Augmented Lyapunov approach to robust stability of discrete-time stochastic neural networks with time-varying delays

Shu Lü, Shouming Zhong, Zixin Liu

Abstract—In this paper, the robust exponential stability problem of discrete-time uncertain stochastic neural networks with time-varying delays is investigated. By introducing a new augmented Lyapunov function, some delay-dependent stable results are obtained in terms of linear matrix inequality (LMI) technique. Compared with some existing results in the literature, the conservatism of the new criteria is reduced notably. Three numerical examples are provided to demonstrate the less conservatism and effectiveness of the proposed method.

Keywords—Robust exponential stability, delay-dependent stability, discrete-time neural networks, stochastic, time-varying delays.

I. INTRODUCTION

RECENTLY, recurrent neural networks (see [1]-[7]), such as Hopfield neural networks, cellular neural networks and other networks have attracted considerable attention because of their potential applications in pattern recognition, image processing, fixed-point computation, and so on. However, because of the finite switching speed of neurons and amplifiers, time delays, both constant and time-varying, are often unavoidable in various engineering, neural networks, large-scale, biological, and economic systems. Since the occurrence of time delays may cause poor performance or instability, the studies on stability for delayed neural networks are of great significance. There has been a growing research interest on the stability analysis problems for delayed neural networks, and many excellent papers and monographs have been available. On the other hand, during the design of neural network and its hardware implementation, the convergence of a neural network may often be destroyed by its unavoidable uncertainty due to the existence of modeling error, the deviation of vital data, and so on. These unavoidable uncertainty can be classified into two types: that is, stochastic disturbances and parameters uncertainties. As pointed out in [8] that, while modeling real nervous systems, both of the stochastic disturbances and parameters uncertainties are probably the main resources of the performance degradations of the implemented neural networks. Therefore, the studies on robust convergence of stochastic delayed neural network have been a hot research direction. Up to now, many sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the global robust asymptotic or exponential stability for different class of delayed neural networks (see [9]-[17]).

S. Lü, S. Zhong and Z. Liu are with School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, 610054, China. E-mail: sangechengxuezi@yahoo.com.cn

It’s worth pointing out that most neural networks have been assumed to be in continuous time, but few in discrete time. In practice, discrete-time neural networks are more applicable to problems that are inherently temporal in nature or related to biological realities. And they can ideally keep the dynamic characteristics, functional similarity, and even the physical or biological reality of the continuous-time networks under mild restriction. Thus, the stability analysis problems for discrete-time neural networks have received more and more interest, and some stability criteria have been proposed in the literature (see [8],[16]-[26]). For the first time, Liu,Wang and Liu considered the robust stability for discrete-time stochastic neural networks with time-varying delays in [8], and proposed some delay-dependent stability criteria in terms of LMI approach. By using a similar technique to that in [19], [20], the result obtained in [8] has been improved by Luo et al. [17].

In this paper, some new improved delay-dependent stability criteria are obtained via constructing a new augmented Lyapunov-Krasovskii function. These new conditions are less conservative than those obtained in [8], [16]-[21]. Furthermore, three numerical examples are also provided to illuminate the improvement of the proposed criteria.

Notation: The notations are used in our paper except where otherwise specified. $\|\cdot\|$ denotes a vector or a matrix norm; $\mathbb{R}$, $\mathbb{R}^n$ are real and n-dimension real number sets, respectively; $\mathbb{N}^+$ is positive integer set. $I$ is identity matrix; $*$ represents the elements below the main diagonal of a symmetric block matrix; Real matrix $P > 0$ ($< 0$) denotes $P$ is a positive definite (negative definite) matrix; $\mathbb{N}[a, b] = \{a, a+1, \ldots, b\}$; $\lambda_{\min}$ ($\lambda_{\max}$) denotes the minimum (maximum) eigenvalue of a real matrix; $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space with filtration $\mathcal{P}$ satisfying the usual condition; $E(\cdot)$ stands for the mathematical expectation operator with respect to the given probability measure.

II. PRELIMINARIES

Consider the following n-neuron discrete-time stochastic neural network (DSNN) [8] with time delays of the form:

\[ x(k+1) = C(k)x(k) + A(k)f(x(k)) + B(k)g(x(k - \tau(k))) \]

\[ = \delta(k, x(k), x(k - \tau(k)))\omega(k), \quad k \in \mathbb{N}^+ \quad (1) \]

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ denotes the neural state vector; $f(x(k)) = [f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k))]^T$, $g(x(k - \tau(k))) = [g_1(x_1(k - \tau(1)), \ldots, g_n(x_n(k - \tau(n)))].$
\(\tau(k))\), \(g_2(x_2(k - \tau(k)))\), \(\cdots, g_n(x_n(k - \tau(k)))\)^T are the neuron activation functions; Positive integer \(\tau(k)\) represents the transmission delay that satisfies 0 < \(\tau(m)\) ≤ \(\tau(k)\) ≤ \(\tau(M)\), where \(\tau(m), \tau(M)\) are known positive integers representing the lower and upper bounds of the delay. \(C(k) = C + \Delta C(k), A(k) = A + \Delta A(k), B(k) = B + \Delta B(k),\) \(C = diag(c_1, c_2, \ldots, c_n)\) with \(|c_i| < 1\) describes the rate with which the \(i\)th neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; \(A, B \in \mathbb{R}^{n \times n}\) represent the weighting matrices; \(\Delta C(k), \Delta A(k), \Delta B(k)\) denote the time-varying structured uncertainties which are of the following form:

\[\begin{bmatrix} \Delta C(k), \Delta A(k), \Delta B(k) \end{bmatrix} = K F(k)[E_e, E_a, E_b],\]

where \(K, E_e, E_a, E_b\) are known real constant matrices of appropriate dimensions, \(F(k)\) is unknown time-varying matrix function satisfying \(F^T(k)F(k) \leq I, \forall k \in \mathbb{N}^+.\) \(\omega(k)\) is a scalar Wiener process (Brownian Motion) on \((\Omega, \mathcal{F}, \mathbb{P})\) with \(E(\omega(k)) = 0, E(\omega(i)\omega(j)) = 0, \forall i \neq j; \delta\) is the continuous function.

To obtain our main results, we need introduce the following assumptions, definition and lemmas.

**Assumption 1:** For \(i \in \mathbb{N}^+, x \neq y \in \mathbb{R}\), the neuron activation functions in DSNN (1) satisfy

\[l_i^\pm(x) \leq l_i(x) - l_i(y) \leq l_i^\pm(x - y),\]

where \(l_i^+, l_i^-, \sigma_i^+, \sigma_i^-\) are known constant scalars.

**Assumption 2:** The continuous function \(\delta^T(k, x(k), x(k - \tau(k)))\) satisfies

\[\delta^T(k, x(k), x(k - \tau(k))) \leq \rho_1 x^T(k) x(k) + \rho_2 x^T(k - \tau(k)) x(k - \tau(k)),\]

where \(\rho_1 > 0, \rho_2 > 0\) are known constant scalars.

**Assumption 3:**
\[f(0) = g(0) = 0,\]

which means that \(x(k) \equiv 0\) is a trivial solution of the DSNN (1).

**Remark 2.1:** As pointed out in [8], the constants \(l_i^+, l_i^-, \sigma_i^+, \sigma_i^-\) in Assumption 1 are allowed to be positive, negative or zero. Hence, the resulting activation functions could be nonmonotonic, and are more general than the usual sigmoid functions and the recently commonly used Lipschitz conditions.

**Definition 2.1:** The DSNN (1) is said to be robustly exponentially stable in the mean square if there exist constants \(\alpha > 0, \beta \in (0, 1)\) such that every solution of the DSNN (1) satisfies

\[E \| x(k) \|^2 \leq \alpha \cdot \beta^k \sup_{i \in \mathbb{N}^+} E \| x(i) \|^2, \forall k \geq 0,\]

for all parameter uncertainties satisfying the admissible condition.

**Lemma 2.1:** [27](Tchebychev Inequality) For any given vectors \(v_i \in \mathbb{R}^n, i = 1, 2, \ldots, n\), the following inequality holds:

\[\sum_{i=1}^{n} v_i^T \sum_{i=1}^{n} v_i \leq n \sum_{i=1}^{n} v_i^T v_i.\]

**Lemma 2.2:** [28] Given constant symmetric matrices \(\Sigma_1, \Sigma_2, \Sigma_3\) where \(\Sigma_1 = 0 < \Sigma_2 = \Sigma_3^T\), then

\[\Sigma_1 + \Sigma_3^T \Sigma_3^{-1} \Sigma_2 < 0\] if and only if

\[\begin{bmatrix} \Sigma_1 & \Sigma_3 \\ \Sigma_3^T & -\Sigma_2 \end{bmatrix} < 0\] or \[\begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.\]

**Lemma 2.3:** [8] Let \(N\) and \(E\) be real constant matrices with appropriate dimensions, matrix \(F(k)\) satisfying \(F^T(k)F(k) \leq I, \forall k \in \mathbb{N}^+.\) Then, for any \(\epsilon > 0, E \| F(k)N + N^T F(k) E \| \leq \epsilon^2 \|F\| + \epsilon \|N\|^2.\]

**III. MAIN RESULTS**

**Theorem 3.1:** For any given positive integers \(0 < \tau_m < \tau_M\), then, under Assumption 1-3, system (1) is globally robustly and exponentially stable in the mean square for any time-varying delay \(\tau(k)\) satisfying \(\tau_m \leq \tau(k) \leq \tau_M\), if there exist positive matrices \(Q, R, H, \gamma, K\), positive diagonal matrices \(A_1, A_2, A_1, A_2, A_1, A_2\), arbitrary matrices \(M_{11}, P_1, P_2, G_1, G_2\) with appropriate dimensions, and two positive scalars \(\lambda^* > 0, \epsilon > 0\) such that the following LMIs hold:

\[\Xi = [\Xi^{(1)}, [\Xi^{(2)}] < 0,\]

where
\[ \Xi_{15,15} = (1 + \frac{1}{\tau_M - \tau_m})R_{44} - \Lambda_1 + \varepsilon E_{T}^{T} E_{b}, \]

\[ \Xi_{16,16} = -\frac{1}{\tau_M - \tau_m} R_{44} - \Gamma_1, \]

\[ \Xi_{17,17} = -P_2 - P_{T_2}^T - H_2 - H_2^T, \]

\[ H = \begin{pmatrix} H_{11} & H_{12} \\ * & H_{22} \end{pmatrix} > 0, \]

\[ Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{17} \\ \ast & Q_{22} & \cdots & Q_{27} \\ \vdots & \vdots & \ddots & \vdots \\ \ast & \ast & \cdots & Q_{77} \end{pmatrix} > 0, \]

\[ R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ \ast & R_{22} & R_{23} & R_{24} \\ \ast & \ast & R_{33} & R_{34} \\ \ast & \ast & \ast & R_{44} \end{pmatrix} > 0, \]

\[ \gamma = \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \ast \gamma_{22} \end{pmatrix} > 0, \]

\[ L_1 = diag(I_{\tilde{T}_1}, I_{\tilde{T}_1}, \ldots, I_{\tilde{T}_m}), \]

\[ L_2 = diag(I_{\tilde{T}_1} + I_{\tilde{T}_1}, \ldots, I_{\tilde{T}_m} + I_{\tilde{T}_m}), \]

\[ \Pi_1 = diag(\sigma_{1}^{+} \sigma_{1}, \ldots, \sigma_{n}^{+} \sigma_{n}), \]

\[ \Pi_2 = diag(\sigma_{1}^{+} + 1/2, \ldots, \sigma_{m}^{+} + 1/2). \]

**Proof.** Constructing a new augmented Lyapunov-Krasovskii functional candidate as follows:

\[ V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) + V_6(k), \]

where

\[ V_1(k) = X^T(k)QX(k), \]

\[ X^T(k) = [x^T(k), x^T(k - \tau_m), x^T(k - \tau_M), \sum_{i=k-\tau_M}^{k-1} x^T(i), \sum_{i=k-\tau_M}^{k-1} \eta^T(i), \sum_{i=k-\tau_M}^{k-1} \xi^T(i), \eta(k), \xi(k) - (x(k + 1) - x(k))]. \]

\[ V_2(k) = \frac{1}{\tau_M - \tau_m} \sum_{i=k-\tau_M}^{k-1} \lambda^T(i) R \lambda(i), \]

\[ V_3(k) = \frac{1}{\tau_M - \tau_m} \sum_{j=k+1}^{k+1} \sum_{i=j-\tau_m}^{i-1} \lambda^T(i) R \lambda(i), \]

\[ V_4(k) = \sum_{i=k-\tau_m}^{k-1} \xi^T(i) H \xi(i) + \sum_{i=k-\tau_m}^{k-1} \xi^T(i) \gamma \xi(i), \]

where

\[ \lambda(k) = \begin{bmatrix} \eta(k) \\ f'(x(k)) \\ g'(x(k)) \end{bmatrix}, \xi(i) = \begin{bmatrix} x(k) \\ \eta(k) \end{bmatrix}. \]

\[ V_5(k) = \sum_{j=k-\tau_m}^{k-1} \sum_{i=j-\tau_m}^{i-1} \eta^T(i) Z_1 \eta(i) + \sum_{j=k-\tau_m}^{k-1} \sum_{i=j-\tau_m}^{i-1} \eta^T(i) Z_2 \eta(i), \]

\[ V_6(k) = \sum_{j=k-\tau_m}^{k-1} \sum_{i=j-\tau_m}^{i-1} x^T(i) Z_3 x(i) + \sum_{j=k-\tau_m}^{k-1} \sum_{i=j-\tau_m}^{i-1} x^T(i) Z_4 x(i). \]

Note that

\[ \lambda^T(k+1) = [x^T(k), x^T(k - \tau_m), x^T(k - \tau_M), \sum_{i=k-\tau_M}^{k-1} x^T(i), \sum_{i=k-\tau_M}^{k-1} \eta^T(i), \sum_{i=k-\tau_M}^{k-1} \xi^T(i), \eta(k), \xi(k) - (x(k + 1) - x(k))]. \]

Define \( \Delta V(k) = V(k+1) - V(k) \), then along the solution of system (1) we obtain

\[ E(\Delta V_1(k)) = E(X^T(k+1)QX(k+1) - X^T(k)QX(k)) = E[X^T(k)(\tilde{I}_{1}^T Q \tilde{I}_{1} - I_{\tilde{T}_1}^T Q I_{\tilde{T}_1}) \tilde{X}(k)], \]

\[ E(\Delta V_2(k)) = \frac{1}{\tau_M - \tau_m} E[\lambda^T(k) R \lambda(k)], \]

\[ -\lambda^T(k) R \lambda(k) + \sum_{i=k+1}^{k+1} \lambda^T(i) R \lambda(i) \]

\[ + \sum_{i=k+1}^{k+1} \lambda^T(i) R \lambda(i) \leq \frac{1}{\tau_M - \tau_m} E[\lambda^T(k) R \lambda(k)] - \lambda^T(k - \tau(k)) R \lambda(k - \tau(k)) \]

\[ + \sum_{i=k+1}^{k+1} \lambda^T(i) R \lambda(i) \leq \frac{1}{\tau_M - \tau_m} E[\lambda^T(k)(I_{\tilde{T}_1}^T R \tilde{I}_{1} - I_{\tilde{T}_1}^T R I_{\tilde{T}_1}) \tilde{X}(k)] + \frac{1}{\tau_M - \tau_m} E[\lambda^T(i) R \lambda(i)], \]
\[ \mathbb{E}(\Delta V_5(k)) = \frac{1}{\tau_M - \tau_m} \mathbb{E}[\sum_{j=0}^{k-1} \sum_{i=1}^{k} \lambda^T(i) R A(i)] - \sum_{j=k+1}^{k-\tau_M} R A(i) \mathbb{E}[\lambda^T(i)] + \mathbb{E}[\tilde{X}(k)^T R \tilde{A}(k)] \]

\[ = \frac{1}{\tau_M - \tau_m} \mathbb{E}[\sum_{i=k+1}^{k-\tau_M} \lambda^T(i) R A(i)] - \frac{1}{\tau_M - \tau_m} \mathbb{E}[\sum_{i=k+1}^{k-\tau_M} \lambda^T(i) R A(i)]. \]

(9)

\[ \mathbb{E}(\Delta V_4(k)) = \mathbb{E}[\xi^T(k) \gamma + \xi(k)] - \xi^T(k - \tau_M) \gamma \xi(k - \tau_M) ] [\xi(k - \tau_m) \mathbb{E}(\xi(k - \tau_m))]. \]

(10)

By lemma 2.1, we have

\[ \mathbb{E}(\Delta V_5(k)) = \mathbb{E}[\sum_{j=0}^{k-1} \sum_{i=1}^{k} \lambda^T(i) Z_1 \xi(i)] + \mathbb{E}[\sum_{j=0}^{k-1} \sum_{i=1}^{k} \lambda^T(i) \tilde{Z}_1 \xi(i)] + \mathbb{E}[\sum_{j=0}^{k-1} \sum_{i=1}^{k} \lambda^T(i) \tilde{Z}_2 \xi(i)]. \]

(11)

\[ \mathbb{E}(\Delta V_6(k)) = \mathbb{E}[\sum_{j=0}^{k-1} \sum_{i=1}^{k} \lambda^T(i) Z_2 \xi(i)] + \mathbb{E}[\sum_{j=0}^{k-1} \sum_{i=1}^{k} \lambda^T(i) \tilde{Z}_2 \xi(i)]. \]

(12)

Set \( M^T = [M^T_1, M^T_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \), \( a_1 = [C - I, I, 0, 0, 0, 0, 0, 0, 0, 0, 0] \), \( a_2 = [E_{0}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \), \( E_{0} \), \( \mathbb{E}[0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \). From lemma 2.3, we get

\[ \mathbb{E}(\Delta V_5(k)) = \frac{1}{\tau_M - \tau_m} \mathbb{E}[\sum_{j=0}^{k-1} \sum_{i=1}^{k} \lambda^T(i) R A(i)]. \]

Since \( x(k) - \sum_{j=k-1}^{k-\tau_M} (\sum_{i=1}^{k} \lambda^T(i) R A(i)) = 0 \), for arbitrary matrices \( P_1, P_2, G_1, G_2 \) of appropriate dimensions, we can obtain that

\[ 0 = 2 \mathbb{E}[\sum_{j=k-1}^{k-\tau_M} (\sum_{i=1}^{k} \lambda^T(i) R A(i)] + 2 \mathbb{E}[\sum_{j=k-1}^{k-\tau_M} (\sum_{i=1}^{k} \lambda^T(i) R A(i)]. \]

(14)

\[ 0 = 2 \mathbb{E}[\sum_{j=k-1}^{k-\tau_M} (\sum_{i=1}^{k} \lambda^T(i) R A(i)] + 2 \mathbb{E}[\sum_{j=k-1}^{k-\tau_M} (\sum_{i=1}^{k} \lambda^T(i) R A(i)]. \]

(15)

From Assumption 1, for any positive diagonal matrices \( A_1, A_2, A_3, A_4 \) of appropriate dimensions, we have

\[ 2 \mathbb{E}[\sum_{j=k-1}^{k-\tau_M} (\sum_{i=1}^{k} \lambda^T(i) R A(i)] + 2 \mathbb{E}[\sum_{j=k-1}^{k-\tau_M} (\sum_{i=1}^{k} \lambda^T(i) R A(i)]. \]

(16)

Combining (7)-(16), we get

\[ \mathbb{E}(\Delta V(k)) \leq \mathbb{E}[\mathbb{E}(\xi^T(k) \gamma + \xi(k)] - \mathbb{E}[\eta^T(k) + \xi(k)] - \mathbb{E}[\eta^T(k) + \xi(k)]. \]

(17)

From the LMI (5) holds, applying Lemma 2.2, it follows that there exists a sufficient small positive scalar \( \varepsilon > 0 \) such that

\[ \mathbb{E}(\Delta V(k)) \leq -\frac{\varepsilon}{2} \mathbb{E}[\|\xi(k)]^2. \]

(18)

On the other hand, it can easily get that

\[ \mathbb{E}(\Delta V(k)) \leq \mathbb{E}[\|\xi(k)]^2 + \alpha_2 \sum_{j=0}^{k-1} \|\xi(j)]^2. \]

(19)

Summing up both sides of (20) from 0 to \( k - 1 \) we can obtain

\[ \mathbb{E}(\Delta V(k) - \Delta V(0)) \leq \mathbb{E}[\|\xi(k)]^2 + \alpha_2 \sum_{j=0}^{k-1} \|\xi(j)]^2. \]

(21)
where $\mu_1(\theta) = \alpha_2(\theta - 1)\tau_{\Delta}^2\theta^{-\mu_M} + \mu_2(\theta) = \alpha_2(\theta - 1)\tau_M\theta^{-\mu_M} + \alpha_1(\theta - 1) - \theta$. Since $\mu_2(\theta) = -\theta < 0$, there must exist a positive $\theta_0 > 0$ such that $\mu_2(\theta_0) < 0$. Then we have

$$E(V(k)) \leq E[\mu_1(\theta_0)\sup_{j \in [-\tau,0]}\|x(j)\|^2 + \frac{1}{\theta_0}V(0)].$$

On the other hand, note $\sigma = \alpha_1 + (1 + \tau_M)\alpha_2$, we can obtain

$$E(V(0)) \leq \sigma \sup_{j \in [-\tau,0]}E[\|x(j)\|^2],$$

$$E(V(k)) \geq \lambda_{\min}(Q)E[\|x(k)\|^2].$$

It follows that $E[\|x(k)\|^2] \leq \alpha^k\sup_{j \in [-\tau,0]}E[\|x(j)\|^2]$, where $\beta = (\theta_0)^{-1}$, $\alpha = \frac{\lambda_{\min}(P)}{\lambda_{\min}(P)}$. By Definition 1, system (1) is globally robustly and exponentially stable, which complete the proof of Theorem 3.1.

**Remark 3.1:** When $\Delta C(k) = \Delta A(k) = \Delta B(k) = 0$, system (1) becomes

$$x(k + 1) = Cx(k) + Af(x(k)) + Bg(x(k - \tau(k)))$$

$$+ \delta(k, x(k), x(k - \tau(k)))u(k),$$

which studied in [8]. In this case, similar to the proofs of Theorem 3.1, we can obtain the following corollary.

**Corollary 3.1:** For any given positive integers $0 < \tau_m < \tau_M$, then, under Assumption 1.3, system (24) is globally exponentially stable in the mean square for any time-varying delay $\tau(k)$ satisfying $\tau_m \leq \tau(k) \leq \tau_M$, if there exist positive matrices $Q, R, H, \gamma, M_{51}$, positive diagonal matrices $A_1, A_2, \Gamma_1, \Gamma_2, Z_1, Z_2, Z_3$, arbitrary matrices $M_{51}, P_{1}, P_2, G_1, G_2$ with appropriate dimensions, and positive scalar $\lambda > 0$, such that the following LMIs hold:

$$\Xi = [\Xi^{(1)}, \Xi^{(2)}] > 0,$$

$$\Xi^{(1)} = \begin{pmatrix}
\Xi_{11} & \Xi_{12} & 0 & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & \Xi_{19} & \Xi_{10} \\
\Xi_{21} & \Xi_{22} & 0 & \Xi_{25} & \Xi_{26} & \Xi_{27} & \Xi_{28} & \Xi_{29} & \Xi_{20} \\
* & * & 0 & \Xi_{35} & \Xi_{36} & \Xi_{37} & \Xi_{38} & \Xi_{39} & \Xi_{30} \\
* & * & * & 0 & \Xi_{40} & \Xi_{41} & \Xi_{42} & \Xi_{43} & \Xi_{44} \\
* & * & * & * & 0 & \Xi_{50} & \Xi_{51} & \Xi_{52} & \Xi_{53} \\
* & * & * & * & * & 0 & \Xi_{60} & \Xi_{61} & \Xi_{62} \\
* & * & * & * & * & * & 0 & \Xi_{70} & \Xi_{71} \\
* & * & * & * & * & * & * & 0 & \Xi_{80} \\
* & * & * & * & * & * & * & * & \Xi_{90} \\
\end{pmatrix} > 0,$$

$$\Xi^{(2)} = \begin{pmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & \Xi_{19} & \Xi_{10} \\
\Xi_{21} & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & \Xi_{26} & \Xi_{27} & \Xi_{28} & \Xi_{29} & \Xi_{20} \\
\Xi_{31} & \Xi_{32} & \Xi_{33} & 0 & \Xi_{35} & \Xi_{36} & \Xi_{37} & \Xi_{38} & \Xi_{39} & \Xi_{30} \\
\Xi_{41} & \Xi_{42} & \Xi_{43} & 0 & \Xi_{45} & \Xi_{46} & \Xi_{47} & \Xi_{48} & \Xi_{49} & \Xi_{40} \\
\Xi_{51} & \Xi_{52} & \Xi_{53} & 0 & \Xi_{55} & \Xi_{56} & \Xi_{57} & \Xi_{58} & \Xi_{59} & \Xi_{50} \\
\Xi_{61} & \Xi_{62} & \Xi_{63} & 0 & \Xi_{65} & \Xi_{66} & \Xi_{67} & \Xi_{68} & \Xi_{69} & \Xi_{60} \\
\Xi_{71} & \Xi_{72} & \Xi_{73} & 0 & \Xi_{75} & \Xi_{76} & \Xi_{77} & \Xi_{78} & \Xi_{79} & \Xi_{70} \\
\Xi_{81} & \Xi_{82} & \Xi_{83} & 0 & \Xi_{85} & \Xi_{86} & \Xi_{87} & \Xi_{88} & \Xi_{89} & \Xi_{80} \\
\Xi_{91} & \Xi_{92} & \Xi_{93} & 0 & \Xi_{95} & \Xi_{96} & \Xi_{97} & \Xi_{98} & \Xi_{99} & \Xi_{90} \\
\end{pmatrix} > 0.$$
 cas(\tau_m)\right) R_1 + \gamma_1 + H_1 + (1 + \tau_m) Z_3 \\
+ \varepsilon E^T_k E_k + P_1 + P_2^T + G_1 + G_2^T + (C - I)^T M_1, \\
\Xi_{aa} = \frac{1}{\tau_m - \tau} R_1 - \Gamma_1 \Pi_1 - \Gamma_2 \Pi_2 - P_1 - G_2^T - G_2.
\]

**Remark 3.3:** We proposed $V_1, V_2$ which take \[
\sum_{k=-\tau_m}^{k-\tau_m} x(k), \sum_{k=-\tau_m}^{k-\tau_m} \eta(k - 1), \sum_{k=-\tau_m}^{k-\tau_m} f(x(k)), g(x(k)) \]
as augmented states. The proposed augmented Lyapunov functional $V_1, V_2$ do not considered in the previous literature and may improve the feasibility region of delay-dependent stability criterion.

**Remark 3.4:** Zero equations (14) (15) provide us a new method to introduce free-weighting matrix, which do not considered in existing works. And free-weighting matrices $P_1, P_2, G_1, G_2$ make an important role in the reducing of conservativeness of above criteria.

**IV. NUMERICAL EXAMPLES**

In this section, three numerical examples will be presented to show the validity of the main results derived above.

**Example 4.1:** For the convenience of comparison, let's consider a delayed discrete recurrent neural network in (24), where

\[
\begin{align*}
 f_1(s) &= \sin(0.2s), \quad f_2(s) = \tanh(-0.4s), \quad g_1(s) = \tanh(0.83s), \\
 g_2(s) &= \tanh(0.2s), \quad \tau_m = 1, \quad \rho_1 = \rho_2 = 0.2 \\
 C &= \begin{pmatrix} -0.1 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad A = \begin{pmatrix} -0.1 & 0.1 \\ 0 & 0.5 \end{pmatrix}, \\
 B &= \begin{pmatrix} 0.05 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}, \quad L_1 = \begin{pmatrix} -0.64 & 0 \\ 0 & 0 \end{pmatrix}, \\
 L_2 &= \begin{pmatrix} 0 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
 \Pi_2 &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0 \end{pmatrix}. \\
\end{align*}
\]

Applying the Theorem 1 in [8] and the Theorem 1 in [17], the maximum value of $\tau_M$ for globally robustly mean square exponentially stable of system (1) is $\tau_M = 10$ and $\tau_M = 55$, respectively.

**Example 4.2:** As a comparison, let's consider a delayed discrete-time recurrent neural network in (24) with parameters given by $\delta = 0$, where

\[
\begin{align*}
 C &= \begin{pmatrix} 0.8 & 0.1 \\ 0 & 0.7 \end{pmatrix}, \quad A = \begin{pmatrix} 0.01 & 0.00 \\ 0 & 0.005 \end{pmatrix}, \quad B = \begin{pmatrix} -0.1 & 0.01 \\ 0 & -0.2 \end{pmatrix}. \\
\end{align*}
\]

And the activation functions satisfy Assumption 1 with $\sigma_{a_1}^1 = \sigma_{a_2}^1 = 0, \sigma_{a_3}^1 = \sigma_3^1 = 1$. For $\tau_m = 1, 4, 8, 15, 25, \text{references} [8], [25], [18]-[21]$, gave out the allowable upper bound $\tau_M$ of the time-varying delay, respectively. Table 2 shows that our results are less conservative than these previous results.

**Example 4.3:** Consider an uncertain delayed discrete-time recurrent neural network in (27) with parameters given by

\[
\begin{align*}
 C &= \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.12 & 0.24 \\ -0.15 & 0.2 \end{pmatrix}, \\
 B &= \begin{pmatrix} -0.25 & 0.1 \\ 0.02 & 0.09 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix} \\
 E_c &= \begin{pmatrix} 0.15 & 0.1 \\ 0 & -0.7 \end{pmatrix}, \quad E_a = \begin{pmatrix} 0.1 & 0.3 \\ 0 & -0.2 \end{pmatrix}. \\
\end{align*}
\]

And the activation functions satisfy Assumption 1 with $\sigma_{a_1}^1 = \sigma_{a_2}^1 = 0, \sigma_{a_3}^1 = \sigma_3^1 = 0.5$. For $\tau_m = 1, 2, 4, 6, 8, 10, \text{references} [16], [19], [20]$ gave out the allowable upper bound $\tau_M$ of the time-varying delay, respectively. The allowable upper bounds $\tau_M$ for given $\tau_m$ are showed in Table 3. Obviously, our results are less conservative than these previous results.

**V. CONCLUSION**

Combined with linear matrix inequality (LMI) technique, a new augmented Lyapunov-Krasovskii function is constructed, and some new improved sufficient conditions ensuring globally exponential stability or robust exponential stability in the mean square are obtained. Numerical examples show that the new results are less conservative than some previous results.

**ACKNOWLEDGMENT**

This work was supported by the program for New Century Excellent Talents in University (NCET-06-8811) and the Science and technology Foundation of Guangzhou Province of China (2010J2130).  

International Scholarly and Scientific Research & Innovation 6(8) 2012  962

ISNI:0000000091950263
REFERENCES


