Commuting Regular $\Gamma$-Semiring

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Abstract—We introduce the notion of commuting regular $\Gamma$-semiring and discuss some properties of commuting regular $\Gamma$-semiring. We also obtain a necessary and sufficient condition for $\Gamma$-semiring to possess commuting regularity.

Keywords—Commutative $\Gamma$-semiring, Idempotent $\Gamma$-semiring, Rectangular $\Gamma$-band, Commuting regular $\Gamma$-semiring, Clifford $\Gamma$-semiring.

I. INTRODUCTION

Commuting regular rings and semigroups were studied by H. Doostie, L. Pourfaraj in [4] and by Amir H Yami, Sh.A.Safari Sabet in [1]. The notion of $\Gamma$-semiring was introduced by M. Muralikrishna Rao [7]. All definitions and fundamental concepts concerning $\Gamma$-semirings can be found in [5],[7]. In this paper we introduce the notion of commuting regular $\Gamma$-semiring. We also discuss some properties of commuting regular $\Gamma$-semiring and obtain a necessary and sufficient condition for $\Gamma$-semiring to possess commuting regularity.

Let $S$ and $\Gamma$ be two additive commutative semigroups. Then $S$ is called a $\Gamma$-semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$(image to be denoted by $a\alpha b$ for $a,b \in S, \alpha \in \Gamma$) satisfying the following conditions.

(i) $a\alpha(b+c) = a\alpha b + a\alpha c$
(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$
(iii) $(a\alpha + b\alpha) = a\alpha b + a\beta b$
(iv) $(a\alpha b)c = (a\alpha )bc$, for all $a,b,c \in S$ and for all $\alpha, \beta \in \Gamma$.

An non empty subset $A$ of a $\Gamma$-semiring $S$ is called a sub $\Gamma$-semiring of $S$ if $A$ is a sub semigroup of $S$ and $A\Gamma A \subseteq A$. A $\Gamma$-semiring $S$ is said to be commutative if $a\alpha b = boa$ for all $a,b \in S$ and for all $\alpha \in \Gamma$. An element $e$ in a $\Gamma$-semiring $S$ is said to be an idempotent in $S$ if there exists an $\alpha \in \Gamma$ such that $e = e\alpha e$. In this case, we say that $e$ is an idempotent. If every element of $S$ is an idempotent, then $S$ is called an idempotent $\Gamma$-semiring. For an element $a$ in a $\Gamma$-semiring $S$, if there exists an element $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta$ and $b = b\beta a\alpha$, then $b$ is said to be an $(\alpha, \beta)$ pair of $a$.

II. COMMUTING REGULAR $\Gamma$-SEMIRING

Definition II.1. A $\Gamma$-semiring $S$ is called commuting regular if for each $a, b \in S$, there exists an element $s \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a\gamma b = y\alpha x\beta y\gamma x\alpha$. $\Gamma$-semiring $S$ is called commuting regular if for each $a, b \in S$, there exists an element $s \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a\gamma b = y\alpha x\beta y\gamma x\alpha$. Hence $S$ is a commuting regular $\Gamma$-semiring.

Theorem II.2. Let $S$ be a rectangular $\Gamma$-band. Then $S$ is commuting if and only if $S$ is commuting regular.

Proof: If $S$ is a commuting $\Gamma$-semiring, then for each $a, b \in S$, there exists an element $e \in S$ such that $a\alpha b = boa$. Since $S$ is a rectangular $\Gamma$-band, there exists an element $e \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a\alpha b = boa\beta e\gamma b\alpha$. Hence $S$ is a commuting regular $\Gamma$-semiring.

Therefore, $S$ is a commuting regular $\Gamma$-semiring.

Theorem II.3. If $S$ is a commuting regular $\Gamma$-semiring with set $E$ of the idempotents, then $E$ is a regular sub $\Gamma$-semiring of $S$. Moreover, for every element $a \in E$, there exists an element $\alpha \in \Gamma$ such that $a = V_{\alpha}^{\alpha}(a)$.

Proof: If $S$ is a commuting regular $\Gamma$-semiring, then for each $a, b \in S$, there exists an element $s \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a\alpha b = boa$. If $a\alpha b = boa$, then $b\beta a = (a\alpha b\gamma)\gamma\alpha(a\alpha\beta) = a\alpha b\gamma\alpha$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. $\Gamma$-semiring $S$ is commuting if and only if $E$ is a regular sub $\Gamma$-semiring of $S$. Hence $E$ is a commuting regular $\Gamma$-semiring.

Corollary II.4. Let $S$ be a commuting regular $\Gamma$-semiring with set $E$ of the idempotents. Let $a \in S$ and let $\alpha, \beta, \gamma \in \Gamma$. If $b \in V_{\alpha}^{\alpha}(a)$, then for any $\gamma$-idempotent $e$ of $S$, (i) $a\alpha e\gamma b = \beta\alpha\gamma e\alpha$ (ii) $\beta\alpha\gamma e\alpha$ is $\alpha$-idempotent.

Proof: Let $b \in V_{\alpha}^{\alpha}(a)$. Then $a\alpha b = (a\alpha b)\beta = (a\alpha b)\beta = a\alpha b$. Hence $S$ is a commuting regular $\Gamma$-semiring.

Now, $e\gamma b\alpha = e\gamma b\alpha = e\gamma b\alpha$, for all $a, b \in S$ and for all $\alpha, \beta \in \Gamma$. $\Gamma$-semiring $S$ is commuting if and only if $E$ is a regular sub $\Gamma$-semiring of $S$. Hence $e\gamma b\alpha = e\gamma b\alpha$.
Let \( S \) be a commuting regular \( \Gamma \)-semiring. Then the set \( S^0_\alpha(e, f) = \{ g \in V_\alpha^0(eaf) \cap E : gae = fag = g \} \) is a regular sub \( \Gamma \)-semiring of \( S \).

**Proof:** Since \( S \) is a commuting regular \( \Gamma \)-semiring, there exists an element \( s \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( eaf = fae\gamma xae \). Now \( (eaf)a(eaf) = fae\gamma xae \). If \( eaf \neq 0 \), then \( (eaf)a(eaf) = eaf \). This yields \( eaf \in S^0_\alpha(e, f) \) which implies \( S^0_\alpha(e, f) \neq \emptyset \). Let \( x, y \in S^0_\alpha(e, f) \). Since \( S \) is a commuting regular \( \Gamma \)-semiring, there exists an element \( t \in S \) and \( \alpha, \beta, \gamma \in \Gamma \) such that \( xay = yaxb \). \( (xay)a(xay) = xayb \). Hence \( xax = yay \). \( (eaf)x(eaf) = (eaf) \). Since \( x, y \in S^0_\alpha(e, f) \), then \( xeo = eao = eaf \). This shows that \( S^0_\alpha(e, f) \) is a regular sub \( \Gamma \)-semiring of \( S \).

**Remark II.6.** The set \( S^0_\alpha(e, f) \) is called the \( (\alpha, \alpha) \) sandwich set of \( e \) and \( f \). It has an obvious alternative characterization \( S^0_\alpha(e, f) = \{ gag = g \in S : gae = fag = g \} \).

**Lemma II.7.** Let \( S \) be a commuting regular \( \Gamma \)-semiring. Let \( a, b \in S \) and let \( \alpha, \beta, \gamma \in \Gamma \). Suppose \( a' \in V_\alpha^0(a) \) and \( b' \in V_\beta^0(b) \). Then for each \( g \in S^0_\alpha(a', b', b') \), \( b'\alpha\alpha' \in V_\beta^0(a, a) \).

**Proof:** \( (a, b, a, a', b, a, a', b', a, b, a, a', b', a, b) = (a, b, a, a', b, a, a', b', a, b) \). Hence \( b'\alpha\alpha' \in V_\beta^0(a, a) \).

**Theorem II.8.** Let \( S \) be a commuting regular \( \Gamma \)-semiring. Let \( a, b \in S \) and \( \alpha, \beta \in \Gamma \). Then \( V_\alpha^0(b) \bigcap V_\beta^0(a) \subseteq V_{\alpha \beta}(a, b) \).

**Proof:** Let \( a, b \in S \) and \( \alpha, \beta \in \Gamma \). Suppose \( a' \in V_\alpha^0(a) \) and \( b' \in V_\beta^0(b) \). Then by Lemma II.7, \( b'\alpha\alpha' \in V_{\alpha \beta}(a, b) \) for all \( g \in S^0_\beta(a', b', b') \). Now, \( (a'\alpha\alpha')b = (a'\alpha\alpha')b = a' \beta b' \) and \( (a'\alpha\alpha')b = (a'\alpha\alpha')b = b' \beta a' \). Then by Theorem II.5, \( b'\beta a' \subseteq S^0_\beta(a', b', b') \). By Lemma II.7, \( b'\beta a' \subseteq S^0_\beta(a', b', b') \). Hence \( b'\alpha\alpha' \in V_{\alpha \beta}(a, b) \).

**Theorem II.9.** Let \( S \) be a commuting regular \( \Gamma \)-semiring with set \( E \) of idempotents. Let \( \alpha, \beta \in \Gamma \). Then \( V_\alpha^0(e) \subseteq E \) for every \( e \in E \).
Proof: Suppose that \(a \mathcal{D} b\). By lemma III.2, there are \(x\) and \(y\) in \(S\) and \(\alpha, \beta \in \Gamma\) such that \(x\alpha a = b\) and \(y\beta b = a\). So, there are \(t_1, t_2 \in S\) and \(\gamma_1, \gamma_2, \delta_1, \delta_2 \in \Gamma\) such that \(b = x\alpha a = x\alpha \gamma_1 t_1 \gamma_2 a a x\) and \(a = y\beta b = y\beta \delta_1 t_2 \delta_2 y\beta b\) where \(u = x\gamma_1 t_1 \gamma_2 a a x, v = y\delta_1 t_2 \delta_2 y\beta b\). This implies \(a \mathcal{D} b\). Proof of the converse is similar.

Remark III.5. The equivalence \(\mathcal{D}\) is a two sided analogue of \(\mathcal{E}\) and \(\mathcal{R}\). Also, we define the equivalence \(\mathcal{J}\) by the rule \(a \mathcal{J} b\) if and only if \(STa \cup a \gamma S \cup STa \Gamma S \cup \{a\} = STb \cup b \gamma S \cup STb \Gamma S \cup \{b\}\) if and only if there exist \(x, y, u, v \in S\) and \(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma\) such that \(x\alpha_1 a \alpha_2 y = b\) and \(u\beta_1 b \beta_2 v = a\). It is immediate that \(\mathcal{L} \subseteq \mathcal{J}\) and \(\mathcal{R} \subseteq \mathcal{J}\). Hence since \(\mathcal{D}\) is the smallest equivalence containing \(\mathcal{L}\) and \(\mathcal{R}\), we get \(\mathcal{D} \subseteq \mathcal{J}\).

Theorem III.6. If \(S\) is a commuting regular \(\Gamma\)-semiring, then \(\mathcal{D} = \mathcal{J}\).

Proof: By remark III.5, it is enough to show that \(\mathcal{J} \subseteq \mathcal{D}\). For elements \(a\) and \(b\) in \(S\), let \(a \mathcal{J} b\). Then there are \(x, y, u, v \in S\) and \(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma\) such that \(x\alpha_1 a \alpha_2 y = b\) and \(u\beta_1 b \beta_2 v = a\). So there exists an element \(t_1 \in S\) and \(\gamma_1, \gamma_2 \in \Gamma\) such that \(u = u\beta_1 b \beta_2 v = (u\beta_1 x\alpha_1 a) \alpha_2 (y\beta_2 v) = (y\beta_2 x\alpha_2 u) x (x\alpha_1 \gamma_2 y) \gamma_1 t_2 \gamma_2 (x\alpha_1 a) = w_1 \beta_1 c\) where \(w_1 = y\beta_2 x\alpha_2 u\beta_1 x\alpha_1 \gamma_2 y \gamma_2 (x\alpha_1 a) \gamma_1 t_2 \gamma_2\) and \(c = x\alpha_1 a\) and so \(a \mathcal{D} c\). Combining the relations \(x\alpha_1 a \alpha_2 y = b\) and \(c = x\alpha_1 a\), we get \(c\alpha_2 y = b\). Then there exists an element \(t_2 \in S\) and \(\delta_1, \delta_2 \in \Gamma\) such that \(c = x\alpha_1 a = (x\alpha_1 u) \beta_1 (b \beta_2 v) = b \beta_2 (u\beta_1 x\alpha_1 u) \delta_1 t_2 \delta_2 (b \beta_2 v) = b \beta_2 w_2\), where \(w_2 = v\beta_2 x\alpha_1 u \delta_1 t_2 \delta_2 b \beta_2 v\gamma_1 x\alpha_1 u\). This shows that \(c \mathcal{D} b\). Hence \(\mathcal{J} \subseteq \mathcal{D}\).

REFERENCES