of Orlicz function to construct the space \( \ell(M) \). Lindenstrauss and Tzafriri [27] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space \( \ell(M) \) contains a subspace isomorphic to \( \ell_p(1 \leq p < \infty) \). Subsequently different classes of sequence spaces defined by Parashar and Choudhary[28], Mursaleen et al.[29], Bektas and Altin[30], Tripathy et al.[31], Rao and Subramanian[32] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref[33].

Recall[26],[33] an Orlicz function is a function \( M : [0, \infty) \rightarrow [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) = 0, x > 0 \). 

The space \( \ell_M \) with the norm
\[
\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{| x_k |}{\rho} \right) \leq 1 \right\}
\]
becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^{\rho}, 1 \leq \rho < \infty \), the space \( \ell_M \) coincide with the classical sequence space \( \ell_\rho \). Given a sequence \( x = \{ x_k \} \) its \( n^{th} \) section is the sequence \( x^{(n)} = \{ x_1, x_2, \ldots, x_n, 0, 0, \ldots \} \).

\[ \delta^{(n)} = (0, 0, \ldots, 1, 0, 0, \ldots), \] in the \( n^{th} \) place and zero's else where.

II. DEFINITIONS AND PRELIMINARIES:
Let \( w \) denote the set of all fuzzy complex sequences \( x = \{ x_k \}_{k=1}^{\infty} \), and \( M \) be an Orlicz function, or a modulus function. consider
\[
\chi_M = \{ x \in w : \lim k \rightarrow \infty \left( M \left( \frac{\| x_k \|^{1/k}}{\rho} \right) \right) = 0 \text{ for some } \rho > 0 \text{ and }
\]
\[ \Lambda_M = \{ x \in w : \sup \left( M \left( \frac{| x_k |^{1/k}}{\rho} \right) \right) < \infty \text{ for some } \rho > 0 \}
\]
The space \( \chi_M \) is a metric space with the metric
\[
d(x,y) = \inf \left\{ \rho > 0 : \sup \left( M \left( \frac{| x_k - y_k |^{1/k}}{\rho} \right) \right) \leq 1 \right\}
\]
forsome \( \rho > 0 \) and \( x, y \) in \( \Gamma_M \).
The space \( \Lambda_M \) is a metric space with the metric
\[
d(x,y) = \inf \left\{ \rho > 0 : \sup \left( M \left( \frac{| x_k - y_k |^{1/k}}{\rho} \right) \right) \leq 1 \right\}
\]
forsome \( \rho > 0 \) and \( x, y \) in \( \Lambda_M \).

In the present paper we introduce and examine the concepts of \( \lambda_M \)– statistical convergence and strongly \( \lambda_M \)- Cesàro convergence of sequences of fuzzy numbers. Firstly in section 2, we give the definition of \( \lambda_M \) – statistical convergence and strongly \( \lambda_M \)- Cesàro convergence of sequence of fuzzy numbers. In section 3, we establish some inclusion relation between the sequences \( s(\Lambda_M) \) and \( (\lambda_M) \).

A. Definition
Let \( X = (X_k) \) be a sequence of fuzzy numbers. A sequence \( X = (X_k) \) of fuzzy numbers is said to converge to fuzzy number \( X_0 \) if for every \( \epsilon > 0 \) there is a positive integer \( N_0 \) such that \( d(M \left( \frac{\| k \| X_k \|^{1/k}}{\rho} \right), X_0) < \epsilon \) for \( k \geq N_0 \). 

And \( X = (X_k) \) is said to be Cauchy sequence if for every \( \epsilon > 0 \) there is a positive integer \( N_0 \) such that \( d(M \left( \frac{\| k \| X_k \|^{1/k}}{\rho} \right), X_0) < \epsilon \) for \( k, \ell \geq N_0 \).

B. Definition
A sequence \( X = (X_k) \) of fuzzy numbers is said to be analytic if the set \( \{ M \left( \frac{\| X_k \|^{1/k}}{\rho} \right) : k \in N \} \) of fuzzy numbers is analytic.

C. Definition
A sequence \( X = (X_k) \) of fuzzy numbers is said to be \( \lambda_M \)– statistically convergent to a fuzzy number \( X_0 \) if for every \( \epsilon > 0 \), we have
\[
\frac{1}{n} \left| \left\{ k \in I_n : d(M \left( \frac{\| k \| X_k \|^{1/k}}{\rho} \right), X_0) \geq \epsilon \right\} \right| \rightarrow 0 \text{ as } n \rightarrow \infty
\]
In this case we shall write \( S_{\lambda_M} \lim_{k \rightarrow \infty} M \left( \frac{\| k \| X_k \|^{1/k}}{\rho} \right) = X_0 \)
It can be shown that if a sequence \( X = (X_k) \) of fuzzy numbers is convergent to a fuzzy number \( X_0 \), then it is statistically convergent to the fuzzy number \( X_0 \), but the converse does not hold. For example, we define \( X = (X_k) \) such that
\[
M \left( \frac{\| k \| X_k \|^{1/k}}{\rho} \right) = \begin{cases} A & \text{if } k = n^2, n = 1, 2, 3, \cdots \\ 0 & \text{otherwise} \end{cases}
\]
Where \( A \) is a fixed fuzzy number. Then \( X = (X_k) \) is statistically convergent but is not convergent.
D. Definition

A sequence \(X = (x_k)\) of fuzzy numbers is said to be strongly \(\lambda_M\)-summable if there is a fuzzy number \(X_0\) such that
\[
\frac{1}{n} \sum_{k=1}^{n} \left| d\left( M\left( \frac{X_k}{\rho} \right), X_0 \right) \right|^p \rightarrow 0 \text{ as } n \rightarrow \infty
\]

E. Definition

A sequence \(X = (x_k)\) of fuzzy numbers is said to be strongly \(\lambda_M\)-Cesàro summable if there is a fuzzy number \(X_0\) such that
\[
\frac{1}{n} \sum_{k=1}^{n} \left| d\left( M\left( \frac{X_k}{\rho} \right), X_0 \right) \right|^p \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
The set of all strongly \(\lambda_M\)-Cesàro summable sequences of fuzzy numbers is denoted by \(\lambda_M\).

F. Definition

A sequence \(X = (x_k)\) of fuzzy numbers is said to be \(\lambda_M\)-statistically convergent or \(S_{\lambda_M}\) to a fuzzy number \(X_0\) if for every \(\varepsilon > 0\), we have
\[
\frac{1}{n} \left\{ k \in I_n : \left| d\left( M\left( \frac{X_k}{\rho} \right), X_0 \right) \right| \geq \varepsilon \} \rightarrow 0.
\]

In this case, we shall write \(S_{\lambda_M} \left( \lim M\left( \frac{X_k}{\rho} \right) \right) = X_0\).

In the special case \(\lambda_M = n\) for all \(n \in N\), then \(\lambda_M\)-statistically convergent is same as statistically convergent.

III. MAIN RESULTS

A. Theorem

(i) If a sequence \(X = (x_k)\) is strongly \((\lambda_M)\)-Cesàro summable to \(X_0\), then \(\lambda_M\)-statistically convergent to \(X_0\).

(ii) If \(X = (x_k)\) is a sequence \(\lambda_M\)-analytic and \(\lambda_M\)-statistically convergent to \(X_0\), then it is strongly \((\lambda_M)\)-Cesàro summable to \(X_0\), and hence \(X\) is strongly \(\lambda_M\)-Cesàro summable to \(X_0\).

Proof: Let \(\varepsilon > 0\) and \(X = (x_k)\). We have
\[
\sum_{k=1}^{n} d\left( M\left( \frac{X_k}{\rho} \right), X_0 \right) \geq \left| d\left( M\left( \frac{X_k}{\rho} \right), X_0 \right) \right|^p \geq \left| d\left( M\left( \frac{x_k}{\rho} \right), X_0 \right) \right|^p > \varepsilon \quad \forall \varepsilon > 0.
\]
Therefore \(\lambda_M\) is statistically convergent to \(X_0\).

(iii) Suppose that \(X = (x_k)\) is analytic and \(\lambda_M\)-statistically convergent to \(X_0\). Then \(S_{\lambda_M}\) is strongly \((\lambda_M)\)-Cesàro summable to \(X_0\) and \(\lambda_M\)-statistically convergent to \(X_0\), then it is strongly \((\lambda_M)\)-Cesàro summable to \(X_0\), and hence \(X\) is strongly \(\lambda_M\)-Cesàro summable to \(X_0\).

B. Theorem

Let \((X_n)\) and \((Y_n)\) be sequence of fuzzy numbers.

(i) If \(S_{\lambda_M} - \lim M\left( \frac{[X_n]}{\rho} \right) = X_0\) and \(c \in R\), then \(S_{\lambda_M} - \lim c M\left( \frac{[X_n]}{\rho} \right) = c X_0\).

(ii) \(S_{\lambda_M} - \lim M\left( \frac{[X_n]}{\rho} \right) = Y_0\) and \(S_{\lambda_M} - \lim (c M\left( \frac{[X_n]}{\rho} \right)) = X_0\), then \(S_{\lambda_M} - \lim M\left( \frac{[X_n]}{\rho} \right) = X_0 + Y_0\).

Proof: Let \(\alpha, \beta \in [0, 1]\) and \(c \in R\). Let
\[
M\left( \frac{[X_n]}{\rho} \right) = X_0, \quad M\left( \frac{[Y_n]}{\rho} \right) = Y_0, \quad c M\left( \frac{[X_n]}{\rho} \right) = c X_0, \quad c M\left( \frac{[Y_n]}{\rho} \right) = c Y_0.
\]

Therefore \(\lambda_M\) is statistically convergent to \(X_0\), and \(\lambda_M\)-statistically convergent to \(Y_0\).

(iii) Suppose that \(S_{\lambda_M} - \lim M\left( \frac{[X_n]}{\rho} \right) = X_0\) and \(S_{\lambda_M} - \lim M\left( \frac{[Y_n]}{\rho} \right) = Y_0\). Firstly, we have
\[
\delta_{\lambda_M} M\left( \frac{[X_n]}{\rho} \right) + Y_0 \leq \delta_{\lambda_M} M\left( \frac{[X_n]}{\rho} \right) + Y_0.
\]

By Minkowski’s inequality, we get
\[
d\left( M\left( \frac{[X_n]}{\rho} \right), X_0 \right) + \left| d\left( M\left( \frac{[Y_n]}{\rho} \right), Y_0 \right) \right| \leq \left| d\left( M\left( \frac{[X_n]}{\rho} \right), X_0 \right) \right| + \left| d\left( M\left( \frac{[Y_n]}{\rho} \right), Y_0 \right) \right|.
\]
Therefore given \( \epsilon > 0 \) we have
\[
\frac{1}{\lambda_n} \left\{ k \in I_n : d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right) > \epsilon \right\} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{q_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{q_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Thus \( X \in (\lambda_M)_q \).

C. Theorem

If a sequence \( X = (X_k) \) is statistically convergent to \( X_0 \) and \( \liminf_{n \rightarrow \infty} \left( \frac{\lambda_n}{n} \right) > 0 \), then it is \( \lambda_M \)-statistically convergent to \( X_0 \).

**Proof:** For given \( \epsilon > 0 \), we have
\[
\frac{1}{\lambda_n} \left\{ k \leq n : d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right) > \epsilon \right\} \leq \frac{1}{\lambda_n} \sum_{k \leq n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{q_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
This completes the proof.

D. Definition

Let \( p = (p_k) \) be any sequence of positive real numbers. Then we define \( (\lambda_M)_p = (X_k) : \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{p_k} \right), X_0 \right)^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty. \)

If \( p_k \) is a constant for all \( k \), then \( (\lambda_M)_p = \lambda_M \).

E. Theorem

Let \( 0 \leq p_k \leq q_k \) and let \( \left\{ \frac{q_k}{p_k} \right\} \) be bounded. Then \( (\lambda_M)_p \subset (\lambda_M)_q \).

**Proof:** Let \( X \in (\lambda_M)_q \).
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{q_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Let \( t_k = \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k} \) and \( \lambda_k = \frac{p_k}{q_k} \). Since \( p_k \leq q_k \), we have \( 0 \leq \lambda_k \leq 1 \).

Take \( 0 < \lambda < \lambda_k \). Define \( u_k = t_k (t_k \geq 1) \);
\[ u_k = 0 (t_k < 1) \text{ and } v_k = 0 (t_k \geq 1); \]
\[ v_k = t_k (t_k < 1), t_k = u_k + v_k \text{ (i.e) } t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}. \]
Now it follows that
\[ u_k^{\lambda_k} \leq u_k \leq t_k \text{ and } v_k^{\lambda_k} \leq v_k^{\lambda_k}. \]
Since \( t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \), then \( t_k^{\lambda_k} \leq t_k + v_k^{\lambda_k} \).
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{q_k} \lambda_k \leq \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{q_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{[k]!|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k/q_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Therefore \( X \in (\lambda_M)_p \). This completes the proof.
G. Theorem

Let $0 < p_k \leq q_k < \infty$ for each $k$. Then $(\lambda M)^\lambda \subseteq (\lambda M)_q$

Proof: Let

$$X \in (\lambda M)_p$$

$$\frac{1}{\lambda n} \sum_{k \in I_n} \left[ d \left( M \left( \left( k! |X_k|^{1/k} \right) \rho^k, X_0 \right) \right) \right] \to 0 \text{ as } n \to \infty$$

This implies that $\frac{1}{\lambda n} \sum_{k \in I_n} \left[ d \left( M \left( \left( k! |X_k|^{1/k} \right) \rho, X_0 \right) \right) \right] \to 1$, for sufficiently large $n$. Since $M$ is non-decreasing, we get

$$\frac{1}{\lambda n} \sum_{k \in I_n} \left[ d \left( M \left( \left( k! |X_k|^{1/k} \right) \rho^k, X_0 \right) \right) \right] \leq \left[ d \left( M \left( \left( k! |X_k|^{1/k} \right) \rho, X_0 \right) \right) \right]$$

$$\Rightarrow \frac{1}{\lambda n} \sum_{k \in I_n} \left[ d \left( M \left( \left( k! |X_k|^{1/k} \right) \rho^k, X_0 \right) \right) \right] \to 0 \text{ as } n \to \infty$$

(by using 12). Hence

$$X \in (\lambda M)_q$$

From (12) and (14) we get $(\lambda M)_p \subseteq (\lambda M)_q$. This completes the proof.

IV. Conclusion

The above results are constructed with the concept of strongly $(\lambda M)_p$- Cesáro summability of a gai sequence of fuzzy numbers and strongly $\lambda M$- statistically convergent sequences of fuzzy numbers.

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References