Commuting Regular Γ-Semiring

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Abstract—We introduce the notion of commuting regular Γ-semiring and discuss some properties of commuting regular Γ-semiring. We also obtain a necessary and sufficient condition for Γ-semiring to possess commuting regularity.

Keywords—Commutative Γ-semiring, Idempotent Γ-semiring, Rectangular Γ-band, Commuting regular Γ-semiring, Clifford Γ-semiring.

I. INTRODUCTION

Commuting regular rings and semigroups were studied by H. Doostie, L.Pourfaraj in [4] and by Amir H Yamin, Sh.A.Safari Sabet in [1]. The notion of Γ-semiring was introduced by M.Muralikrishna Rao [7]. All definitions and fundamental concepts concerning Γ-semirings can be found in [5],[7]. In this paper we introduce the notion of commuting regular Γ-semiring. We also discuss some properties of commuting regular Γ-semiring and obtain a necessary and sufficient condition for Γ-semiring to possess commuting regularity.

Let S and Γ be two additive commutative semigroups. Then S is called Γ-semiring if there exists a mapping S × Γ × S → S(image to be denoted by aab for a, b ∈ S, a ∈ Γ) satisfying the following conditions.

(i) aab(b + c) = aab + aac
(ii) (a + b)bc = aac + bcc
(iii) a(a + b)c = aab + abc
(iv) a(ab)c = (a(cb))c, for all a, b, c ∈ S and for all α, β, γ ∈ Γ.

A non empty subset A of a Γ-semiring S is called a sub Γ-semiring of S if A is a sub semigroup of S and AΓA ⊆ A. A Γ-semiring S is said to be commutative if aab = boa for all a, b ∈ S and for all α ∈ Γ. An element e in a Γ-semiring S is said to be an idempotent in S if there exists an α ∈ Γ such that e = eae. In this case, we say that e is an α-idempotent. If every element of S is an idempotent, then S is called an idempotent Γ-semiring. For an element a in a Γ-semiring, if there exists an element b ∈ S and α, β ∈ Γ such that a = aabβa and b = bαab, then b is said to be an (α, β) inverse of a. In this case, we write b ∈ V_α^β(a). We also denote it by a^{-1}_αβ, i.e., a^{-1}_αβ ∈ V_α^β(a).

II. COMMUTING REGULAR Γ-SEMIRING

Definition II.1. A Γ-semiring S is called commuting regular if for each x, y ∈ S, there exists an element s ∈ S and α, β, γ ∈ Γ such that xoy = yoxβγyox.

Theorem II.2. Let S be a rectangular Γ-band. Then S is commuting if and only if S is commuting regular.

Proof: If S is a commuting Γ-semiring, then for each a, b ∈ S, there exists an α ∈ Γ such that aab = boa. Since S is a rectangular Γ-band, there exists an element e ∈ S and β, γ ∈ Γ such that aab = boaβγbaha. Hence S is a commuting regular Γ-semiring. Conversely if S is a commuting regular Γ-semiring, for each x, y ∈ S, there exists an element z ∈ S and α, β, γ ∈ Γ such that xoy = yoxβγyox. Since S is a rectangular Γ-band, xoy = yoxβγyox = yox. Hence S is a commuting Γ-semiring.

Theorem II.3. If S is a commuting regular Γ-semiring with set E of the idempotents, then E is a regular sub Γ-semiring of S. Moreover for every element a of E, there exists an element α ∈ Γ such that a ∈ V_α a(a).

Proof: If S is a commuting regular Γ-semiring, then for each a ∈ S, there exists an element s ∈ S and α, β, γ ∈ Γ such that aaaa = aaaaαβγaaa. If aaaaβα = b, then bαb = (aaaaβαγaaaβα) = (aaaaβαγaaaβα) = bα = b. Hence E is not empty. For elements xox = x and yoy = y, there exists an element t ∈ S and α, β, γ ∈ Γ such that xoy = yoxβγyox. Now (xoy)δ(xyoy) = xoy(yδxyoy) = xoy(γxyoy). Hence E is a sub Γ-semiring of S and xoxayx = x yields that E is a regular sub Γ-semiring of S. Hence x ∈ V_α^δ(x).

Corollary II.4. Let S be a commuting regular Γ-semiring with set E of the idempotents. Let a ∈ S and let α, β, γ ∈ Γ. If b ∈ V_α^δ(a), then for any γ-idempotent e of S, (i) aaeγ = β-idealpunkt (ii) bβεγa is an α-idempotent.

Proof: Let b ∈ V_α^δ(a). Then (aaab)(ab) = (aaaβa)ab = aab and (bβa)(b3a) = (b3ab)βa = b3a. Hence S is a commuting regular Γ-semiring with set E of the idempotents. Let a ∈ S and let α, β, γ ∈ Γ. If b ∈ V_α^δ(a), then for any γ-idempotent e of S, (i) aaeγb = β-idealpunkt (ii) bβεγa is an α-idempotent.

Proof: Let b ∈ V_α^δ(a). Then (aaab)(ab) = (aaaβa)ab = aab and (bβa)(b3a) = (b3ab)βa = b3a. Since (eγb3a)β(εγb3a) = eγβ3a and (eγεab)β(εγεab) = eγεab, by theorem II.3, eγβ3a, eγεab ∈ E. Now, (aaεγb)β(εaeγb) = aaeγεb3aεaeεb3aab =
\[ aae\gamma(b\beta aab) = aae\gamma b. \] Moreover, \((b\beta \gamma a)a(b\beta \gamma a) = b\beta (e\beta aab\gamma e\beta aab)\beta a = b\beta (e\beta aab\gamma a\beta aab)\beta a = b\beta \gamma a. \]

**Theorem II.5.** Let \( S \) be a commuting regular \( \Gamma \)-semiring with set \( E \) of \( \alpha \)-idempotents. Let \( e, f \in E \) and \( \alpha \in \Gamma \). Then the set \( S_{\alpha}^E(e, f) = \{g \in V_{\alpha}^E(eaf) \cap E; gae = fag = g \} \) is a regular sub \( \Gamma \)-semiring of \( S \).

**Proof:** Since \( S \) is a commuting regular \( \Gamma \)-semiring, there exists an element \( s \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( eaf = foeb \gamma fis \). Then \( (eaf)(a)(eaf) = foeb\gamma fis = (eaf)(a)(eaf) = eaf \). Now, \((eaf)(a)(eaf) = eaf \), \((eaf)(a)(eaf) = eaf \) and \( foeb\gamma fis = foeb \gamma fis \). Hence \( eaf \). This yields \( eaf \) and we can also prove that \( foeb \gamma fis \) which implies \( S_{\alpha}^E(e, f) \neq \emptyset \). Let \( x, y \in S_{\alpha}^E(e, f) \). Since \( S \) is a commuting regular \( \Gamma \)-semiring, there exists an element \( t \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( xy = y\gamma b, \gamma ax \). Then \((xy)(ax) = x(y\gamma b, \gamma ax) = (x\gamma b, \gamma ax) = x\gamma b, \gamma ax \). Since \( xaf = xaf \) and \( foeb \gamma fis = foeb \gamma fis \), and so \( foeb\gamma fis = foeb \gamma fis \). Hence \( x\gamma b, \gamma ax = x\gamma b, \gamma ax \) for some \( v \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( xy = y\gamma b, \gamma ax \).\]

**Remark II.6.** The set \( S_{\alpha}^E(e, f) \) is called the \((\alpha, \alpha)\) sandwich set of \( e, f \) and it has an obvious alternative characterization \( S_{\alpha}^E(e, f) \).\]

**Lemma II.7.** Let \( S \) be a commuting regular \( \Gamma \)-semiring. Let \( a, b \in S \) and let \( \alpha, \beta, \gamma \in \Gamma \). Suppose \( a' \in V_{\alpha}^3(a) \) and \( b' \in V_{\beta}^3(b) \). Then for each \( g \in S_{\alpha}^3(a'\beta a, b'\beta a) \), \( b'\beta a' \in V_{\beta}^3(a) \).

**Proof:** \((a'\beta a)\beta b' = a\beta ab' = a\beta ab' = a\beta ab' = (a\beta ab')\beta b' = (a\beta ab')\beta b' = b'\beta a' \in V_{\beta}^3(a) \). Hence \( b'\beta a' \in V_{\beta}^3(a) \).

**Theorem II.8.** Let \( S \) be a commuting regular \( \Gamma \)-semiring. Let \( a, b \in S \) and \( \alpha, \beta, \gamma \in \Gamma \). Then \( V_{\alpha}^3(b) = V_{\beta}^3(b) \).

**Proof:** Let \( a, b \in S \) and let \( \alpha, \beta, \gamma \in \Gamma \). Suppose \( a' \in V_{\alpha}^3(a) \) and \( b' \in V_{\beta}^3(b) \). Then \( V_{\alpha}^3(a) \subseteq V_{\beta}^3(b) \) and \( V_{\beta}^3(b) \subseteq V_{\alpha}^3(a) \).

**Theorem II.9.** Let \( S \) be a commuting regular \( \Gamma \)-semiring with set \( E \) of idempotents. Let \( \alpha, \beta \in \Gamma \). Then \( V_{\alpha}^3(e) \subseteq E \) for every \( e \in E \).
Proof: Suppose that \( a \Omega b \). By lemma III.2, there are \( x \) and \( y \) in \( S \) and \( \alpha, \beta \in \Gamma \) such that \( x\alpha a = b \) and \( y\beta b = a \).

So, there are \( t_1, t_2 \) in \( S \) and \( \gamma_1, \gamma_2, \delta_1, \delta_2 \) in \( \Gamma \) such that \( b = x\alpha a = a\alpha_1 \gamma_1 t_1 \gamma_2 a\alpha x \) and \( a = y\beta b = b\beta_1 t_2 \delta_2 y\beta b \) where \( u = x\gamma_1 t_1 \gamma_2 a\alpha x, v = y\beta t_2 \delta_2 y\beta b \). This implies \( a \Omega b \). Proof of the converse is similar.

Remark III.5. The equivalence \( \Theta \) is a two sided analogue of \( \Sigma \) and \( \mathcal{R} \). Also, we define the equivalence \( \tilde{\Theta} \) by the rule \( a \tilde{\Theta} b \) if and only if \( STa \cup a\Gamma S \cup STa\Gamma S \cup \{a\} = STb \cup b\Gamma S \cup STb\Gamma S \cup \{b\} \) if and only if there exist \( x, y, u, v \in S \) and \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma \) such that \( x\alpha_1 a\alpha_2 y = b \) and \( u\beta_1 b\beta_2 y = a \).

It is immediate that \( \Sigma \subseteq \tilde{\Theta} \) and \( \mathcal{R} \subseteq \tilde{\Theta} \). Hence since \( \Theta \) is the smallest equivalence containing \( \Sigma \) and \( \mathcal{R} \), we get \( \Theta \subseteq \tilde{\Theta} \).

Theorem III.6. If \( S \) is a commuting regular \( \Gamma \)-semiring, then \( \Theta = \tilde{\Theta} \).

Proof: By remark III.5, it is enough to show that \( \tilde{\Theta} \subseteq \Theta \). For elements \( a \) and \( b \) in \( S \), let \( a \tilde{\Theta} b \). Then there are \( x, y, u, v \in S \) and \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma \) such that \( x\alpha_1 a\alpha_2 y = b \) and \( u\beta_1 b\beta_2 y = a \).

So there exists an element \( t_1 \) in \( S \) and \( \gamma_1, \gamma_2 \in \Gamma \) such that \( a = u\beta_1 b\beta_2 y = (u\beta_1 x\alpha_1 a)\alpha_2 (y\beta_2 b) = (y\beta_2 c) \in \Gamma \) and \( x\alpha_1 a\alpha_2 y = b \) and \( u\beta_1 b\beta_2 y = a \).

Combining the relations \( x\alpha_1 a\alpha_2 y = b \) and \( u\beta_1 b\beta_2 y = a \), we get \( x\alpha_1 a = b \). Then there exists an element \( t_2 \) in \( S \) and \( \delta_1, \delta_2 \in \Gamma \) such that \( b = x\alpha_1 a = (x\alpha_1 u)\beta_1 (b\beta_2 u) = b\beta_2 w_2, \) where \( x\alpha_1 u = v_3 x\alpha_1 u \delta_1 t_2 \delta_2 b\beta_2 v_3 \).

This shows that \( a \Omega b \). Hence \( \tilde{\Theta} \subseteq \Theta \).

REFERENCES