A note on the convergence of the generalized AOR iterative method for linear systems

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I. INTRODUCTION

Consider the linear system

\[ Hy = f \]

where

\[ H = \begin{bmatrix} I - B_1 & D \\ C & I - B_2 \end{bmatrix}, \]

is an invertible matrix. For example, in the generalized least squares problem [3], [4], we must solve the generalized least squares problem

\[ \min_{x \in \mathbb{R}^n} (Ax - b)^T W^{-1} (Ax - b), \]

where \( W \) is the variance-covariance matrix [5]. If \( I - B_1 \) for \( i = 1, 2 \) are nonsingular, we can apply the regular SOR method, or the regular AOR method [6] to solve (1). However, \( I - B_1 \) for \( i = 1, 2 \) sometimes are singular. In fact, even if \( I - B_1 \) are nonsingular, it is also not easy to solve linear system (1) because we have to find the inverses of \( I - B_1 \) for \( i = 1, 2 \), or to solve two subsystems

\[ (I - B_1)x_i = d_i, i = 1, 2. \]

Hence a generalized SOR (GSOR) method was proposed by Yuan to solve linear system (1) in [3], afterwards, Yuan and Jin [4] established a generalized AOR (GAOR) method to solve linear system (1) as follows.

\[ y^{k+1} = G_{\omega, \gamma} y^k + \omega k, \]

where

\[ G_{\omega, \gamma} = (1 - \omega) I + \omega J + \omega \gamma K, \]

\[ k = \begin{bmatrix} I \\ -\gamma C & 0 \end{bmatrix} f, \]

\[ J = \begin{bmatrix} B_1 & -D \\ -C & B_2 \end{bmatrix}, \]

\[ K = \begin{bmatrix} 0 & 0 \\ C(I - B_1) & CD \end{bmatrix} = \begin{bmatrix} 0 \\ C \end{bmatrix} \begin{bmatrix} I - B_1 & D \end{bmatrix}. \]

From the above, we know that the GAOR method does not need any inverse of \( I - B_i \) for \( i = 1, 2 \). It is easy to check that the GAOR method is the GSOR method when \( \omega = \gamma \); the generalized Jacobi method when \( \gamma = 0 \); and the regular AOR method [5] when \( B_1 = B_2 = 0 \).

Throughout this paper, we shall employ the same notations as in [1], [2]. For instance, \( N \triangleq \{1, 2, \ldots, n\} \), denote the class of all complex matrices by \( C^{n,n} \), and denote \( \rho(G_{\omega, \gamma}) \) by the spectral radius of iterative matrix \( G_{\omega, \gamma} \).

For \( A = (a_{ij}) \in C^{n,n} \), let

\[ R_i(A) = \sum_{i \neq j} |a_{ij}|. \]

Recall that \( A \) is said to be strictly diagonally dominant (\( A \in SD \)), if

\[ |a_{ii}| > \sum_{i \neq j} |a_{ij}|, \forall i \in N, \]

and if

\[ |a_{ii}| > \sum_{i \neq j} |a_{ij}|, \forall i, j \in N, i \neq j, \]

we call that \( A \) is strictly doubly diagonally dominant (\( A \in SDD \)). Obviously, \( SD \subseteq SDD \).

In [1], [2], the following main results are presented:

**Theorem 1.1** (1) Let \( H \in SD \), then \( \rho(G_{\omega, \gamma}) \) satisfies the following inequality

\[ |\omega - 1| + \min_i \{ |\omega| J_i + |\omega| K_i \} \leq \rho(G_{\omega, \gamma}) \leq |\omega - 1| + \max_i \{ |\omega| J_i + |\omega| K_i \}, \]

where \( J_i \) and \( K_i \) are the \( i \)-row sums of the modulus of the entries of \( J \) and \( K \), respectively.

**Theorem 1.2** (2) Let \( H \in SD \), then \( \rho(G_{\omega, \gamma}) \) satisfies the following inequality

\[ \min_i \{ |\omega - 1| - |\omega| (J + \gamma K)_i \} \leq \rho(G_{\omega, \gamma}) \leq \max_i \{ |\omega - 1| + |\omega| (J + \gamma K)_i \}, \]

where \( (J + \gamma K)_i \) denotes the \( i \)-row sums of the modulus of the entries of matrix \( J + \gamma K \).
In this note, we will continue to study this problem and obtain some new inequalities which improves the corresponding results in [1], [2].

The paper is organized as follows. In Section 2, based on our results [8], [9], we obtain new upper and lower bounds for the spectral radius of $G_{\omega,\gamma}$ when $H \in SDD$, which is better than one of Theorem 1.1 and Theorem 1.2. In Section 3, we discuss the convergence of the GAOR method for SDD. In Section 4, we present numerical examples to show that our results are better than Theorem 1.1 and 1.2.

II. UPPER AND LOWER BOUNDS FOR $\rho(G_{\omega,\gamma})$

**Theorem 2.1** Let $H \in SDD$. Then $\rho(G_{\omega,\gamma})$ satisfies the following inequality

$$\rho(G_{\omega,\gamma}) \geq \max \left\{ 0, \min_{i \neq j} |\omega - 1| - |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \right\},$$

or

$$\rho(G_{\omega,\gamma}) \leq \max_{i \neq j} \left\{ |\omega - 1| + |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \right\}.$$ 

**Proof.** Let $\lambda$ be an arbitrary eigenvalue of iterative matrix $G_{\omega,\gamma}$, then

$$\det(\lambda I - G_{\omega,\gamma}) = 0.$$ 

(7)

We can show that Eq.(7) holds if and only if

$$\det((\lambda + \omega - 1)I - \omega J - \omega \gamma K) = 0.$$ 

If we take the parameter $\gamma$, $\omega$, and $\lambda$ in order that

$$(\lambda + \omega - 1)I - \omega J - \omega \gamma K \in SDD,$$

i.e., for any $i, j \in N, i \neq j$,

$$\omega^2((J + \gamma K_i - (J + \gamma K)_{ii})((J + \gamma K)_{ij} - (J + \gamma K)_{jj}) < |\lambda + \omega - 1 - \omega (J + \gamma K)_{ii}| |\lambda + \omega - 1 - \omega (J + \gamma K)_{jj}|$$

then $\lambda$ is not an eigenvalue of $G_{\omega,\gamma}$, where $(J + \gamma K)_{ii}$ denotes the diagonal element of matrix $J + \gamma K$.

Obviously, especially when

$$\omega^2(J + rK)_{ii}(J + \gamma K)_{jj} < |\lambda + \omega - 1|^2, \forall i, j \in N, i \neq j,$$

i.e.,

$$|\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} < |\lambda + \omega - 1|$$

then $\lambda$ can not an eigenvalue of $G_{\omega,\gamma}$. Hence if $\lambda$ is an eigenvalue of $G_{\omega,\gamma}$, we must have

$$|\lambda + \omega - 1| \leq |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)},$$

especially,

$$||\lambda| - |\omega| - 1| \leq |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)},$$

or

$$|\omega - 1| - |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \leq |\lambda| \leq |\omega - 1| + |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)},$$

i.e.,

$$\rho(G_{\omega,\gamma}) \geq \max_{i \neq j} \left\{ 0, \min_{i \neq j} |\omega - 1| - |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \right\},$$

or

$$\rho(G_{\omega,\gamma}) \leq \max_{i \neq j} \left\{ |\omega - 1| + |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \right\}.$$ 

So the assertion holds. The proof is completed. □

**Remark 2.1** The results (5) and (6) of Theorem 2.1 are better than ones of Theorem 1.1 and Theorem 1.2, since for any $i, j \in N, i \neq j$

$$\min_{j , j \neq i} \{J + \gamma K_i \} \leq \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \leq \max_{i \neq j} \{J + \gamma K_i \}.$$ 

(8)

In addition, for some values of $\gamma$ and $\omega$, the GAOR method reduces to the well-known methods, i.e.,

1) The GAOR method reduces to the GSOR method when $\omega = \gamma$, thus

$$\rho(G_{\omega,\omega}) \geq \min_{i \neq j} \left\{ 0, |\omega - 1| - |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \right\},$$

$$\rho(G_{\omega,\omega}) \leq \max_{i \neq j} \left\{ |\omega - 1| + |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} \right\}.$$ 

2) The GAOR method reduces to the generalized Jacobi method when $\gamma = 0$, thus

$$\rho(G_{\omega,0}) \geq \min_{i \neq j} \left\{ |\omega - 1| - |\omega| \sqrt{J_i J_j} \right\},$$

$$\rho(G_{\omega,0}) \leq \max_{i \neq j} \left\{ |\omega - 1| + |\omega| \sqrt{J_i J_j} \right\}.$$ 

III. CONVERGENCE OF THE GAOR METHOD

**Theorem 3.1** Let $H \in SDD$ and assume that $\gamma$ and $\omega$ satisfy

$$\max_{i \neq j} (J + \gamma K_i)(J + \gamma K_j) < 1$$

and

$$0 < \omega < \frac{2}{1 + \max_{i \neq j} \sqrt{(J + \gamma K_i)(J + \gamma K_j)}} \forall i, j \in N.$$

then the GAOR is convergent.

**Proof.** By Theorem 2.1, we see that $\rho(G_{\omega,\gamma}) < 1$ if

$$|\omega - 1| + |\omega| \sqrt{(J + \gamma K_i)(J + \gamma K_j)} < 1, \forall i, j \in N, i \neq j.$$ 

Hence, $\omega$ must satisfy $0 < \omega < 2$.

Next, we consider the following two cases:

Case 1: If $0 < \omega \leq 1$, i.e.,

$$(J + \omega K_i)(J + \omega K_j) < 1, \forall i, j \in N.$$ 

Case 2: If $1 < \omega < 2$, i.e.,

$$0 < \omega < \frac{2}{1 + \sqrt{(J + \gamma K_i)(J + \gamma K_j)}} \forall i, j \in N, i \neq j.$$ 

which implies

$$(J + \omega K_i)(J + \omega K_j) < 1, \forall i, j \in N, i \neq j.$$ 

Combining Case 1 with 2, we get

$$\max_{i \neq j} (J + \gamma K_i)(J + \gamma K_j) < 1$$
and

$$0 < \omega < \frac{2}{1 + \sqrt{\max_{i \neq j}(J + \gamma K_i)(J + \gamma K_j)}}, \forall i, j \in N, i \neq j.$$ 

Hence the assertion holds. The proof is completed. □

According to the inequality (8), our results are obviously better than the ones in [1], [2]. In addition, some other conclusions in [1], [2] may be also obtained similarly. Here, we can not describe these results in detail.

IV. NUMERICAL EXAMPLE

The following two simple examples show that the results of Theorem 2.1 are better than ones of Theorem 1.1 and 1.2.

Example 4.1

Let

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I - B_1 & D \\ C & I - B_2 \end{bmatrix}.$$

Clearly, $H \in SD$. Meanwhile, $H \in SDD$. For convenient, supposing that $\omega = \gamma = 1$. By Theorem 2.1, we have

$$0 \leq \rho_{(\omega, \gamma)} \leq 0.5046,$$

but we have, by Theorem 1.2,

$$0 \leq \rho_{(\omega, \gamma)} \leq 0.8333.$$ 

In fact, $\rho_{(\omega, \gamma)} = 0.2392$. These show that our results are better than ones of Theorem 1.1 and 1.2.

Example 4.2

Let

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I - B_1 & D \\ C & I - B_2 \end{bmatrix}.$$

Obviously, $H \in SDD$, but $H \notin SD$, therefore Theorem 1.1 and Theorem 1.2 are not valid. For convenient, supposing that $\omega = \gamma = 1$. By Theorem 2.1, we have

$$0 \leq \rho_{(\omega, \gamma)} \leq \frac{\sqrt{35}}{6} < 1,$$

which shows that our conclusions are valid.

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