Finite-time stability analysis of fractional-order with multi-state time delay

Liqiong Liu and Shouming Zhong

Abstract—In this paper, the finite-time stabilization of a class of multi-state time delay of fractional-order system is proposed. First, we define finite-time stability with the fractional-order system. Second, by using Generalized Gronwall’s approach and the methods of the inequality, we get some conditions of finite-time stability for the fractional system with multi-state delay. Finally, a numerical example is given to illustrate the result.

Keywords—Finite-time stabilization, Fractional-order system, Gronwall inequality.

I. INTRODUCTION

In recent years, many studies focus on a lot in fractional order systems. They study aspects of fractional order systems. For instance, in [1], the author study the existence of solutions for fractional differential equations, and in [2], the author study existence and uniqueness of solutions for the linear time-delay differential equations of fractional order systems. It comes to time-delay systems, time-delays are often present in various engineering systems such as biological, economical systems, chemical processes. Time-delays are described by differential-difference equations which belong to a class of functional differential equations [3]. Stability analysis is one of the most important issues for control systems, although this problem has been investigated for time-delay systems over many years in [4]. Recently, for the first time, finite-time stability analysis of fractional time-delay systems is presented and reported on paper [5]. And in [6], a stability test procedure is proposed for linear nonhomogeneous fractional order systems with a pure time delay using a recently obtained generalized Gronwall’s inequality. Here, the finite-time stabilization of a class of multi-state time delay of fractional-order system using Gronwall’s approach is proposed. The main contribution of this paper is to introduce multi-state time delay of fractional-order system, and when $\tau_i = 0$ [6] is the special circumstances of this paper.

II. FUNDAMENTALS OF FRACTIONAL DERIVATIVE

There are many ways to define the fractional integral and derivative, and three definitions are generally used in recent studies, they are Riemann-Liouville definition, Grünwald-Letnikov definition and Caputo definition. Given, Riemann-Liouville definition of q-th order fractional derivative operator $0 < q < 1$ is given by [7]

\[
D^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-\tau)^{-q} f(\tau) d\tau
\]

(1)

Where $\Gamma(\cdot)$ is the Gamma function generalizing factorial for non-integer arguments

\[
\Gamma(q) = \int_0^{+\infty} e^{-t} t^{q-1} dt
\]

(2)

The Grünwald-Letnikov fractional derivative definition is given by [8]

\[
aD^q f(t) = \lim_{h \rightarrow 0} \frac{1}{h^q} \sum_{j=0}^{\left\lfloor \frac{t}{h} \right\rfloor} (-1)^j \binom{q}{j} f(t-jh)
\]

(3)

where $\lfloor \cdot \rfloor$ is a floor operator. And Caputo definition:

\[
aD^q f(t) = \begin{cases} 
\frac{d}{dt} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau, & m - 1 < q < m \\
\frac{d}{dt} f(t), & q = m 
\end{cases}
\]

(4)

A linear time-invariant function-order system can be represented in the following state-space form:

\[
D^q x(t) = Ax(t) + Bu(t)
\]

(5)

Where $D^q x(t)$ denotes the Riemann-Liouville fractional derivative of order $q \in R$, $x(t) \in R^n$, $u(t) \in R^m$, $A \in R^{n \times n}$, $B \in R^{n \times m}$, $q = (q_1, q_2, \cdots, q_n) \in R^n$.

III. MULTI-STATE TIME DELAY OF FRACTIONAL-ORDER SYSTEM WITH INPUT DELAY

Consider the following fractional order system:

\[
\left\{ \begin{array}{cl}
D^q x(t) &= A_0 x(t) + \sum_{i=1}^{\tau_i} A_i x(t-\tau_i) + B_0 u(t), t \geq 0 \\
x(t) &= \Psi_x(t), t \in [-\tau, 0]
\end{array} \right.
\]

(6)

where $D^q$ denotes Riemann-Liouville derivative of order $q$, $0 < q < 1$, $\Psi_x(\cdot)$ is a given continuous function on $[-\tau, 0]$, $\tau = \max(\tau_1, \tau_2, \cdots, \tau_n)$, and $\tau_i$ is a constant with $\tau_i > 0$. In Eq.(6), $x(t) \in R^n$ is a state vector, $u(t) \in R^m$ is a input control vector, $A_0$, $A_i$, $B_0$ are constant system matrices of appropriate dimensions, and the system is defined over time interval $J = [0, T]$, where $T$ is a positive number, $u(t)$ is a given continuous function on $[0, T]$. 

This work was supported by the National Basic Research Program of China (2010CB732501).

Liqiong Liu and Shouming Zhong are with the School of Mathematics Science, University Electronic Science and Technology of China, Chengdu 611731, PR China.

Shouming Zhong is with Key Laboratory for NeuroInformation of Ministry of Education, University of Electronic Science and Technology of China, Chengdu 611731, PR China.

Email address: lilq664335@126.com.
Let us denote by \( C([a,b]) \) the space of all continuous real functions defined on \([a,b]\) and by \( C([a,b],R^n) \) the Banach space of continuous functions mapping the interval \([a,b]\) into \( R^n \) with the topology of uniform convergence. Let \( C=C([-\tau,0],R^n) \), if \([a,b]=[\tau,0]\) and designate the norm of an element \( ||\Psi_s||_C \) in \( C \) by

\[
||\Psi_s||_C = \sup_{-\tau \leq t \leq 0} ||\Psi(t)||
\]

Before proceeding further, we will introduce the following definition and lemmas which will be used in the next section.

**Definition 3.1** The system given by homogeneous state equation (6) \((u(t) \equiv 0, \forall t)\), satisfying initial condition \( x(t) = \Psi_s(t) \), \(-\tau \leq t \leq 0\) is finite-time stable w.r.t. \((\delta, \varepsilon, J)\), if and only if:

\[
||\Psi_s||_C < \delta
\]

imply:

\[
||x(t)|| < \varepsilon, \forall t \in J
\]

Where \( J \) denotes time interval \([0,T]\).

**Definition 3.2** The system given by (6) satisfying initial condition \( x(t) = \Psi_s(t) \), \(-\tau \leq t \leq 0\) is finite-time stable w.r.t. \((\delta, \varepsilon, q_u, J)\), if and only if:

\[
||\Psi_s||_C < \delta
\]

and

\[
||x(t)|| < q_u, \forall t \in J
\]

imply:

\[
||x(t)|| < \varepsilon, \forall t \in J
\]

Where \( J \) denotes time interval \([0,T]\). Let

\[
f(t) = \int_0^t (t-s)^p ||x(l)||dl, \forall t \in J, p > 0
\]

we have the following definition.

**Lemma 3.1** (9) Generalized Gronwall Inequality) Suppose \( x(t), a(t) \) are nonnegative and local integrable on \( 0 \leq t < T \), some \( T \leq +\infty \), and \( g(t) \) is a nonnegative, non-decreasing continuous function defined on \( 0 \leq t < T \), \( g(t) \leq M = \text{const}, q > 0 \),

\[
x(t) \leq a(t) + g(t) \int_0^t (t-s)^{q-1} x(s) ds
\]

on this interval. Then

\[
x(t) \leq a(t) + \int_0^t \left[ \sum_{i=1}^{\infty} \frac{(g(t)\Gamma(q))}{k(q+1)} (t-s)^{q-1} a(s) \right] ds
\]

where \( 0 \leq t < T \).

**Lemma 3.2** (9) Under the hypothesis of Lemma 3.1, let \( a(t) \) be a nondecreasing function on \([0,T]\). Then holds:

\[
x(t) \leq a(t) E_q \left( g(t) \cdot \Gamma(q) \cdot t^q \right)
\]

where \( E_q \) is the Mittag-Leffler function defined by

\[
E_q (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq + 1)}
\]

**IV. MAIN RESULTS**

**Theorem 4.1** The system given by (6) satisfying initial condition \( x(t) = \Psi_s(t), -\tau \leq t \leq 0 \) is time-stable w.r.t. \((\delta, \varepsilon, q_u, J)\), if the following condition is satisfied:

\[
\left[ 1 + \frac{(n+1)\sigma q}{\Gamma(q+1)} \right] + \frac{q_u \cdot b_0 \cdot t^q}{\delta \Gamma(q+1)} \leq \frac{\varepsilon}{\delta}
\]

where \( \sigma_{\text{max}}(\cdot) \) being the largest singular value of matrix \((\cdot)\)

\[
\sigma_i = \max_{1 \leq i \leq n} \{ \sigma_{\text{max}}(A_i) \}
\]

Proof: In accordance with the property of the fractional order 

\[
0 < q < 1, \text{ one can obtain a solution in the form of the equivalent Volterra integral equation:}
\]

\[
x(t) = x(0) + \int_0^t (t-s)^{q-1} A_0 x(s) ds + \sum_{i=1}^{n} A_i x(s-\tau_i) + B_0 u(s) ds
\]

Applying the norm \( || \cdot || \) on Eq.(20) and using appropriate property of the norm, it follows that

\[
||x(t)|| \leq ||x(0)|| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds
\]

\[
\leq ||\Psi_s||_C + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ||A_0|| ||x(s)|| ds + \sum_{i=1}^{n} ||A_i|| ||x(s-\tau_i)|| + ||B_0|| ||u(s)|| ds
\]

\[
\leq ||\Psi_s||_C + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sigma(n+1) \times \frac{1}{\Gamma(q+1)} \int_0^t (s-t)^{q-1} \sup_{s-\tau \leq t \leq s} ||x(t)|| ds + \frac{1}{\Gamma(q+1)} \int_0^t (s-t)^{q-1} \sup_{s-\tau \leq t \leq s} ||x(t)|| ds
\]

let

\[
a(t) = ||\Psi_s||_C \left[ 1 + \frac{\sigma q(t-\tau)^{q-1}}{\Gamma(q+1)} \right] + \frac{q_u \cdot b_0 \cdot t^q}{\delta \Gamma(q+1)}
\]

\[
g(t) = \frac{(n+1)\sigma q(t-\tau)^{q-1}}{\Gamma(q+1)} + \frac{b_0 \cdot t^q}{\delta \Gamma(q+1)}
\]

by (21), we have

\[
||x(t)|| \leq a(t) + g(t) \int_0^t (t-s)^{q-1} \sup_{s-\tau \leq t \leq s} ||x(t)|| ds
\]
Theorem 4.1 applying the same procedure taking into account

Proof: The proof immediately follows from the proof of

\[ \| x(t) \| \leq a(t) \cdot E_q (g(t) \cdot \Gamma(q) \cdot t^q) \]

and

\[ \| x(t) \| \leq \left[ \delta \left( 1 + \frac{(n+1) \cdot \tau t}{\Gamma(q+1)} \right) + \frac{\delta}{q+1} \right] \cdot E_q ((n+1) \sigma \cdot \Gamma(q) \cdot t^q) \]

Hence, using the basic condition of Theorem 4.1, relation (18) yields:

\[ \| x(t) \| < \varepsilon, \forall t \in J_0 \]

This is a proof of the theorem.

When \( u(t) = 0 \), we can get Theorem 4.2.

Theorem 4.2 The linear autonomous system given by (6) satisfying initial condition \( x(t) = \Psi_x(t), -\tau \leq t \leq 0 \) is finite-time stable w.r.t. \( (\delta, \varepsilon, J) \), \( \forall t \in J \) if the following condition is satisfied:

\[ \left( 1 + \frac{(n+1) \cdot \sigma \cdot t^q}{\Gamma(q+1)} \right) E_q ((n+1) \sigma \cdot \Gamma(q) \cdot t^q) \leq \frac{\varepsilon}{\delta} \]

Proof: The proof immediately follows from the proof of Theorem 4.1 applying the same procedure taking into account Eqs.(8) and (28).

V. AN ILLUSTRATIVE EXAMPLE

Using a time-delay \( PD^q \) compensator on a linear system of equations with respect to the small perturbation \( \dot{z}(t) = g(t) - y_{id}(t) \), one can obtain:

\[ \dot{z}(t) + \omega z(t) = K_{p_1} (z(t) - \tau_1) + K_{d_1} \cdot \frac{dz(t)}{dt} (z(t) - \tau_1) + K_{p_2} \cdot \frac{dz(t)}{dt} \cdot \frac{dz(t)}{dt} = u(t) \]

Where \( \frac{1}{2} \omega = 2, K_{p_1} = 3, K_{d_1} = 4, K_{p_2} = 0.1, K_{d_2} = 0.2 \), and \( u(t) \) is feed forward control. \( K_P, K_D \) are gain matrix. Also, all initial values are zero. Introducing:

\[ x_1(t) = z_1(t), x_2(t) = \frac{dz(t)}{dt} \]

and

\[ D^q_{1/2} x_1(t) = D^q_{1/2} z_1(t) = x_2(t) \]

\[ D^q_{1/2} x_2(t) = D^q_{1/2} \left( D^q_{1/2} z(t) \right) = z(t) = -2x_1(t) + 3x_1(t - \tau_1) + 4x_2(t - \tau_1) + 0.1x_1(t - \tau_2) + 0.2x_2(t - \tau_2) + u(t) \]

Or, in condensed form, where \( x(t) = (x_1, x_2)^T \), we can obtain this as:

\[ D^q_{1/2} x(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x_1(t) + \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} x_2(t - \tau_1) + \begin{bmatrix} 0 & 1 \end{bmatrix} u(t) \]

with the initial state of the function:

\[ x(t) = \psi_x(t) = 0, -\tau \leq t \leq 0 \]

And now, we check the finite-time stability w.r.t. \( \{\tau_0 = 0, J = [0, 10], \delta = 0.1, \varepsilon = 100, \tau_1 = 0.1, \tau_2 = 0.01, \gamma_0 = 1\} \)

Then we can obtain: \( \sigma = 5 \).

Applying the condition of Theorem 4.1 we can get:

\[ \left( 1 + \frac{(2 + 1) \cdot T^0.5}{T^{0.5} + \frac{1}{0.05} T^0.5} \right) \cdot E_0.5 \left( (2 + 1) \cdot 5. T^0.5 \right) < \frac{100}{2.5} \]

and then \( T \approx 0.15 \), \( T \) being “estimated time” of finite time stability.

REFERENCES


Liqiong Liu was born in Hunan Province, China in 1985. She received the B.S. degree from Southwest University of Nationals, Chengdu, in 2009.
She is currently pursuing the M.S. degree with UESTC. Her research interests are fractional-order systems.
Shouming Zhong was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from UESTC, Chengdu, China, in 1982. From 1984 to 1986, he studied at the Department of Mathematics in Sun Yat-sen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the Department of Mathematics in University of Waterloo, Waterloo, Canada. He is currently as a full professor with School of Applied Mathematics, UESTC. His current research interests include differential equations, neural networks, biomathematics and robust control. He has authored more than 80 papers in reputed journals such as the International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics.