On $\lambda-$ Summable of Orlicz Space of Entire Sequences of Fuzzy Numbers

N.Subramanian, U.K.Misra and M.S.Panda

Abstract—In this paper the concept of strongly $(\lambda M)_\delta$ – Cesàro summability of a sequence of fuzzy numbers and strongly $\lambda M$– statistically convergent sequences of fuzzy numbers is introduced.

Keywords—Fuzzy numbers, statistical convergence, Orlicz space, entire sequence.

I. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced Zadeh[18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka[10] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Matloka[10] also has shown that every convergent sequence of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed by Nanda[12], Nuray[14], Kwon[9], Savas[15], Wu and Wang[17], Bilgin[3] Basirar and Mursaleen [2,11], Ay-

II. THEORETICAL RESULTS

The existing literature on summable sequences of fuzzy numbers is quite extensive and has been introduced by Fast[6] and Schoenberg[16] independently. Asymptotic equivalence was introduced by Nanda[12], Nuray [14], Kwon[9], Savas[15], Wu and Wang[17], Bilgin[3] Basirar and Mursaleen [2,11], Ay-

The Hausdorff distance between $A$ and $B$ of $C(R^n)$ is defined as

$$d_\infty(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \}$$

It is well known that $(C(R^n), d_\infty)$ is a complete metric space.

The fuzzy number is a function $X$ from $R^n$ to $[0,1]$ which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the $\alpha$–level set $[X]_\alpha = \{x \in R^n : X(x) \geq \alpha\}$ is a nonempty convex compact subset of $R^n$, with support $X^0 = \{x \in R^n : X(x) > 0\}$. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$, in terms of $\alpha$–level sets, by $[X + Y]_\alpha = [X]_\alpha + [Y]_\alpha, [\lambda X]_\alpha = \lambda [X]_\alpha$ for each $0 \leq \alpha \leq 1$. Define, for each $1 \leq q < \infty$,

$$d_q(X, Y) = \left( \int_0^1 \delta_\infty((X^\alpha, Y^\alpha)^q) \, d\alpha \right)^{1/q}, \text{ and } d_\infty = \sup_{0 \leq \alpha \leq 1} \delta_\infty((X^\alpha, Y^\alpha)^q),$$

where $\delta_\infty$ is the Hausdorff metric. Clearly $d_\infty(X, Y) = \lim_{q \to \infty} d_q(X, Y)$ with $d_q \leq d_r$, if $q \leq r$ [4]. Throughout the paper, $d$ will denote $d_q$ with $1 \leq q \leq \infty$. Let $w$ be a set of all sequences of fuzzy numbers. The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a nondecreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1 = 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. A sequence $x(k)$ is said to be $(V, \lambda)$ – summable to a number $L$ [8] if $t_n(x(k)) \to L$ as $n \to \infty$. $(V, \lambda)$ – summability reduces to $(C, 1)$ summability when $\lambda_n = n$ for all $n$. A complex sequence,
whose \( k^{th} \) terms is \( x_k \) is denoted by \( \{x_k\} \) or simply \( x \). Let \( \phi \) be the set of all finite sequences. Let \( \ell_{c_0}, c, c_0 \) be the sequence spaces of bounded, convergent, and null sequences \( x = (x_k) \) respectively. In respect of \( \ell_{c_0}, c, c_0 \) we have
\[
\|x\| = \sup |x_k|, \quad \text{where } x = (x_k) \in c_0 \subset c \subset \ell_{c_0}.
\]
A sequence \( x = (x_k) \) is said to be analytic if \( \sup |x_k|^{1/k} < \infty \). The vector space of all analytic sequences will be denoted by \( \Lambda \). A sequence \( x \) is called entire sequence if \( \lim_{k \to \infty} |x_k|^{1/k} = 0 \). The vector space of all entire sequences will be denoted by \( \Gamma \).

Orlicz [26] used the idea of Orlicz function to construct the space \( (\ell_{c_0}) \). Lindenstrauss and Tzafriri [27] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space \( \ell_{\phi} \) contains a subspace isomorphic to \( \ell_p(1 \leq p < \infty) \). Subsequently different classes of sequence spaces defined by Parashar and Choudhary [28], Mursaleen et al. [29], Bektas and Altin [30], Tripathy et al. [31], Rao and subramaniam [32] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref. [33].

Recall [26], [33] an Orlicz function is a function \( M : [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x+y) \leq M(x)+M(y) \) then this function is called modulus function introduced by Nakano [34] and further discussed by Ruckle [35] and Maddox [36] and many others.

An Orlicz function \( M \) is said to satisfy \( \Delta_2 \)–condition for all values of \( u \), if there exists a constant \( K > 0 \) such that \( M(2u) \leq KM(u)(u \geq 0) \). The \( \Delta_2 \)–condition is equivalent to \( K\Delta_2 \)–condition for all values of \( u \) and for \( \ell > 1 \). Lindenstrauss and Tzafriri [27] used the idea of Orlicz function to construct Orlicz sequence space
\[
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{for some } \rho > 0 \right\}.
\]

The space \( \ell_M \) with the norm
\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}
\]
becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p, 1 \leq p < \infty \), the space \( \ell_M \) coincide with the classical sequence space \( \ell_p \). Given a sequence \( x = (x_k) \) its \( n^{th} \) section is the sequence \( x(n) = (x_n, x_{n+1}, \ldots) \) \( \delta^{(n)} = (0, 0, \ldots, 1, 0, 0, \ldots) \), 1 in the \( n^{th} \) place and zero’s else where.

**II. DEFINITIONS AND PRELIMINARIES:**

Let \( w \) denote the set of all fuzzy complex sequences \( x = (x_k)_{k=1}^\infty \), and \( M \) be an Orlicz function, or a modulus function. consider
\[
\Gamma_M = \{ x \in w : \lim_k \infty M \left( \frac{|x_k|^{1/k}}{\rho} \right) = 0 \text{ for some } \rho > 0 \} \text{ and } \Lambda_M = \{ x \in w : \sup_k M \left( \frac{|x_k|^{1/k}}{\rho} \right) < \infty \text{ for some } \rho > 0 \}. \]

The space \( \Gamma_M \) and \( \Lambda_M \) is a metric space with the metric
\[
d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right) \right) \leq 1 \right\}
\]
for all \( x = (x_k) \) and \( y = (y_k) \) in \( \Gamma_M \).

In the present paper we introduce and examine the concepts of \( \lambda_M \)–statistical convergence and strongly \( \lambda_M \)–Cesàro convergence of sequences of fuzzy numbers. Firstly in section 2, we give the definition of \( \lambda_M \)–statistical convergence and strongly \( \lambda_M \)–Cesàro convergence of sequence of fuzzy numbers. In section 3, we establish some inclusion relation between the sequences \( s(\lambda_M) \) and \( \lambda_M \). We now give the following new definitions which will be needed in the sequel.

**A. Definition**

Let \( X = (X_k) \) be a sequence of fuzzy numbers. A sequence \( X = (X_k) \) of fuzzy numbers is said to converge to fuzzy number \( X_0 \) if for every \( \varepsilon > 0 \) there is a positive integer \( N_0 \) such that \( d \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) < \varepsilon \) for \( k \geq N_0 \). And \( X = (X_k) \) is said to be Cauchy sequence if for every \( \varepsilon > 0 \) there is a positive integer \( N_0 \) such that \( d \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) < \varepsilon \) for \( k, \ell \geq N_0 \).

**B. Definition**

A sequence \( X = (X_k) \) of fuzzy numbers is said to be analytic if the set \( \{ M \left( \frac{|X_k|^{1/k}}{\rho} \right) : k \in N \} \) of fuzzy numbers is analytic.

**C. Definition**

A sequence \( X = (X_k) \) of fuzzy numbers is said to be \( \lambda_M \)–statistically convergent to a fuzzy number \( X_0 \) if for every \( \varepsilon > 0 \), we have
\[
\frac{1}{n} \left\{ \sum_{i=1}^{n} M \left( \frac{|X_i|^{1/k}}{\rho} \right) \geq \varepsilon \right\} \to 0 \quad \text{as } n \to \infty.
\]
In this case we shall write \( S_{\lambda_M} \lim k \to \infty M \left( \frac{|X_k|^{1/k}}{\rho} \right) = X_0 \)

It can be shown that if a sequence \( X = (X_k) \) of fuzzy numbers is convergent to a fuzzy number \( X_0 \), then it is statistically convergent to the fuzzy number \( X_0 \), but the converse does not hold. For example, we define \( X = (X_k) \) such that
\[
M \left( \frac{|X_k|^{1/k}}{\rho} \right) = \left\{ \begin{array}{ll} A & \text{if } k = n^2, n = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{array} \right.
\]
Where \( A \) is a fixed fuzzy number. Then \( X = (X_k) \) is statistically convergent but not convergent.

**D. Definition**

A sequence \( X = (X_k) \) of fuzzy numbers is said to be strongly \( \lambda_M \)–summable if there is a fuzzy number \( X_0 \) such that
\[
\frac{1}{n} \sum_{k=1}^{n} M \left( \frac{|X_k|^{1/k}}{\rho} \right) \to 0 \quad \text{as } n \to \infty
\]
E. Definition

A sequence \( X = (X_k) \) of fuzzy numbers is said to be strongly \( \lambda_M \)-Cesàro summable if there is a fuzzy number \( X_0 \) such that \( \frac{1}{n^p} \sum_{k=1}^{n} d \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \to 0 \) as \( n \to \infty \). The set of all strongly \( (\lambda_M)_p \)-Cesàro summable sequences of fuzzy numbers is denoted by \( \lambda(M)_p \).

F. Definition

A sequence \( X = (X_k) \) of fuzzy numbers is said to be \( \lambda_M \)-statistically convergent or \( S_M \lambda(x) \) to a fuzzy number \( X_0 \) if for every \( \epsilon > 0 \), we have

\[
\frac{1}{n^p} \sum_{k=1}^{n} \left| \left\{ k \in I_n : d \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \geq \epsilon \right\} \right| \to 0 \text{ as } n \to \infty
\]

In this case we shall write \( S_M \lambda(x) \to X_0 \).

In the special case \( (\lambda_M)_n \) is a sequence \( n \) for all \( n \in N \), then \( \lambda_M \)-statistically convergent is same as statistically convergent.

III. MAIN RESULTS

A. Theorem

(i) If a sequence \( X = (X_k) \) is strongly \( \lambda(M)_p \)-Cesàro summable to \( X_0 \), then it is \( \lambda_M \)-statistically convergent to \( X_0 \).

(ii) Suppose that \( X = (X_k) \) is a sequence \( \lambda(M)_p \)-analytic and \( \lambda(M)_p \)-statistically convergent to \( X_0 \), then it is strongly \( \lambda_M \)-Cesàro summable to \( X_0 \), and hence \( X \) is strongly \( \lambda_M \)-Cesàro summable to \( X_0 \).

B. Theorem

Let \( (X_k) \) and \( (Y_k) \) be sequence of fuzzy numbers.

(i) If \( S_M \lambda(x) \to X_0 \) and \( c \in R \), then \( S_M \lambda \left( \frac{|X_k|^{1/k}}{\rho} \right) \to cX_0 \).

(ii) Suppose that \( S_M \lambda(x) \to X_0 \) and \( \lambda_M \lambda \left( \frac{|Y_k|^{1/k}}{\rho} \right) \to Y_0 \) respectively. Since \( \delta \left( \frac{|X_k|^{1/k}}{\rho} \right), \frac{|Y_k|^{1/k}}{\rho} \) are \( \alpha \)-level sets of \( M \left( \frac{|X_k|^{1/k}}{\rho} \right), M \left( \frac{|Y_k|^{1/k}}{\rho} \right) \) and \( X_0, Y_0 \) respectively, we have

\[
d \left( \frac{|X_k|^{1/k}}{\rho}, X_0 \right) \leq \delta \left( \frac{|X_k|^{1/k}}{\rho} \right) \leq \delta \left( \frac{|X_k|^{1/k}}{\rho} \right)
\]

Hence \( S_M \lambda \left( \frac{|X_k|^{1/k}}{\rho} \right) \to X_0 \) and \( S_M \lambda \left( \frac{|Y_k|^{1/k}}{\rho} \right) \to Y_0 \).

Proof: Let \( M \left( \frac{|x|^{1/k}}{\rho} \right), M \left( \frac{|y|^{1/k}}{\rho} \right) \) be \( \alpha \)-level sets of \( M \left( \frac{|x|^{1/k}}{\rho} \right), M \left( \frac{|y|^{1/k}}{\rho} \right) \) and \( X_0, Y_0 \) respectively. Then \( \delta \left( \frac{|X_k|^{1/k}}{\rho} \right) \leq \delta \left( \frac{|X_k|^{1/k}}{\rho} \right) \leq \delta \left( \frac{|X_k|^{1/k}}{\rho} \right) \leq \delta \left( \frac{|X_k|^{1/k}}{\rho} \right) \leq \delta \left( \frac{|X_k|^{1/k}}{\rho} \right) \).

Hence \( S_M \lambda \left( \frac{|X_k|^{1/k}}{\rho} \right) \to X_0 \) and \( S_M \lambda \left( \frac{|Y_k|^{1/k}}{\rho} \right) \to Y_0 \).

(ii) Suppose that \( S_M \lambda(x) \to X_0 \) and \( \lambda_M \lambda \left( \frac{|Y_k|^{1/k}}{\rho} \right) \to Y_0 \).

By Minkowski’s inequality we get

\[
d \left( \frac{|X_k|^{1/k}}{\rho} + \frac{|Y_k|^{1/k}}{\rho} \right) \to X_0 + Y_0
\]

Hence \( S_M \lambda \left( \frac{|X_k|^{1/k}}{\rho} \right) \to X_0 + Y_0 \).

\[
d \left( \frac{|X_k|^{1/k}}{\rho} \right) + d \left( \frac{|Y_k|^{1/k}}{\rho} \right) \to X_0 + Y_0
\]

Therefore given \( \epsilon > 0 \) we have

\[
\frac{1}{n} \sum_{k=1}^{n} \left| \left\{ k \in I_n : d \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) + d \left( M \left( \frac{|Y_k|^{1/k}}{\rho} \right), Y_0 \right) \geq \epsilon \right\} \right| \to 0 \text{ as } n \to \infty
\]
This completes the proof.

C. Theorem

If a sequence $X = (X_k)$ is statistically convergent to $X_0$ and $\lim inf_{n}(\frac{1}{n}\mathcal{M}(\frac{1}{n})) > 0$, then it is $\lambda_m$-statistically convergent to $X_0$.

Proof: For given $\frac{1}{n} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) < \epsilon$, therefore

$$\frac{1}{n} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) < \epsilon.$$ 

Taking limits as $n \to \infty$ and using $\lim inf_{n}(\frac{1}{n}\mathcal{M}(\frac{1}{n})) > 0$, we get $X = (X_k)$ is $\lambda_m$-statistically convergent to $X_0$. This completes the proof.

D. Definition

Let $p = (p_k)$ be any sequence of positive real numbers. Then we define $(\lambda_m)_p = X = (X_k)$:

$$\frac{1}{p_k} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right) \geq 0 \quad \text{as} \quad n \to \infty.$$

Suppose that $p_k$ is a constant for all $k$, then $(\lambda_m)_p = \lambda_m$.

E. Theorem

Let $0 \leq p_k \leq q_k$ and let $\{\frac{q_k}{p_k}\}$ be bounded. Then $(\lambda_m)_p \subset (\lambda_m)_q$.

Proof: Let

$$X \in \lambda_m$$

Hence

$$\frac{1}{p_k} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k} \to 0 \quad \text{as} \quad n \to \infty.$$ 

From (4) and (6) we get $(\lambda_m)_q \subset (\lambda_m)_p$. This completes the proof.

F. Theorem

(a) Let $0 < \inf p_k \leq p_k \leq 1$. Then $(\lambda_m)_p \subset \lambda_m$.

(b) Let $1 \leq p_k \leq \sup p_k < \infty$. Then $\lambda_m \subset (\lambda_m)_p$.

Proof: (a) Let

$$X \in \lambda_m$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\frac{1}{\lambda_m} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$(\lambda_m)_p \subset \lambda_m.$$ 

This completes the proof.

Proof: (b) Let $p_k \geq 1$ for each $k$ and $\sup p_k < \infty$. Let $X \in \lambda_m$.

Hence

$$\frac{1}{\lambda_m} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Since $1 \leq p_k \leq \sup p_k < \infty$ we have

$$\frac{1}{\lambda_m} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k} \leq \frac{1}{\lambda_m} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Therefore $X \in (\lambda_m)_p$. This completes the proof.

G. Theorem

Let $0 < q_k < \infty$ for each $k$. Then $(\lambda_m)_p \subset (\lambda_m)_q$.

Proof: Let

$$X \in \lambda_m$$

Hence

$$\frac{1}{\lambda_m} \sum_{k \in I_n} \left( M \left( \frac{|X_k|^{1/k}}{\rho} \right), X_0 \right)^{p_k} \to 0 \quad \text{as} \quad n \to \infty.$$
This implies that \( \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{|X_n|^k}{\rho} \right), X_0 \right) \leq 1 \), for sufficiently large \( n \). Since \( M \) is non-decreasing, we get
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{|X_n|^k}{\rho} \right), X_0 \right) \leq \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{|X_n|^k}{\rho} \right), X_0 \right) \leq \lambda \implies \frac{1}{\lambda_n} \sum_{k \in I_n} d \left( M \left( \frac{|X_n|^k}{\rho} \right), X_0 \right) \to 0 \text{ as } n \to \infty \] 
(by using 12). Hence
\[
X \in (\lambda M)_\rho
\] 
(13)

From (11) and (13) we get \((\lambda M)_\rho \subseteq (\lambda M)_\rho \). This completes the proof.

IV. Conclusion

The above results are constructed with the concept of strongly \((\lambda M)_\rho \) — Cesàro summability of a sequence of fuzzy numbers and strongly \(\lambda M \) — statistically convergent sequences of fuzzy numbers.

Acknowledgment

I wish to thank the referees for their several remarks and valuable suggestions that improved the presentation of the paper.

References