

# Augmented Lyapunov approach to robust stability of discrete-time stochastic neural networks with time-varying delays

Shu Lü, Shouming Zhong, Zixin Liu

**Abstract**—In this paper, the robust exponential stability problem of discrete-time uncertain stochastic neural networks with time-varying delays is investigated. By introducing a new augmented Lyapunov function, some delay-dependent stable results are obtained in terms of linear matrix inequality (LMI) technique. Compared with some existing results in the literature, the conservatism of the new criteria is reduced notably. Three numerical examples are provided to demonstrate the less conservatism and effectiveness of the proposed method.

**Keywords**—Robust exponential stability, delay-dependent stability, discrete-time neural networks, stochastic, time-varying delays.

## I. INTRODUCTION

**R**ECENTLY, recurrent neural networks (see [1]-[7]), such as Hopfield neural networks, cellular neural networks and other networks have attracted considerable attention because of their potential applications in pattern recognition, image processing, fixed-point computation, and so on. However, because of the finite switching speed of neurons and amplifiers, time delays, both constant and time-varying, are often unavoidable in various engineering, neural networks, large-scale, biological, and economic systems. Since the occurrence of time delays may cause poor performance or instability, the studies on stability for delayed neural networks are of great significance. There has been a growing research interest on the stability analysis problems for delayed neural networks, and many excellent papers and monographs have been available. On the other hand, during the design of neural network and its hardware implementation, the convergence of a neural network may often be destroyed by its unavoidable uncertainty due to the existence of modeling error, the deviation of vital data, and so on. These unavoidable uncertainty can be classified into two types: that is, stochastic disturbances and parameters uncertainties. As pointed out in [8] that, while modeling real nervous systems, both of the stochastic disturbances and parameters uncertainties are probably the main resources of the performance degradations of the implemented neural networks. Therefore, the studies on robust convergence of stochastic delayed neural network have been a hot reach direction. Up to now, many sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the global robust asymptotic or exponential stability for different class of delayed neural networks (see [9]-[17]).

S. Lü, S. Zhong and Z. Liu are with School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, 610054, China. E-mail:sangchengxuezi@yahoo.com.cn

It's worth pointing out that most neural networks have been assumed to be in continuous time, but few in discrete time. In practice, discrete-time neural networks are more applicable to problems that are inherently temporal in nature or related to biological realities. And they can ideally keep the dynamic characteristics, functional similarity, and even the physical or biological reality of the continuous-time networks under mild restriction. Thus, the stability analysis problems for discrete-time neural networks have received more and more interest, and some stability criteria have been proposed in the literature (see [8],[16]- [26]). For the first time, Liu,Wang and Liu considered the robust stability for discrete-time stochastic neural networks with time-varying delays in [8], and proposed some delay-dependent stability criteria in terms of LMI approach. By using a similar technique to that in [19], [20], the result obtained in [8] has been improved by Luo et al. [17].

In this paper, some new improved delay-dependent stability criteria are obtained via constructing a new augmented Lyapunov-Krasovskii function. These new conditions are less conservative than those obtained in [8], [16]-[21]. Furthermore, three numerical examples are also provided to illuminate the improvement of the proposed criteria.

**Notation:** The notations are used in our paper except where otherwise specified.  $\|\cdot\|$  denotes a vector or a matrix norm;  $\mathbb{R}, \mathbb{R}^n$  are real and n-dimension real number sets, respectively;  $\mathbb{N}^+$  is positive integer set.  $I$  is identity matrix;  $*$  represents the elements below the main diagonal of a symmetric block matrix; Real matrix  $P > 0 (< 0)$  denotes  $P$  is a positive definite (negative definite) matrix;  $\mathbb{N}[a, b] = \{a, a + 1, \dots, b\}$ ;  $\lambda_{min}(\lambda_{max})$  denotes the minimum (maximum) eigenvalue of a real matrix;  $(\Omega, \mathcal{R}, \mathcal{P})$  is a complete probability space with filtration  $\mathcal{P}$  satisfying the usual condition;  $\mathbb{E}(\cdot)$  stands for the mathematical expectation operator with respect to the given probability measure.

## II. PRELIMINARIES

Consider the following n-neuron discrete-time stochastic neural network (DSNN) [8] with time delays of the form:

$$\begin{aligned} x(k+1) &= C(k)x(k) + A(k)f(x(k)) + B(k)g(x(k-\tau(k))) \\ &= +\delta(k, x(k), x(k-\tau(k)))\omega(k), \quad k \in \mathbb{N}^+ \end{aligned} \quad (1)$$

where  $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^T \in \mathbb{R}^n$  denotes the neural state vector;  $f(x(k)) = [f_1(x_1(k)), f_2(x_2(k)), \dots, f_n(x_n(k))]^T$ ,  $g(x(k-\tau(k))) = [g_1(x_1(k-$

$\tau(k)), g_2(x_2(k - \tau(k))), \dots, g_n(x_n(k - \tau(k)))^T$  are the neuron activation functions; Positive integer  $\tau(k)$  represents the transmission delay that satisfies  $0 < \tau(m) \leq \tau(k) \leq \tau(M)$ , where  $\tau(m), \tau(M)$  are known positive integers representing the lower and upper bounds of the delay.  $C(k) = C + \Delta C(k)$ ,  $A(k) = A + \Delta A(k)$ ,  $B(k) = B + \Delta B(k)$ .  $C = \text{diag}(c_1, c_2, \dots, c_n)$  with  $|c_i| < 1$  describes the rate with which the  $i$ th neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs;  $C, A, B \in \mathbb{R}^{n \times n}$  represent the weighting matrices;  $\Delta C(k), \Delta A(k), \Delta B(k)$  denote the time-varying structured uncertainties which are of the following form:

$$[\Delta C(k), \Delta A(k), \Delta B(k)] = KF(k)[E_c \ E_a \ E_b],$$

where  $K, E_c, E_a, E_b$  are known real constant matrices of appropriate dimensions,  $F(k)$  is unknown time-varying matrix function satisfying  $F^T(k)F(k) \leq I, \forall k \in \mathbb{N}^+$ .  $\omega(k)$  is a scalar Wiener process (Brownian Motion) on  $(\Omega, \mathcal{R}, \mathcal{P})$  with  $E(\omega(k)) = 0, E(\omega^2(k)) = 1, E(\omega(i)\omega(j)) = 0, \forall i \neq j$ ;  $\delta$  is the continuous function.

To obtain our main results, we need introduce the following assumptions, definition and lemmas.

**Assumption 1:** For  $i \in \mathbb{N}^+, x \neq y \in \mathbb{R}$ , the neuron activation functions in DSNN (1) satisfy

$$l_i^- \leq \frac{f_i(x) - f_i(y)}{x - y} \leq l_i^+, \sigma_i^- \leq \frac{g_i(x) - g_i(y)}{x - y} \leq \sigma_i^+, \quad (2)$$

where  $l_i^-, l_i^+, \sigma_i^-, \sigma_i^+$  are known constant scalars.

**Assumption 2:** The continuous function  $\delta$  satisfies that

$$\begin{aligned} & \delta^T(k, x(k), x(k - \tau(k)))\delta(k, x(k), x(k - \tau(k))) \\ & \leq \rho_1 x^T(k)x(k) + \rho_2 x^T(k - \tau(k))x(k - \tau(k)), \end{aligned} \quad (3)$$

where  $\rho_1 > 0, \rho_2 > 0$  are known constant scalars.

**Assumption 3:**

$$f(0) = g(0) = 0, \quad (4)$$

which means that  $x(k) \equiv 0$  is a trivial solution of the DSNN (1).

**Remark 2.1:** As pointed out in [8], the constants  $l_i^-, l_i^+, \sigma_i^-, \sigma_i^+$  in Assumption 1 are allowed to be positive, negative or zero. Hence, the resulting activation functions could be nonmonotonic, and are more general than the usual sigmoid functions and the recently commonly used Lipschitz conditions.

**Definition 2.1:** The DSNN (1) is said to be robustly exponentially stable in the mean square if there exist constants  $\alpha > 0$  and  $\beta \in (0, 1)$  such that every solution of the DSNN (1) satisfies that

$$\mathbb{E}\|x(k)\|^2 \leq \alpha \cdot \beta^k \sup_{i \in \mathbb{N}[-\tau_M, 0]} \mathbb{E}\|x(i)\|^2, \forall k \geq 0,$$

for all parameter uncertainties satisfying the admissible condition.

**Lemma 2.1:** [27](Tchebychev Inequality) For any given vectors  $v_i \in \mathbb{R}^n, i = 1, 2, \dots, n$ , the following inequality holds:

$$\left[ \sum_{i=1}^n v_i \right]^T \left[ \sum_{i=1}^n v_i \right] \leq n \sum_{i=1}^n v_i^T v_i.$$

**Lemma 2.2:** [28] Given constant symmetric matrices  $\Sigma_1, \Sigma_2, \Sigma_3$  where  $\Sigma_1^T = \Sigma_1$  and  $0 < \Sigma_2 = \Sigma_2^T$ , then  $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$  if and only if

$$\begin{pmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{pmatrix} < 0 \text{ or } \begin{pmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{pmatrix} < 0.$$

**Lemma 2.3:** [8] Let  $N$  and  $E$  be real constant matrices with appropriate dimensions, matrix  $F(k)$  satisfying  $F^T(k)F(k) \leq I$ , then, for any  $\epsilon > 0$ ,  $EF(k)N + N^T F^T(k)E^T \leq \epsilon^{-1}EE^T + \epsilon NN^T$ .

### III. MAIN RESULTS

**Theorem 3.1:** For any given positive integers  $0 < \tau_m < \tau_M$ , then, under Assumption 1-3, system (1) is globally robustly and exponentially stable in the mean square for any time-varying delay  $\tau(k)$  satisfying  $\tau_m \leq \tau(k) \leq \tau_M$ , if there exist positive matrices  $Q, R, H, \gamma, M_{51}$ , positive diagonal matrices  $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, Z_1, Z_2, Z_3, Z_4$ , arbitrary matrices  $M_{11}, P_1, P_2, G_1, G_2$  with appropriate dimensions, and two positive scalars  $\lambda^* > 0, \epsilon > 0$  such that the following LMIs hold:

$$\Xi = [\Xi^{(1)}, \Xi^{(2)}] < 0, \quad (5)$$

where

$$\Xi^{(1)} = \begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & \Xi_{15} & \Xi_{16} & \Xi_{17} & 0 & \Xi_{19} & \Xi_{1,10} \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & \Xi_{26} & \Xi_{27} & 0 & \Xi_{29} & \Xi_{2,10} \\ * & * & \Xi_{33} & 0 & \Xi_{35} & \Xi_{36} & \Xi_{37} & 0 & \Xi_{39} & \Xi_{3,10} \\ * & * & * & \Xi_{44} & 0 & 0 & 0 & \Xi_{48} & 0 & 0 \\ * & * & * & * & \Xi_{55} & \Xi_{56} & \Xi_{57} & 0 & \Xi_{59} & \Xi_{5,10} \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} & 0 & \Xi_{69} & \Xi_{6,10} \\ * & * & * & * & * & * & \Xi_{77} & 0 & \Xi_{79} & \Xi_{7,10} \\ * & * & * & * & * & * & * & \Xi_{88} & \Xi_{89} & \Xi_{8,10} \\ * & * & * & * & * & * & * & * & \Xi_{99} & \Xi_{9,10} \\ * & * & * & * & * & * & * & * & * & \Xi_{10,10} \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \end{pmatrix},$$

$$\Xi^{(2)} = \begin{pmatrix} \Xi_{1,11} & \Xi_{1,12} & \Xi_{1,13} & 0 & \Xi_{1,15} & 0 & \Xi_{1,17} & M_{11}K \\ \Xi_{2,11} & \Xi_{2,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi_{3,11} & \Xi_{3,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_{4,14} & 0 & \Xi_{4,16} & \Xi_{4,17} & 0 \\ \Xi_{5,11} & \Xi_{5,12} & \Xi_{5,13} & 0 & \Xi_{5,15} & 0 & \Xi_{5,17} & M_{51}K \\ \Xi_{6,11} & \Xi_{6,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi_{7,11} & \Xi_{7,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi_{8,11} & \Xi_{8,12} & 0 & \Xi_{8,14} & 0 & \Xi_{8,16} & 0 & 0 \\ \Xi_{9,11} & \Xi_{9,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi_{10,11} & \Xi_{10,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi_{11,11} & \Xi_{11,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Xi_{12,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Xi_{13,13} & 0 & \Xi_{13,15} & 0 & 0 & 0 \\ * & * & * & \Xi_{14,14} & 0 & \Xi_{14,16} & 0 & 0 \\ * & * & * & * & \Xi_{15,15} & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{16,16} & 0 & 0 \\ * & * & * & * & * & * & \Xi_{17,17} & \epsilon I \end{pmatrix},$$

$$M_{51} < \lambda^* I,$$

$$\Xi_{11} = Q_{14} + Q_{14}^T + Q_{15} + Q_{15}^T + Q_{45} + Q_{45}^T$$

$$\begin{aligned}
 & + + Q_{44} + Q_{55} \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{11} + \gamma_{11} \\
 & + H_{11} + (1 + \tau_m) Z_3 + (1 + \tau_M) Z_4 - \Lambda_1 L_1 - \Lambda_2 L_2 \\
 & + M_{11}(C - I) + (C - I)^T M_{11}^T + \varepsilon E_c^T E_c \\
 & + 2\lambda^* \rho_1 I + P_1 + P_1^T + G_1 + G_1^T, \\
 \Xi_{12} & = Q_{24}^T + Q_{25}^T - Q_{14} - Q_{44} - Q_{45}^T, \\
 \Xi_{13} & = Q_{34}^T + Q_{35}^T - Q_{15} - Q_{45} - Q_{55}, \\
 \Xi_{14} & = P_2^T - P_1 - G_1 + G_2^T, \\
 \Xi_{15} & = Q_{11} + Q_{14}^T + Q_{15}^T + Q_{14} + Q_{44} \\
 & + Q_{45}^T + Q_{15} + Q_{45} + Q_{55} + Q_{16} + Q_{46} + Q_{56} \\
 & + P_1^T + Q_{17} + Q_{47} + Q_{57} + \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{12} \\
 & + \gamma_{12} + H_{12} - M_{11} + (C - I)^T M_{51}^T, \\
 \Xi_{16} & = Q_{12} - Q_{16} + Q_{24}^T - Q_{46} + Q_{25}^T - Q_{56}, \\
 \Xi_{17} & = Q_{13} - Q_{17} + Q_{34}^T - Q_{47} + Q_{35}^T - Q_{57}, \\
 \Xi_{19} & = Q_{44} + Q_{45}^T, \quad \Xi_{1,10} = Q_{45} + Q_{55}, \\
 \Xi_{1,11} & = Q_{46} + Q_{56}, \quad \Xi_{1,12} = Q_{47} + Q_{57}, \\
 \Xi_{1,13} & = \Lambda_2 L_2 + \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{13} + M_{11} A + \varepsilon E_c^T E_a, \\
 \Xi_{1,15} & = \Lambda_1 L_2 + \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{14} + M_{11} B + \varepsilon E_c^T E_b, \\
 \Xi_{1,17} & = -P_1 - G_1 + P_2^T + G_2^T, \\
 \Xi_{22} & = -Q_{24}^T - Q_{24} + Q_{44} - \gamma_{11}, \\
 \Xi_{23} & = -Q_{34}^T - Q_{25} + Q_{45}, \\
 \Xi_{25} & = Q_{21} - Q_{14}^T + Q_{24} - Q_{44} + Q_{25} \\
 & - Q_{45} + Q_{26} - Q_{46} + Q_{27} - Q_{47}, \\
 \Xi_{26} & = Q_{22} - Q_{24}^T + Q_{46} - Q_{26} - \gamma_{12}, \\
 \Xi_{27} & = Q_{23} - Q_{43} + Q_{47} - Q_{27}, \\
 \Xi_{29} & = -Q_{44}, \Xi_{2,10} = -Q_{45}, \Xi_{2,11} = -Q_{46}, \\
 \Xi_{2,12} & = -Q_{47}, \Xi_{33} = -Q_{35}^T - Q_{35} + Q_{55} - H_{11}, \\
 \Xi_{35} & = Q_{13}^T - Q_{15}^T + Q_{34} - Q_{45}^T + Q_{35} \\
 & - Q_{55} + Q_{36} - Q_{56} + Q_{37} - Q_{57}, \\
 \Xi_{36} & = Q_{23}^T - Q_{25}^T + Q_{56} - Q_{36}, \\
 \Xi_{37} & = Q_{33} - Q_{35}^T + Q_{57} - Q_{37} - H_{12}, \\
 \Xi_{39} & = -Q_{54}, \Xi_{3,10} = -Q_{55}, \\
 \Xi_{3,11} & = -Q_{56}, \Xi_{3,12} = -Q_{57}, \\
 \Xi_{44} & = -\frac{1}{\tau_M - \tau_m} R_{11} - \Gamma_1 \Pi_1 - \Gamma_2 \Pi_1 \\
 & + 2\lambda^* \rho_2 I - P_2^T - P_2 - G_2^T - G_2, \\
 \Xi_{45} & = -P_1^T, \quad \Xi_{48} = -\frac{1}{\tau_M - \tau_m} R_{12},
 \end{aligned}$$

$$\begin{aligned}
 \Xi_{4,14} & = -\frac{1}{\tau_M - \tau_m} R_{13} + \Gamma_2 \Pi_2, \\
 \Xi_{4,16} & = -\frac{1}{\tau_M - \tau_m} R_{14} + \Gamma_1 \Pi_2, \\
 \Xi_{4,17} & = -P_2 - P_2^T - H_2, \\
 \Xi_{55} & = Q_{11} + Q_{14}^T + Q_{15}^T + Q_{16}^T + Q_{17}^T + Q_{14} + Q_{44} \\
 & + Q_{45}^T + Q_{46}^T + Q_{47}^T + Q_{15} + Q_{45} + Q_{55} + Q_{65} + Q_{75} \\
 & + Q_{16} + Q_{46} + Q_{56} + Q_{66} + Q_{76} + Q_{17} + Q_{47} + Q_{57} \\
 & + Q_{67} + Q_{77} + \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{22} + \gamma_{22} \\
 & + H_{22} + \tau_m Z_1 + \tau_M Z_2 - M_{51} - M_{51}^T, \\
 \Xi_{56} & = Q_{12} + Q_{24}^T + Q_{25}^T + Q_{26}^T + Q_{27}^T \\
 & - Q_{16} - Q_{46} - Q_{56} - Q_{66} - Q_{76}, \\
 \Xi_{57} & = Q_{13} + Q_{34}^T + Q_{35}^T + Q_{36}^T + Q_{37}^T \\
 & - Q_{17} - Q_{47} - Q_{57} - Q_{67} - Q_{77}, \\
 \Xi_{59} & = Q_{14} + Q_{44} + Q_{45}^T + Q_{46}^T + Q_{47}^T, \\
 \Xi_{5,10} & = Q_{16} + Q_{46} + Q_{56} + Q_{66} + Q_{76}, \\
 \Xi_{5,11} & = Q_{17} + Q_{47} + Q_{57} + Q_{67} + Q_{77}, \\
 \Xi_{5,12} & = Q_{16} + Q_{46} + Q_{56} + Q_{66} + Q_{76}, \\
 \Xi_{5,13} & = \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{23} + M_{51} A, \\
 \Xi_{5,15} & = \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{24} + M_{51} B, \Xi_{5,17} = -P_1, \\
 \Xi_{66} & = Q_{22} - Q_{26}^T - Q_{26} + Q_{66} - \gamma_{22}, \\
 \Xi_{67} & = Q_{23} - Q_{36}^T - Q_{27} + Q_{67}, \\
 \Xi_{69} & = Q_{24} - Q_{46}^T, \Xi_{6,10} = Q_{25} - Q_{65}, \\
 \Xi_{6,11} & = Q_{26} - Q_{66}, \Xi_{6,12} = Q_{27} - Q_{67}, \\
 \Xi_{77} & = Q_{33} - Q_{37}^T - Q_{37} + Q_{77} - H_{22}, \\
 \Xi_{79} & = Q_{34} - Q_{47}^T, \Xi_{7,10} = Q_{35} - Q_{75}, \\
 \Xi_{7,11} & = Q_{36} - Q_{76}, \Xi_{7,12} = Q_{37} - Q_{77}, \\
 \Xi_{88} & = -\frac{1}{\tau_M - \tau_m} R_{22}, \Xi_{8,14} = -\frac{1}{\tau_M - \tau_m} R_{23}, \\
 \Xi_{8,16} & = -\frac{1}{\tau_M - \tau_m} R_{24}, \Xi_{9,9} = -\frac{Z_3}{(1 + \tau_m)} \\
 \Xi_{10,10} & = -\frac{Z_4}{(1 + \tau_M)}, \quad \Xi_{11,11} = -\frac{Z_2}{\tau_m}, \quad \Xi_{12,12} = -\frac{Z_1}{\tau_M}, \\
 \Xi_{13,13} & = \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{33} - \Lambda_2 + \varepsilon E_a^T E_a, \\
 \Xi_{13,15} & = \left(1 + \frac{1}{\tau_M - \tau_m}\right) R_{34} + \varepsilon E_a^T E_b, \\
 \Xi_{14,14} & = -\frac{1}{\tau_M - \tau_m} R_{33} - \Gamma_2, \\
 \Xi_{14,16} & = -\frac{1}{\tau_M - \tau_m} R_{34},
 \end{aligned}$$

$$\Xi_{15,15} = (1 + \frac{1}{\tau_M - \tau_m})R_{44} - \Lambda_1 + \varepsilon E_b^T E_b,$$

$$\Xi_{16,16} = -\frac{1}{\tau_M - \tau_m}R_{44} - \Gamma_1,$$

$$\Xi_{17,17} = -P_2 - P_2^T - H_2 - H_2^T,$$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ * & H_{22} \end{pmatrix} > 0,$$

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{17} \\ * & Q_{22} & \cdots & Q_{27} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & Q_{77} \end{pmatrix} > 0,$$

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ * & R_{22} & R_{23} & R_{24} \\ * & * & R_{33} & R_{34} \\ * & * & * & R_{44} \end{pmatrix} > 0,$$

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ * & \gamma_{22} \end{pmatrix} > 0,$$

$$L_1 = \text{diag}(l_1^+ l_1^-, \dots, l_n^+ l_n^-),$$

$$L_2 = \text{diag}(\frac{l_1^+ + l_1^-}{2}, \dots, \frac{l_n^+ + l_n^-}{2}),$$

$$\Pi_1 = \text{diag}(\sigma_1^+ \sigma_1^-, \dots, \sigma_n^+ \sigma_n^-),$$

$$\Pi_2 = \text{diag}(\frac{\sigma_1^+ + \sigma_1^-}{2}, \dots, \frac{\sigma_n^+ + \sigma_n^-}{2}).$$

**Proof.** Constructing a new augmented Lyapunov-Krasovskii functional candidate as follows:

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) + V_6(k),$$

where

$$V_1(k) = X^T(k)QX(k),$$

$$X^T(k) = [x^T(k), x^T(k - \tau_m), x^T(k - \tau_M), \sum_{i=k-\tau_m}^k x^T(i), \sum_{i=k-\tau_M}^k x^T(i), \sum_{i=k-\tau_m}^{k-1} \eta^T(i), \sum_{i=k-\tau_M}^{k-1} \eta^T(i)], \eta(k) = x(k+1) - x(k),$$

$$V_2(k) = \frac{1}{\tau_M - \tau_m} \sum_{i=k-\tau(k)}^{k-1} \lambda^T(i)R\lambda(i),$$

$$V_3(k) = \frac{1}{\tau_M - \tau_m} \sum_{j=k+1-\tau_M}^{k-\tau_m} \sum_{i=j}^{k-1} \lambda^T(i)R\lambda(i),$$

$$V_4(k) = \sum_{i=k-\tau_m}^{k-1} \xi^T(i)H\xi(i) + \sum_{i=k-\tau_M}^{k-1} \xi^T(i)\gamma\xi(i),$$

$$\text{where } \lambda(k) = \begin{bmatrix} x(k) \\ \eta(k) \\ f(x(k)) \\ g(x(k)) \end{bmatrix}, \xi(i) = \begin{bmatrix} x(k) \\ \eta(k) \end{bmatrix},$$

$$V_5(k) = \sum_{j=-\tau_M}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i)Z_1\eta(i) + \sum_{j=-\tau_m}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i)Z_2\eta(i),$$

$$V_6(k) = \sum_{j=k-\tau_m}^k \sum_{i=j}^{k-1} x^T(i)Z_3x(i) + \sum_{j=k-\tau_M}^k \sum_{i=j}^{k-1} x^T(i)Z_4x(i).$$

Note that  $X^T(k+1) = [x^T(k) + \eta^T(k), x^T(k - \tau_m) + \eta^T(k - \tau_m), x^T(k - \tau_M) + \eta^T(k - \tau_M), x^T(k) + \eta^T(k) + \sum_{i=k-\tau_m}^k x^T(i) - x^T(k - \tau_m), x^T(k) + \eta^T(k) + \sum_{i=k-\tau_M}^k x^T(i) - x^T(k - \tau_M), \eta^T(k) + \sum_{i=k-\tau_m}^{k-1} \eta^T(i) - \eta^T(k - \tau_m), \eta^T(k) + \sum_{i=k-\tau_M}^{k-1} \eta^T(i) - \eta^T(k - \tau_M)]$ , and set  $\tilde{X}^T(k) = [x^T(k), x^T(k - \tau_m), x^T(k - \tau_M), x^T(k - \tau(k)), \eta^T(k), \eta^T(k - \tau_m), \eta^T(k - \tau_M), \eta^T(k - \tau(k)), \sum_{i=k-\tau_m}^k x^T(i), \sum_{i=k-\tau_M}^k x^T(i), \sum_{i=k-\tau_m}^{k-1} \eta^T(i), \sum_{i=k-\tau_M}^{k-1} \eta^T(i), f^T(x(k)), f^T(x(k - \tau(k))), g^T(x(k)), g^T(x(k - \tau(k)))]$ ,

$$I_1^T = \begin{pmatrix} I & 0 & 0 & I & I & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & I & I & I & I \\ 0 & I & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & I & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{I}_2^T = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{I}_3^T = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{I}_4^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Define  $\Delta V(k) = V(k+1) - V(k)$ , then along the solution of system (1) we obtain

$$\mathbb{E}(\Delta V_1(k)) = \mathbb{E}(X^T(k+1)QX(k+1) - X^T(k)QX(k)) = \mathbb{E}[\tilde{X}^T(k)(\tilde{I}_1^T Q \tilde{I}_1 - \tilde{I}_2^T Q \tilde{I}_2)\tilde{X}(k)], \tag{7}$$

$$\mathbb{E}(\Delta V_2(k)) = \frac{1}{\tau_M - \tau_m} \mathbb{E}[\lambda^T(k)R\lambda(k) - \lambda^T(k - \tau(k))R\lambda(k - \tau(k)) + \sum_{i=k+1-\tau(k+1)}^{k-\tau_m} \lambda^T(i)R\lambda(i) + \sum_{i=k+1-\tau_m}^{k-1} \lambda^T(i)R\lambda(i) - \sum_{i=k+1-\tau(k)}^{k-1} \lambda^T(i)R\lambda(i)] \leq \frac{1}{\tau_M - \tau_m} \mathbb{E}[\lambda^T(k)R\lambda(k) - \lambda^T(k - \tau(k))R\lambda(k - \tau(k)) + \sum_{i=k+1-\tau_m}^{k-\tau_m} \lambda^T(i)R\lambda(i)] \leq \frac{1}{\tau_M - \tau_m} \mathbb{E}[\tilde{X}^T(k)(\tilde{I}_3^T R \tilde{I}_3 - \tilde{I}_4^T R \tilde{I}_4)\tilde{X}(k)] + \frac{1}{\tau_M - \tau_m} \mathbb{E}[\sum_{i=k+1-\tau_M}^{k-\tau_m} \lambda^T(i)R\lambda(i)], \tag{8}$$

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$$\begin{aligned} \mathbb{E}(\Delta V_3(k)) &= \frac{1}{\tau_M - \tau_m} \mathbb{E} \left[ \sum_{j=k+2-\tau_M}^{k+1-\tau_m} \sum_{i=j}^k \lambda^T(i) R \lambda(i) \right. \\ &\quad \left. - \sum_{j=k+1-\tau_M}^{k-\tau_m} \sum_{i=j}^{k-1} \lambda^T(i) R \lambda(i) \right] \\ &= \mathbb{E}[\tilde{X}^T(k) \tilde{I}_3^T R \tilde{I}_3 \tilde{X}(k)] \\ &\quad - \frac{1}{\tau_M - \tau_m} \mathbb{E} \left[ \sum_{i=k+1-\tau_M}^{k-\tau_m} \lambda^T(i) R \lambda(i) \right], \end{aligned} \quad (9)$$

$$\begin{aligned} \mathbb{E}(\Delta V_4(k)) &= \mathbb{E}[\xi^T(k)(\gamma + H)\xi(k) \\ &\quad - \xi^T(k - \tau_M)\gamma\xi(k - \tau_M - \xi^T(k - \tau_M)H\xi(k - \tau_M))] \end{aligned} \quad (10)$$

By lemma 2.1, we have

$$\begin{aligned} \mathbb{E}(\Delta V_5(k)) &= \mathbb{E}[\tau_M \eta^T(k) Z_1 \eta(k) - \sum_{i=k-\tau_M}^{k-1} \eta^T(i) Z_1 \eta(i) \\ &\quad + \tau_m \eta^T(k) Z_2 \eta(k) - \sum_{i=k-\tau_m}^{k-1} \eta^T(i) Z_2 \eta(i)] \\ &= \mathbb{E}[\tau_M \eta^T(k) Z_1 \eta(k) + \tau_m \eta^T(k) Z_2 \eta(k) \\ &\quad - \sum_{i=k-\tau_M}^{k-1} (\sqrt{Z_1} \eta(i))^T \sqrt{Z_1} \eta(i) \\ &\quad - \sum_{i=k-\tau_m}^{k-1} (\sqrt{Z_2} \eta(i))^T \sqrt{Z_2} \eta(i)] \\ &\leq \mathbb{E}[\tau_M \eta^T(k) Z_1 \eta(k) + \tau_m \eta^T(k) Z_2 \eta(k) \\ &\quad - \frac{1}{\tau_M} \left( \sum_{i=k-\tau_M}^{k-1} \eta(i) \right)^T Z_1 \left( \sum_{i=k-\tau_M}^{k-1} \eta(i) \right) \\ &\quad - \frac{1}{\tau_m} \left( \sum_{i=k-\tau_m}^{k-1} \eta(i) \right)^T Z_2 \left( \sum_{i=k-\tau_m}^{k-1} \eta(i) \right)], \end{aligned} \quad (11)$$

$$\begin{aligned} \mathbb{E}(\Delta V_6(k)) &= \mathbb{E}[(1 + \tau_m) x^T(k) Z_3 x(k) - \sum_{i=k-\tau_m}^k x^T(i) Z_3 x(i) \\ &\quad + (1 + \tau_M) x^T(k) Z_4 x(k) - \sum_{i=k-\tau_M}^k x^T(i) Z_4 x(i)] \\ &\leq \mathbb{E}\{ (1 + \tau_m) x^T(k) Z_3 x(k) \\ &\quad - \frac{1}{1 + \tau_m} \left[ \sum_{i=k-\tau_m}^k x(i) \right]^T Z_3 \left[ \sum_{i=k-\tau_m}^k x(i) \right] \\ &\quad + (1 + \tau_M) x^T(k) Z_4 x(k) \\ &\quad - \frac{1}{1 + \tau_M} \left[ \sum_{i=k-\tau_M}^k x(i) \right]^T Z_4 \left[ \sum_{i=k-\tau_M}^k x(i) \right] \}. \end{aligned} \quad (12)$$

Set  $M^T = [M_{11}^T, 0, 0, 0, M_{51}^T, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ ,  $\alpha_1 = [C - I, 0, 0, 0, -I, 0, 0, 0, 0, 0, 0, 0, 0, A, 0, B, 0]$ ,  $\alpha_2 = [E_a, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, E_a, 0, E_b, 0]^T$ . From lemma 2.3, we get

$$\begin{aligned} 0 &= \mathbb{E}\{2\tilde{X}^T(k) M [(C(k) - I)x(k) + A(k)f(x(k)) \\ &\quad + B(k)g(x(k - \tau(k))) + \delta(k, x(k), x(k - \tau(k)))\omega(k) - \eta(k)]\} \\ &\leq \mathbb{E}[\tilde{X}^T(k) (M\alpha_1 + \alpha_1^T M^T + \varepsilon\alpha_2\alpha_2^T + \frac{1}{\varepsilon} M K K^T M^T) \tilde{X}(k)] \\ &\quad + \mathbb{E}[2\tilde{X}^T(k) M \delta(k, x(k), x(k - \tau(k)))\omega(k)] \\ &\leq \mathbb{E}[\tilde{X}^T(k) (M\alpha_1 + \alpha_1^T M^T + \varepsilon\alpha_2\alpha_2^T + \frac{1}{\varepsilon} M K K^T M^T) \tilde{X}(k)] \\ &\quad + 2\lambda^* [\rho_1 \mathbb{E}(x^T(k)x(k)) + \rho_2 \mathbb{E}(x^T(k - \tau(k))x(k - \tau(k)))]. \end{aligned} \quad (13)$$

Since  $x(k) - \sum_{i=k-\tau(k)}^{k-1} \eta(i) - x(k - \tau(k)) = 0$ , for arbitrary matrices  $P_1, P_2, G_1, G_2$  of appropriate dimensions, we can obtain that

$$\begin{aligned} 0 &= 2\mathbb{E}\{[\eta^T(k) + x^T(k), \sum_{i=k-\tau(k)}^{k-1} \eta^T(i) + x^T(k - \tau(k))] \\ &\quad \begin{bmatrix} 0 & P_1 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \eta(k) + x(k) \\ x(k) - \sum_{i=k-\tau(k)}^{k-1} \eta(i) - x(k - \tau(k)) \end{bmatrix}\}, \end{aligned} \quad (14)$$

$$\begin{aligned} 0 &= 2\mathbb{E}\{[x^T(k), \sum_{i=k-\tau(k)}^{k-1} \eta^T(i) + x^T(k - \tau(k))] \\ &\quad \begin{bmatrix} 0 & -G_1 \\ 0 & -G_2 \end{bmatrix} \begin{bmatrix} \eta(k) + x(k) \\ -x(k) + \sum_{i=k-\tau(k)}^{k-1} \eta(i) + x(k - \tau(k)) \end{bmatrix}\}. \end{aligned} \quad (15)$$

From Assumption 1, for any positive diagonal matrices  $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2$  of appropriate dimensions, we have

$$\begin{aligned} 2x^T(k)\Lambda_1 L_2 f(x(k)) - x^T(k)\Lambda_1 L_1 x(k) - f^T(x(k))\Lambda_1 f(x(k)) &\geq 0, \\ -x^T(k - \tau(k))\Lambda_2 L_1 x(k - \tau(k)) - f^T(x(k - \tau(k))) \times \\ \Lambda_2 f(x(k - \tau(k))) + 2x^T(k - \tau(k))\Lambda_2 L_2 f(x(k - \tau(k))) &\geq 0, \\ 2x^T(k)\Gamma_1 \Pi_2 g(x(k)) - x^T(k)\Gamma_1 \Pi_1 x(k) - g^T(x(k))\Gamma_1 g(x(k)) &\geq 0, \\ -x^T(k - \tau(k))\Gamma_2 \Pi_1 x(k - \tau(k)) + 2x^T(k - \tau(k))\Gamma_2 \times \\ \Pi_2 g(x(k - \tau(k))) - g^T(x(k - \tau(k)))\Gamma_2 g(x(k - \tau(k))) &\geq 0, \end{aligned} \quad (16)$$

Combining (7)-(16), we get

$$\mathbb{E}(\Delta V(k)) \leq \mathbb{E}\{\hat{X}^T(k) [\Xi' + \frac{1}{\varepsilon} M K K^T M^T] \hat{X}(k)\}, \quad (17)$$

where  $\hat{X}^T(k) = [\tilde{X}^T(k), \sum_{i=k-\tau(k)}^{k-1} \eta^T(i)]$ ,  $\Xi' \in \mathbb{R}^{17n \times 17n}$  and the elements are the same with the foreword seventeen rows and columns in matrix  $\Xi$ .

If the LMI (5) holds, applying Lemma 2.2, it follows that there exists a sufficient small positive scalar  $\varepsilon > 0$  such that

$$\mathbb{E}(\Delta V(k)) \leq -\varepsilon \mathbb{E}\|x(k)\|^2. \quad (18)$$

On the other hand, it can easily to get that

$$\mathbb{E}(V(k)) \leq \mathbb{E}(\alpha_1 \|x(k)\|^2 + \alpha_2 \sum_{i=k-\tau_M}^{k-1} \|x(i)\|^2), \quad (19)$$

where  $\alpha_1 = \alpha_2 + 7\lambda_{max}(Q)$ ,  $\alpha_2 = 4\lambda_{max}(R)(5 + \|L\|^2 + \|\Pi\|^2)(\frac{1}{\tau_M - \tau_m} + 1) + 10(\lambda_{max}(H) + \lambda_{max}(\gamma)) + 2\tau_M \lambda_{max}(Z_1) + 2\tau_m \lambda_{max}(Z_2) + (1 + \tau_M)\lambda_{max}(Z_4) + (1 + \tau_m)\lambda_{max}(Z_3)$ ,  $\|L\|^2 = \max\{\|L_1\|^2, \|L_2\|^2\}$ ,  $\|\Pi\|^2 = \max\{\|\Pi_1\|^2, \|\Pi_2\|^2\}$ .

For any  $\theta > 1$ , it follows from (19) that

$$\begin{aligned} &\theta^{j+1} \mathbb{E}(V(j+1) - \theta^j V(j)) \\ &= \theta^{j+1} \Delta \mathbb{E}(V(j)) + \theta^j (\theta - 1) \mathbb{E}(V(j)) \\ &\leq \mathbb{E}[\theta^j (-\varepsilon \theta \|x(j)\|^2 + (\theta - 1) \alpha_1 \|x(j)\|^2 \\ &\quad + (\theta - 1) \alpha_2 \sum_{i=j-\tau_M}^{j-1} \|x(i)\|^2)]. \end{aligned} \quad (20)$$

Summing up both sides of (20) from 0 to  $k - 1$  we can obtain

$$\begin{aligned} \mathbb{E}(\theta^k V(k) - V(0)) &\leq \mathbb{E}\{[\alpha_1 (\theta - 1) - \varepsilon \theta] \sum_{j=0}^{k-1} \theta^j \|x(j)\|^2 \\ &\quad + \alpha_2 (\theta - 1) \sum_{j=0}^{k-1} \sum_{i=j-\tau_M}^{j-1} \theta^j \|x(i)\|^2\} \\ &\leq \mathbb{E}\{\mu_1(\theta) \sup_{j \in \mathbb{N}[-\tau_M, 0]} \|x(j)\|^2 \\ &\quad + \mu_2(\theta) \sum_{j=0}^k \theta^k \|x(k)\|^2\}, \end{aligned} \quad (21)$$

where  $\mu_1(\theta) = \alpha_2(\theta - 1)\tau_M^2\theta^{\tau_M}$ ,  $\mu_2(\theta) = \alpha_2(\theta - 1)\tau_M\theta^{\tau_M} + \alpha_1(\theta - 1) - \varepsilon\theta$ . Since  $\mu_2(1) = -\varepsilon < 0$ , there must exist a positive  $\theta_0 > 1$  such that  $\mu_2(\theta_0) < 0$ . Then we have

$$\mathbb{E}(V(k)) \leq \mathbb{E}[\mu_1(\theta_0)\left(\frac{1}{\theta_0}\right)^k \sup_{j \in \mathbb{N}[-\tau_M, 0]} \|x(j)\|^2 + \left(\frac{1}{\theta_0}\right)^k V(0)], \quad (22)$$

On the other hand, note  $\varpi = \alpha_1 + (1 + \tau_M)\alpha_2$ , we can obtain

$$\mathbb{E}(V(0)) \leq \varpi \sup_{j \in \mathbb{N}[-\tau_M, 0]} \mathbb{E}\|x(j)\|^2, \quad \mathbb{E}(V(k)) \geq \lambda_{\min}(Q)\mathbb{E}\|x(k)\|^2. \quad (23)$$

It follows that  $\mathbb{E}\|x(k)\|^2 \leq \alpha \cdot \beta^k \sup_{j \in \mathbb{N}[-\tau_M, 0]} \mathbb{E}\|x(j)\|^2$ , where  $\beta = (\theta_0)^{-1}$ ,  $\alpha = \frac{\mu_1(\theta_0) + \varpi}{\lambda_{\min}(P)}$ . By Definition 1, system (1) is globally robustly and exponentially stable, which complete the proof of Theorem 3.1.

*Remark 3.1:* When  $\Delta C(k) = \Delta A(k) = \Delta B(k) = 0$ , system (1) becomes

$$x(k+1) = Cx(k) + Af(x(k)) + Bg(x(k - \tau(k))) + \delta(k, x(k), x(k - \tau(k)))\omega(k), \quad k \in \mathbb{N}^+, \quad (24)$$

which studied in [8]. In this case, similar to the proofs of Theorem 3.1, we can obtain the following corollary.

*Corollary 3.1:* For any given positive integers  $0 < \tau_m < \tau_M$ , then, under Assumption 1-3, system (24) is globally exponentially stable in the mean square for any time-varying delay  $\tau(k)$  satisfying  $\tau_m \leq \tau(k) \leq \tau_M$ , if there exist positive matrices  $Q, R, H, \gamma, M_{51}$ , positive diagonal matrices  $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, Z_1, Z_2, Z_3, Z_4$ , arbitrary matrices  $M_{11}, P_1, P_2, G_1, G_2$  with appropriate dimensions, and positive scalar  $\lambda^* > 0$ , such that the following LMIs hold:

$$\hat{\Xi} = [\hat{\Xi}^{(1)}, \hat{\Xi}^{(2)}] < 0, \quad (25)$$

where

$$\hat{\Xi}^{(1)} = \begin{pmatrix} \hat{\Xi}_{11} & \hat{\Xi}_{12} & \hat{\Xi}_{13} & 0 & \hat{\Xi}_{15} & \hat{\Xi}_{16} & \hat{\Xi}_{17} & 0 & \hat{\Xi}_{19} & \hat{\Xi}_{1,10} \\ * & \hat{\Xi}_{22} & \hat{\Xi}_{23} & 0 & \hat{\Xi}_{25} & \hat{\Xi}_{26} & \hat{\Xi}_{27} & 0 & \hat{\Xi}_{29} & \hat{\Xi}_{2,10} \\ * & * & \hat{\Xi}_{33} & 0 & \hat{\Xi}_{35} & \hat{\Xi}_{36} & \hat{\Xi}_{37} & 0 & \hat{\Xi}_{39} & \hat{\Xi}_{3,10} \\ * & * & * & \hat{\Xi}_{44} & 0 & 0 & 0 & \hat{\Xi}_{48} & 0 & 0 \\ * & * & * & * & \hat{\Xi}_{55} & \hat{\Xi}_{56} & \hat{\Xi}_{57} & 0 & \hat{\Xi}_{59} & \hat{\Xi}_{5,10} \\ * & * & * & * & * & \hat{\Xi}_{66} & \hat{\Xi}_{67} & 0 & \hat{\Xi}_{69} & \hat{\Xi}_{6,10} \\ * & * & * & * & * & * & \hat{\Xi}_{77} & 0 & \hat{\Xi}_{79} & \hat{\Xi}_{7,10} \\ * & * & * & * & * & * & * & \hat{\Xi}_{88} & \hat{\Xi}_{89} & \hat{\Xi}_{8,10} \\ * & * & * & * & * & * & * & * & \hat{\Xi}_{99} & \hat{\Xi}_{9,10} \\ * & * & * & * & * & * & * & * & * & \hat{\Xi}_{10,10} \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \end{pmatrix},$$

$$\hat{\Xi}^{(2)} = \begin{pmatrix} \hat{\Xi}_{1,11} & \hat{\Xi}_{1,12} & \hat{\Xi}_{1,13} & 0 & \hat{\Xi}_{1,15} & 0 & \hat{\Xi}_{1,17} \\ \hat{\Xi}_{2,11} & \hat{\Xi}_{2,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{3,11} & \hat{\Xi}_{3,12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{\Xi}_{4,14} & 0 & \hat{\Xi}_{4,16} & \hat{\Xi}_{4,17} \\ \hat{\Xi}_{5,11} & \hat{\Xi}_{5,12} & \hat{\Xi}_{5,13} & 0 & \hat{\Xi}_{5,15} & 0 & \hat{\Xi}_{5,17} \\ \hat{\Xi}_{6,11} & \hat{\Xi}_{6,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{7,11} & \hat{\Xi}_{7,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{8,11} & \hat{\Xi}_{8,12} & 0 & \hat{\Xi}_{8,14} & 0 & \hat{\Xi}_{8,16} & 0 \\ \hat{\Xi}_{9,11} & \hat{\Xi}_{9,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{10,11} & \hat{\Xi}_{10,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{11,11} & \hat{\Xi}_{11,12} & 0 & 0 & 0 & 0 & 0 \\ * & \hat{\Xi}_{12,12} & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Xi}_{13,13} & 0 & \hat{\Xi}_{13,15} & 0 & 0 \\ * & * & * & \hat{\Xi}_{14,14} & 0 & \hat{\Xi}_{14,16} & 0 \\ * & * & * & * & \hat{\Xi}_{15,15} & 0 & 0 \\ * & * & * & * & * & \hat{\Xi}_{16,16} & 0 \\ * & * & * & * & * & * & \hat{\Xi}_{17,17} \end{pmatrix},$$

$$M_{51} < \lambda^* I, \quad (26)$$

$$\begin{aligned} \hat{\Xi}_{11} &= Q_{14} + Q_{14}^T + Q_{15} + Q_{15}^T + Q_{45} + Q_{45}^T + Q_{44} + Q_{55} \\ &+ \left(1 + \frac{1}{\tau_M - \tau_m}\right)R_{11} + \gamma_{11} + H_{11} + (1 + \tau_m)Z_3 - \Lambda_1 L_1 \\ &+ (C - I)^T M_{11}^T + (1 + \tau_M)Z_4 - \Lambda_2 L_2 + M_{11}(C - I) \\ &+ 2\lambda^* \rho_1 I + P_1 + P_1^T + G_1 + G_1^T, \end{aligned}$$

$$\begin{aligned} \hat{\Xi}_{1,13} &= \Lambda_2 L_2 + \left(1 + \frac{1}{\tau_M - \tau_m}\right)R_{13} + M_{11}A, \quad \hat{\Xi}_{1,15} = \Lambda_1 L_2 + \left(1 + \frac{1}{\tau_M - \tau_m}\right)R_{14} + M_{11}B, \\ \hat{\Xi}_{13,13} &= \left(1 + \frac{1}{\tau_M - \tau_m}\right)R_{33} - \Lambda_2, \quad \hat{\Xi}_{13,15} = \left(1 + \frac{1}{\tau_M - \tau_m}\right)R_{34}, \quad \hat{\Xi}_{15,15} = \left(1 + \frac{1}{\tau_M - \tau_m}\right)R_{44} - \Lambda_1. \end{aligned}$$

*Remark 3.2:* When  $\delta(k, x(k), x(k - \tau(k))) = 0$ , system (1) becomes

$$x(k+1) = C(k)x(k) + A(k)f(x(k)) + B(k)g(x(k - \tau(k))), \quad (27)$$

which studied in [8], [16]-[18]. In this case, similar to the proofs of Theorem 3.1, we can obtain the following corollary.

*Corollary 3.2:* For any given positive integers  $0 < \tau_m < \tau_M$ , then, under Assumption 1,3, system (27) is globally robustly and exponentially stable for any time-varying delay  $\tau(k)$  satisfying  $\tau_m \leq \tau(k) \leq \tau_M$ , if there exist positive matrices  $Q, R, H, \gamma, M_{51}$ , positive diagonal matrices  $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2, Z_1, Z_2, Z_3, Z_4$ , arbitrary matrices  $M_{11}, P_1, P_2, G_1, G_2$  with appropriate dimensions, and positive scalar  $\varepsilon > 0$ , such that the following LMI hold:

$$\hat{\Xi} = [\hat{\Xi}^{(1)}, \hat{\Xi}^{(2)}] < 0, \quad (28)$$

where

$$\hat{\Xi}^{(1)} = \begin{pmatrix} \hat{\Xi}_{11} & \hat{\Xi}_{12} & \hat{\Xi}_{13} & 0 & \hat{\Xi}_{15} & \hat{\Xi}_{16} & \hat{\Xi}_{17} & 0 & \hat{\Xi}_{19} & \hat{\Xi}_{1,10} \\ * & \hat{\Xi}_{22} & \hat{\Xi}_{23} & 0 & \hat{\Xi}_{25} & \hat{\Xi}_{26} & \hat{\Xi}_{27} & 0 & \hat{\Xi}_{29} & \hat{\Xi}_{2,10} \\ * & * & \hat{\Xi}_{33} & 0 & \hat{\Xi}_{35} & \hat{\Xi}_{36} & \hat{\Xi}_{37} & 0 & \hat{\Xi}_{39} & \hat{\Xi}_{3,10} \\ * & * & * & \hat{\Xi}_{44} & 0 & 0 & 0 & \hat{\Xi}_{48} & 0 & 0 \\ * & * & * & * & \hat{\Xi}_{55} & \hat{\Xi}_{56} & \hat{\Xi}_{57} & 0 & \hat{\Xi}_{59} & \hat{\Xi}_{5,10} \\ * & * & * & * & * & \hat{\Xi}_{66} & \hat{\Xi}_{67} & 0 & \hat{\Xi}_{69} & \hat{\Xi}_{6,10} \\ * & * & * & * & * & * & \hat{\Xi}_{77} & 0 & \hat{\Xi}_{79} & \hat{\Xi}_{7,10} \\ * & * & * & * & * & * & * & \hat{\Xi}_{88} & \hat{\Xi}_{89} & \hat{\Xi}_{8,10} \\ * & * & * & * & * & * & * & * & \hat{\Xi}_{99} & \hat{\Xi}_{9,10} \\ * & * & * & * & * & * & * & * & * & \hat{\Xi}_{10,10} \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \end{pmatrix},$$

$$\hat{\Xi}^{(2)} = \begin{pmatrix} \hat{\Xi}_{1,11} & \hat{\Xi}_{1,12} & \hat{\Xi}_{1,13} & 0 & \hat{\Xi}_{1,15} & 0 & \hat{\Xi}_{1,17} M_{11} K \\ \hat{\Xi}_{2,11} & \hat{\Xi}_{2,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{3,11} & \hat{\Xi}_{3,12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{\Xi}_{4,14} & 0 & \hat{\Xi}_{4,16} & \hat{\Xi}_{4,17} \\ \hat{\Xi}_{5,11} & \hat{\Xi}_{5,12} & \hat{\Xi}_{5,13} & 0 & \hat{\Xi}_{5,15} & 0 & \hat{\Xi}_{5,17} M_{51} K \\ \hat{\Xi}_{6,11} & \hat{\Xi}_{6,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{7,11} & \hat{\Xi}_{7,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{8,11} & \hat{\Xi}_{8,12} & 0 & \hat{\Xi}_{8,14} & 0 & \hat{\Xi}_{8,16} & 0 \\ \hat{\Xi}_{9,11} & \hat{\Xi}_{9,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{10,11} & \hat{\Xi}_{10,12} & 0 & 0 & 0 & 0 & 0 \\ \hat{\Xi}_{11,11} & \hat{\Xi}_{11,12} & 0 & 0 & 0 & 0 & 0 \\ * & \hat{\Xi}_{12,12} & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Xi}_{13,13} & 0 & \hat{\Xi}_{13,15} & 0 & 0 \\ * & * & * & \hat{\Xi}_{14,14} & 0 & \hat{\Xi}_{14,16} & 0 \\ * & * & * & * & \hat{\Xi}_{15,15} & 0 & 0 \\ * & * & * & * & * & \hat{\Xi}_{16,16} & 0 \\ * & * & * & * & * & * & \hat{\Xi}_{17,17} \varepsilon I \end{pmatrix},$$

where

$$\hat{\Xi}_{11} = Q_{14} + Q_{14}^T + Q_{15} + Q_{15}^T + Q_{45} + Q_{45}^T + Q_{44} + Q_{55}$$

$$\begin{aligned}
 &+(1 + \text{frac}1\tau_M - \tau_m)R_{11} + \gamma_{11} + H_{11} + (1 + \tau_m)Z_3 \\
 &+(1 + \tau_M)Z_4 - \Lambda_1 L_1 - \Lambda_2 L_2 + M_{11}(C - I) \\
 &+\varepsilon E_c^T E_c + P_1 + P_1^T + G_1 + G_1^T + (C - I)^T M_{11}^T, \\
 \hat{E}_{44} = &-\frac{1}{\tau_M - \tau_m} R_{11} - \Gamma_1 \Pi_1 - \Gamma_2 \Pi_1 - P_2^T - P_2 - G_2^T - G_2.
 \end{aligned}$$

*Remark 3.3:* We proposed  $V_1, V_2$  which take  $\sum_{i=k-\tau_m}^k x(k)$ ,  $\sum_{i=k-\tau_M}^k x(k)$ ,  $\sum_{i=k-\tau_m}^k \eta(k-1)$ ,  $\sum_{i=k-\tau_M}^{k-1} \eta(k)$ ,  $f(x(k))$ ,  $g(x(k))$  as augmented states. The proposed augmented Lyapunov functional  $V_1, V_2$  do not considered in the previous literature and may improve the feasibility region of delay-dependent stability criterion.

*Remark 3.4:* Zero equations (14) (15) provide us a new method to introduce free-weighting matrix, which do not considered in existing works. And free-weighting matrices  $P_1, P_2, G_1, G_2$  make an important role in the reducing of conservatism for above criteria.

#### IV. NUMERICAL EXAMPLES

In this section, three numerical examples will be presented to show the validity of the main results derived above.

*Example 4.1:* For the convenience of comparison, let's consider a delayed discrete-time recurrent neural network in (1), where

$$\begin{aligned}
 f_1(s) &= \sin(0.2s), \quad f_2(s) = \tanh(-0.4s), \quad g_1(s) = \tanh(0.83s), \\
 g_2(s) &= \tanh(0.2s), \quad \tau_m = 1, \rho_1 = \rho_2 = 0.2 \\
 C &= \begin{pmatrix} -0.1 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad A = \begin{pmatrix} -0.1 & 0.1 \\ -0.1 & 0.5 \end{pmatrix}, \\
 B &= \begin{pmatrix} 0.05 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}, \quad L_1 = \begin{pmatrix} -0.64 & 0 \\ 0 & 0 \end{pmatrix}, \\
 L_2 &= \begin{pmatrix} 0 & 0 \\ 0 & -0.2 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} -0.6 & 0 \\ 0 & 0 \end{pmatrix}, \\
 \Pi_2 &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad K = E_c = E_a = E_b \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.
 \end{aligned}$$

Applying the Theorem 1 in [8] and the Theorem 1 in [17], the maximum value of  $\tau_M$  for globally robustly mean square exponentially stable of system (1) is  $\tau_M = 10$  and  $\tau_M = 55$ , respectively. While by using Theorem 3.1 obtained in this paper, the allowable upper bound  $\tau_M$  of the time-varying delay is infinity (details see Table 1), which means that our result is less conservative than that obtained in [8], [17].

When  $\tau_1 = 1, \tau_M = 100$ , by the MATLAB LMI control toolbox, we find a solution to the LMIs (5) as follows

$$\begin{aligned}
 Q^{(1)} &= \begin{pmatrix} 45.06 & 1.50 & -13.77 & -3.08 & -1.66 & -0.00 & -5.67 \\ * & 39.88 & -2.72 & -20.01 & 0.00 & -1.65 & -0.40 \\ * & * & 12.97 & 2.48 & -0.80 & 0.00 & 3.81 \\ * & * & * & 18.13 & -0.00 & -0.82 & 0.39 \\ * & * & * & * & 3.57 & -0.25 & 0.00 \\ * & * & * & * & * & 3.46 & -0.01 \\ * & * & * & * & * & * & 1.73 \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix} \\
 Q^{(2)} &= \begin{pmatrix} -0.39 & -0.01 & -0.00 & -13.75 & -3.06 & -1.64 & -0.11 \\ -7.65 & -0.00 & -0.01 & -2.83 & -19.97 & 0.11 & -1.64 \\ 0.42 & 0.00 & 0.00 & 0.00 & 0.00 & -0.81 & 0.11 \\ 5.29 & 0.00 & 0.01 & 0.00 & 0.01 & -0.11 & -0.81 \\ 0.01 & 0.00 & 0.00 & -0.81 & 0.00 & 2.46 & 0.00 \\ 0.00 & -0.00 & 0.00 & -0.01 & -0.82 & -0.00 & 2.46 \\ -0.09 & -0.00 & 0.00 & 3.84 & 0.44 & -0.00 & -0.00 \\ 2.15 & 0.00 & -0.00 & 0.42 & 5.35 & -0.00 & -0.00 \\ * & 0.01 & -0.00 & 0.01 & 0.00 & 0.00 & 0.00 \\ * & * & 0.01 & 0.00 & 0.01 & 0.00 & 0.00 \\ * & * & * & 11.29 & 2.68 & -0.81 & 0.1159 \\ * & * & * & * & 16.28 & -0.11 & -0.81 \\ * & * & * & * & * & 2.46 & -0.00 \\ * & * & * & * & * & * & 2.46 \end{pmatrix}
 \end{aligned}$$

$$Q = [Q^{(1)}, Q^{(2)}],$$

$$R = \begin{pmatrix} 2.71 & 1.91 & 2.16 & 1.16 & -1.27 & -1.84 & 2.34 & -1.27 \\ * & 10.83 & 1.89 & 8.81 & 1.29 & -3.57 & -5.94 & -7.40 \\ * & * & 1.85 & 1.64 & 1.22 & -1.46 & -0.87 & -1.46 \\ * & * & * & 8.00 & 1.00 & -5.18 & -5.22 & -5.19 \\ * & * & * & * & 2.32 & -0.21 & -0.17 & -0.21 \\ * & * & * & * & * & 13.07 & 0.76 & 0.79 \\ * & * & * & * & * & * & 6.38 & 0.77 \\ * & * & * & * & * & * & * & 14.41 \end{pmatrix},$$

$$H = \begin{pmatrix} 2.58 & -0.66 & 1.79 & -0.48 \\ * & 2.56 & -0.38 & 1.73 \\ * & * & 1.80 & -0.41 \\ * & * & * & 1.63 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 3.47 & -0.54 & 2.30 & -0.47 \\ * & 4.00 & -0.22 & 2.53 \\ * & * & 2.16 & -0.34 \\ * & * & * & 2.19 \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 0.0037 & 0 \\ 0 & 0.0030 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 2.9562 & 0 \\ 0 & 3.8065 \end{pmatrix},$$

$$Z_3 = \begin{pmatrix} 2.2837 & 0 \\ 0 & 2.8459 \end{pmatrix}, \quad Z_4 = \begin{pmatrix} 0.0099 & 0 \\ 0 & 0.0088 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} 11.2062 & 0 \\ 0 & 27.2168 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 1.2155 & 0 \\ 0 & 6.0336 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 4.3341 & 0 \\ 0 & 24.5398 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 1.2270 & 0 \\ 0 & 4.9688 \end{pmatrix},$$

$$M_{11} = \begin{pmatrix} 18.4085 & -0.2803 \\ 2.3140 & 18.0917 \end{pmatrix}, \quad M_{55} = \begin{pmatrix} 17.5754 & 0.9462 \\ -0.8644 & 16.0139 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0.6030 & -0.1628 \\ -0.1749 & 0.5666 \end{pmatrix}, \quad P_2 = a * \begin{pmatrix} 0.0150 & 0.0417 \\ -0.0488 & -4.4629 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 1.3491 & 0.2686 \\ -0.2979 & 1.0429 \end{pmatrix}, \quad G_2 = a * \begin{pmatrix} -0.0070 & -0.0416 \\ 0.0489 & 4.4699 \end{pmatrix},$$

$$a = 1.0e + 003, \varepsilon = 9.0456, \lambda^* = 17.9133.$$

*Example 4.2:* For the convenience of comparison, let's consider a delayed discrete-time recurrent neural network in (24) with parameters given by  $\delta = 0$ ,

$$C = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.7 \end{pmatrix}, \quad A = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.005 \end{pmatrix}, \quad B = \begin{pmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{pmatrix}.$$

And the activation functions satisfy Assumption 1 with  $l_1^- = l_2^- = 0, \sigma_1^+ = \sigma_2^+ = 1$ . For  $\tau_m = 1, 4, 8, 15, 25$ , references [8], [25], [18]-[21] gave out the allowable upper bound  $\tau_M$  of the time-varying delay, respectively. Table 2 shows that our results are less conservative than these previous results.

*Example 4.3:* Consider an uncertain delayed discrete-time recurrent neural network in (27) with parameters given by

$$C = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.12 & 0.24 \\ -0.15 & 0.2 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.25 & 0.1 \\ 0.02 & 0.09 \end{pmatrix}, \quad K = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix},$$

$$E_c = \begin{pmatrix} 0.15 & 0.1 \\ 0 & -0.7 \end{pmatrix}, \quad E_a = \begin{pmatrix} 0.1 & 0.3 \\ -0.2 & 0.05 \end{pmatrix},$$

$E_b = \begin{pmatrix} 0.13 & 0.06 \\ -0.05 & 0.15 \end{pmatrix}$ . And the activation functions satisfy Assumption 1 with  $l_1^- = -0.5, l_2^- = 0, \sigma_1^+ = 1, \sigma_2^+ = 0.5$ . For  $\tau_m = 1, 2, 4, 6, 8, 10$ , references [16], [19], [20] gave out the allowable upper bound  $\tau_M$  of the time-varying delay, respectively. The allowable upper bounds  $\tau_M$  for given  $\tau_m$  are showed in Table 3. Obviously, our results are less conservative than these previous results.

#### V. CONCLUSION

Combined with linear matrix inequality (LMI) technique, a new augmented Lyapunov-Krasovskii function is constructed, and some new improved sufficient conditions ensuring globally exponential stability or robust exponential stability in the mean square are obtained. Numerical examples show that the new results are less conservative than some previous results.

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**Shu Lü** was born in 1963 in Jilin, China. She received B.S. degree in Mathematics Department of Northeast Normal University, Changchun, China, in 1984 and the M.S. degree from the School of mathematical statistics, China University of Technology. She is now a Associate professor with School of Applied Mathematics, University of Electronic Science and Technology of China (UESTC). Her research interests include the theory and application of economics mathematics, differential equations, neural networks, biomathematics and robust control..

**Shouming Zhong** was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from the University of Electronic Science and Technology of China (UESTC), Chengdu, China, in 1982. From 1984 to 1986, he studied at the Department of Mathematics in Sun Yat-sen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the Department of Mathematics in University of Waterloo, Waterloo, Canada. He is currently as a full professor with School of Applied Mathematics, UESTC. His current research interests include differential equations, neural networks, biomathematics and robust control. He has authored more than 80 papers in reputed journals such as the In International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics.

**Zixin Liu** was born in Sichuan Province, China, in 1977. He received the B.S. degree from China West Normal University, Sichuan in 1999. The M.S. and Ph.D. degree from the University of Electronic Science and Technology of China (UESTC), Sichuan, in 2006 and 2010, respectively. He is currently as a full professor with School of Applied Mathematics and Statistics, Guizhou University of Finance and Economics. His research interests include neural networks, chaos synchronization and stochastic delayed dynamic systems.