On $\lambda-$ Summable of Orlicz Space of Entire Sequences of Fuzzy Numbers

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Abstract—In this paper the concept of strongly $(\lambda a.k)_{\lambda}$— Cesáro summability of a sequence of fuzzy numbers and strongly $\lambda a.k$—statistically convergent sequences of fuzzy numbers is introduced.

Keywords—Fuzzy numbers, statistical convergence, Orlicz space, entire sequence.

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced Zadeh[18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka[10] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Matloka[10] also has shown that every convergent sequence of fuzzy numbers is summable of Orlicz space of entire sequences of fuzzy numbers. The generalized de la Válle-Pousin mean is defined by

$$\delta(A) = \lim_{n \to \infty} n^{-1} \sum_{k=n}^{\infty} \{k \leq n : k \in A\},$$

where $\{k \leq n : k \in A\}$ denotes the number of elements of $A \subseteq N$ not exceeding $n$ [13]. It is clear that any finite subset of $N$ have zero natural density and $\delta(A^c) = 1 - \delta(A)$.

If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that $P$ holds for almost all $k$, we abbreviate this by "n.a.k.". A sequence $(x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0, \delta(\{k \in N : |x_k - L| > \varepsilon\}) = 0$. In this case we write $S - \lim x_k = L$. The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but in [9] Kwon, Nuray [14] and Savas[15] extended the idea to apply to sequences of fuzzy numbers.

Let $C(R^n) = \{A \subseteq R^n : A$ compact and convex $\}$. The space $C(R^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \subseteq C(R^n)$ and $\lambda \in R$. The Hausdorff distance between $A$ and $B$ of $C(R^n)$ is defined as

$$\delta_\infty(A, B) = \max \{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}.$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete metric space.

The fuzzy number is a function $X$ from $R^n$ to $[0,1]$ which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the $\alpha$—level set $\{X \leq \alpha\} = \{x \in R^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of $R^n$, with support $X^0 = \{x \in R^n : X(x) > 0\}$. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$, in terms of $\alpha$—level sets, by $[X + Y]^\alpha = [X^\alpha + Y^\alpha], [\lambda X]^\alpha = \lambda [X]^\alpha$ for each $0 \leq \alpha \leq 1$. Define, for each $1 \leq q < \infty$,

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty(X^\alpha,Y^\alpha)^q \, d\alpha\right)^{1/q},$$

and $d_\infty = \sup_{0 \leq \alpha \leq 1} \delta_\infty(X^\alpha,Y^\alpha)$,

where $\delta_\infty$ is the Hausdorff metric. Clearly $d_\infty(X,Y) = \lim_{q \to \infty} d_q(X,Y)$ with $d_q \leq d_r, \text{ if } q \leq r$ [4]. Throughout the paper, $d$ will denote $d_q$ with $1 \leq q \leq \infty$. Let $w$ be set of all sequences of fuzzy numbers. The generalized de la Vallé-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a nondecreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1 = 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n := [n - \lambda_n + 1, n]$. A sequence $x(x_k)$ is said to be $(V, \lambda)$—summable to a number $L$ [8] if $t_n(x) \to L$ as $n \to \infty$. $(V, \lambda)$—summability reduces to $(C, 1)$ summability when $\lambda_n = n$ for all $n$. A complex sequence,
whose \( k^{th} \) terms is \( x_k \) is denoted by \( \{x_k\} \) or simply \( x \). Let \( \phi \)
be the set of all finite sequences. Let \( \ell_\infty, c, c_0 \) be the sequence
spaces of bounded, convergent and null sequences \( x = (x_k) \) respectively.
In respect of \( \ell_\infty, c, c_0 \) we have
\[ ||x|| = \sup_k |x_k|, \]
where \( x = (x_k) \in c_0 \subset c \subset \ell_\infty \). A sequence \( x = \{x_k\} \) is said to be analytic if \( \sup_k |x_k|^{1/k} < \infty \). The vector space of all analytic sequences will be denoted by \( \Lambda \). A sequence \( x \) is called entire sequence if \( \lim_{k \to \infty} |x_k|^{1/k} = 0 \). The vector space of all entire sequences will be denoted by \( \Gamma \).

The space \( \Gamma_M \) and \( \Lambda_M \) is a metric space with the metric
\[ d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right) \right) \leq 1 \right\} \]
for all \( x = \{x_k\} \) and \( y = \{y_k\} \) in \( \Gamma_M \).

In the present paper we introduce and examine the concepts of \( \lambda_M \)—statistical convergence and strongly \( (\lambda_M)_\mu \)– Cesàro convergence of sequences of fuzzy numbers. Firstly in section 2, we give the definition of \( \lambda_M \)—statistical convergence and strongly \( (\lambda_M)_\mu \)– Cesàro convergence of sequence of fuzzy numbers. In section 3, we establish some inclusion relation between the sequences \( s(\lambda_M) \) and \( (\lambda_M)_\mu \). We now give the following new definitions which will be needed in the sequel.

**A. Definition**

Let \( X = (X_k) \) be a sequence of fuzzy numbers. A sequence \( X = (X_k) \) of fuzzy numbers is said to converge to fuzzy number \( X_0 \) if for every \( \epsilon > 0 \) there is a positive integer \( N_0 \) such that \( d\left( M\left(\frac{|x_k|^{1/k}}{\rho}\right), X_0\right) < \epsilon \) for \( k \geq N_0 \) and \( X = (X_k) \) is said to be Cauchy sequence if for every \( \epsilon > 0 \) there is a positive integer \( N_0 \) such that \( d\left( M\left(\frac{|x_k|^{1/k}}{\rho}\right), X_0\right) < \epsilon \) for \( k, \ell \geq N_0 \).

**B. Definition**

A sequence \( X = (X_k) \) of fuzzy numbers is said to be analytic if the set \( \left\{ M\left(\frac{|X_k|^{1/k}}{\rho}\right) : k \in N \right\} \) of fuzzy numbers is analytic.

**C. Definition**

A sequence \( X = (X_k) \) of fuzzy numbers is said to be \( \lambda_M \)—statistically convergent to a fuzzy number \( X_0 \) if for every \( \epsilon > 0 \) there is a positive integer \( N_0 \) such that \( \frac{1}{n} \left| \left\{ k \in I_n : d\left( M\left(\frac{|X_k|^{1/k}}{\rho}\right), X_0\right) \geq \epsilon \right\} \right| \to 0 \) as \( n \to \infty \).

In this case we shall write \( S_{\lambda_M} \lim k \rightarrow \infty \left( M\left(\frac{|X_k|^{1/k}}{\rho}\right)\right) = X_0 \)

It can be shown that if a sequence \( X = (X_k) \) of fuzzy numbers is convergent to a fuzzy number \( X_0 \), then it is statistically convergent to the fuzzy number \( X_0 \), but the converse does not hold. For example, we define \( X = (X_k) \) such that
\[ M\left(\frac{|X_k|^{1/k}}{\rho}\right) = \left\{ \begin{array}{ll} A & \text{if } k = n^2, n = 1, 2, 3, \ldots \text{ otherwise} \\ 0 & \end{array} \right. \]
Where \( A \) is a fixed fuzzy number. Then \( X = (X_k) \) is statistically convergent but not convergent.

**D. Definition**

A sequence \( X = (X_k) \) of fuzzy numbers is said to be strongly \( \lambda_M \)–summable if there is a fuzzy number \( X_0 \) such that \( \frac{1}{n} \sum_{k \in I_n} d\left( M\left(\frac{|X_k|^{1/k}}{\rho}\right), X_0\right) \to 0 \) as \( n \to \infty \).
E. Definition

A sequence \( X = (X_k) \) of fuzzy numbers is said to be strongly \( \lambda_M\)-\( \text{Cesáro} \) summable if there is a fuzzy number \( X_0 \) such that
\[
\frac{1}{n} \sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \rightarrow 0 \quad \text{as} \ n \rightarrow \infty
\]
The set of all strongly \( (\lambda_M)_p \)-\( \text{Cesáro} \) summable sequences of fuzzy numbers is denoted by \( \lambda(M_p) \).

F. Definition

A sequence \( X = (X_k) \) of fuzzy numbers is said to be \( \lambda_M\)-statistically convergent or \( S_{\lambda_M} \) to a fuzzy number \( X_0 \) if for every \( \epsilon > 0 \), we have
\[
\frac{1}{n} \left\{ \sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \geq \epsilon \right\} \rightarrow 0 \quad \text{as} \ n \rightarrow \infty
\]
In this case we shall write \( S_{\lambda_M} \) \( \lim \left( M \left( \frac{|X_k|^1}{\rho} \right) \right) = X_0 \).

III. MAIN RESULTS

A. Theorem

(i) If a sequence \( X = (X_k) \) is strongly \( \lambda(M)_p \)-\( \text{Cesáro} \) summable to \( X_0 \), then it is \( \lambda_M\)-statistically convergent to \( X_0 \).

(ii) If \( X = (X_k) \) is a sequence \( \lambda_M\)-analytic and \( \lambda_M\)-statistically convergent to \( X_0 \), then it is strongly \( \lambda(M)_p \)-\( \text{Cesáro} \) summable to \( X_0 \) and hence \( X \) is strongly \( \lambda_M\)-\( \text{Cesáro} \) summable to \( X_0 \).

Proof: Let \( \epsilon > 0 \) and \( X \in \lambda(M)_p \). We have
\[
\sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \geq \left\{ \sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \geq \epsilon \right\} \geq \epsilon^p
\]
Therefore \( \lambda_M \) is statistically convergent \( X_0 \).

B. Theorem

Let \( (X_k) \) and \( (Y_k) \) be sequence of fuzzy numbers.

(i) If \( S_{\lambda_M} \) \( \lim \left( M \left( \frac{|X_k|^1}{\rho} \right) \right) = X_0 \) and \( c \in R \), then \( S_{\lambda_M} \) \( \lim \left( cM \left( \frac{|X_k|^1}{\rho} \right) \right) = cX_0 \) and \( S_{\lambda_M} \) \( \lim \left( M \left( \frac{|Y_k|^1}{\rho} \right) \right) = X_0 \) and \( S_{\lambda_M} \) \( \lim \left( M \left( \frac{|X_k|^1}{\rho} \right) + \left( M \left( \frac{|Y_k|^1}{\rho} \right) \right) \right) = X_0 + Y_0 \).

Proof: Let \( \alpha \in [0, 1] \) and \( c \in R \). Let \( M \left( \frac{|X_k|^1}{\rho} \right) \), \( M \left( \frac{|Y_k|^1}{\rho} \right) \), \( X_0 \) and \( Y_0 \) be \( \alpha \)-level sets of \( M \left( \frac{|X_k|^1}{\rho} \right) \), \( M \left( \frac{|Y_k|^1}{\rho} \right) \) respectively. Since \( \delta_\alpha \left( cM \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) = |\alpha - \delta_\alpha \left( M \left( \frac{|X_k|^1}{\rho} \right) , X_0 \right) | \) we have
\[
d \left( cM \left( \frac{|X_k|^1}{\rho} \right), cX_0 \right) = |\alpha - \delta_\alpha \left( M \left( \frac{|X_k|^1}{\rho} \right) , X_0 \right) |.
\]
For given \( \epsilon > 0 \) we have
\[
\frac{1}{n} \left\{ \sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \geq \epsilon \right\} \leq \frac{1}{n} \left\{ \sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \geq \epsilon \right\}.
\]
Hence
\[
S_{\lambda_M} \) \( \lim \left( M \left( \frac{|X_k|^1}{\rho} \right) \right) = X_0 \).
\]
Therefore \( \epsilon > 0 \) we have
\[
\frac{1}{n} \left\{ \sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \geq \epsilon \right\} \leq \frac{1}{n} \left\{ \sum_{k=1}^{n} \left( d \left( M \left( \frac{|X_k|^1}{\rho} \right), X_0 \right) \right)^p \geq \epsilon \right\}.
\]
Hence \( S_{\lambda_M} \) \( \lim \left( M \left( \frac{|X_k|^1}{\rho} \right) \right) = X_0 + Y_0 \).
This completes the proof.

C. Theorem

If a sequence \( X = (X_k) \) is statistically convergent to \( X_0 \) and \( \liminf_{n} \left( \frac{1}{n} \sum_{k \in I_n} d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right) > 0 \), then it is \( \lambda_M \)-statistically convergent to \( X_0 \).

Proof: For given \( \epsilon > 0 \), we have
\[
\left\{ k \in n : d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \geq \epsilon \right\} \supset \left\{ k \in I_n : d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \geq \epsilon \right\}.
\]
Therefore
\[
\frac{1}{n} \sum_{k \in I_n} d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \geq \epsilon > 0.
\]
Taking \( \lim \) as \( n \to \infty \) and using \( \liminf_{n} \left( \frac{1}{n} \sum_{k \in I_n} d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right) > 0 \), we get \( X = (X_k) \) is \( \lambda_M \)-statistically convergent to \( X_0 \). This completes the proof.

D. Definition

Let \( p = (p_k) \) be any sequence of positive real numbers. Then we define \( \lambda_{M_p} = X = (X_k) : \frac{1}{n} \sum_{k \in I_n} d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \to 0 \) as \( n \to \infty \).

E. Theorem

Let \( 0 \leq p_k \leq q_k \) and let \( \left\{ \frac{p_k}{q_k} \right\} \) be bounded. Then \( \lambda_{M_p} \subseteq \lambda_{M_q} \).

Proof: Let
\[
\frac{1}{n} \sum_{k \in I_n} \left[ d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.
\]

Hence
\[
\frac{1}{n} \sum_{k \in I_n} \left[ d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.
\]

From (4) and (6) we get \( \lambda_{M_q} \subseteq \lambda_{M_p} \). This completes the proof.

F. Theorem

(a) Let \( 0 < \inf p_k \leq p_k \leq 1 \). Then \( \lambda_{M_p} \subseteq \lambda_{M} \) (b) Let \( 1 \leq p_k \leq \sup p_k < \infty \). Then \( \lambda_{M} \subseteq \lambda_{M_p} \).

Proof: (a) Let
\[
X \in \lambda_{M_p}
\]
\[
\frac{1}{n} \sum_{k \in I_n} d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \to 0 \text{ as } n \to \infty.
\]
Since \( 0 < \inf p_k \leq p_k \leq 1 \)
\[
\frac{1}{n} \sum_{k \in I_n} \left[ d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right]^{p_k} \leq
\]
\[
\frac{1}{n} \sum_{k \in I_n} \left[ d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right] \to 0 \text{ as } n \to \infty.
\]
So \( \lambda_{M} \subseteq \lambda_{M_p} \).

Thus
\[
\lambda_{M_p} \subseteq \lambda_{M}.
\]

This completes the proof.

Proof: (b) Let \( p_k \geq 1 \) for each \( k \) and \( \sup p_k < \infty \). Let \( X \in \lambda_{M} \)
\[
\frac{1}{n} \sum_{k \in I_n} \left[ d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right] \to 0 \text{ as } n \to \infty
\]
Since \( 1 \leq p_k \leq \sup p_k < \infty \) we have
\[
\frac{1}{n} \sum_{k \in I_n} \left[ d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right]^{p_k}
\]
\[
\leq \frac{1}{n} \sum_{k \in I_n} \left[ d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \right] \to 0 \text{ as } n \to \infty
\]
(11)

Therefore \( X \in \lambda_{M_p} \). This completes the proof.

G. Theorem

Let \( 0 < p_k \leq q_k < \infty \) for each \( k \). Then \( \lambda_{M_p} \subseteq \lambda_{M_q} \).

Proof: Let
\[
X \in \lambda_{M_p}
\]
\[
\frac{1}{n} \sum_{k \in I_n} d \left( \frac{|X_k|^{1/k}}{\rho} , X_0 \right) \to 0 \text{ as } n \to \infty
\]
This implies that \( \frac{1}{n} \sum_{k \in I_n} d\left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \leq 1 \), for sufficiently large \( n \). Since \( M \) is non-decreasing, we get
\[
\frac{1}{n} \sum_{k \in I_n} d\left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \leq \frac{1}{n} \sum_{k \in I_n} d\left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \leq \lambda_n \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \Rightarrow \frac{1}{n} \sum_{k \in I_n} d\left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \rightarrow 0 \text{as } n \rightarrow \infty \] 
(by using 12). Hence
\[
X \in (\lambda_M)_q \tag{13}
\]
From (11) and (13) we get \( (\lambda_M)_p \subseteq (\lambda_M)_q \). This completes the proof.

IV. CONCLUSION

The above results are constructed with the concept of strongly \( (\lambda_M)_p \) — Cesàro summability of a entire sequence of fuzzy numbers and strongly \( \lambda_M \) — statistically convergent sequences of fuzzy numbers.

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REFERENCES