Numerical Calculation of Coils Filled With Biaxial Media

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Abstract—Recently, biaxial media again received increasing importance in electromagnetic theory because of advances in material science which enable the manufacturing of complex biaxial media. By using Maxwell’s equations and corresponding boundary conditions, the electromagnetic field distribution in biaxial media is determined and the influence of the biaxial media behavior on the impedance and Q-factor is considered. Biaxial media are the largest class of linear media which is able to describe the macroscopic material properties of artificial dielectrics, artificial magnetic fields, artificial chiral materials, left-handed materials, metamaterials, and other composite materials. Several special cases of coils, filled with complex substance, have been analyzed. Results obtained by using the analytical approach are compared with values calculated by numerical methods, especially by our new hybrid EEM/BEM method and FEM.

Keywords—Biaxial medium, impedance and Q-factor, Maxwell’s equations, hybrid EEM/BEM method.

I. INTRODUCTION

The main idea of biaxial media is a combination of two physical notions: (a) the near-field manipulation and (b) chirality. Biaxial media concerns the subject of an intrinsic magnetoelectric (ME) coupling in media. Biaxial media are conceived as artificial structures and placed in an electric field. In a biaxial medium, the electromagnetic field becomes both polarized and magnetized. Almost any media in motion becomes biaxial. The first cases of biaxial media were indeed moving dielectrics and magnetic materials in the presence of electric or magnetic fields.

Historically, the electromagnetic chirality was studied in the optical region. Hence, the most common term for chiral media is optically active media. Usually, the effects of magnetoelectricity in natural crystals are considered without any relations with a symmetry structure of the EM field. The biaxial medium is the most general linear complex medium.

Biaxial media, analytically proposed by Tellegen [1] over sixty years ago, can be nonreciprocal, and is not as popular, possibly because it has not been found in nature. Both chiral and bi-isotropic media are isotropic, and they become polarized when placed in a magnetic field, and magnetized when placed in an electric field.

This effect was theoretically predicted by Landau and Lifshitz in 1957 and Dzyaloshinskii in 1960 [2], and experimentally confirmed for anti-ferromagnetic chromium oxide by Astrov in 1961 [3].

In a chiral material the electric field induces a magnetic polarization with which it has π/2 phase difference, and the magnetic field induces an electric polarization with similar phase difference. However, in the magneto-electric effect, the induced magnetic polarization is in phase with the electric field and the induced electric polarization is in phase with the magnetic field [4].

A biaxial medium provides a coupling between electric and magnetic fields. The field vectors D and H depend on both E, B, and C. The constitutive relations for a biaxial medium is given by [4] - [10]:

\[ D = \varepsilon E + \xi H, \]
\[ B = \xi E + \mu H, \]

where:

- \( E \) is electric field strength, (vector values are in bold).
- \( D \) is electric displacement,
- \( H \) is magnetic field strength and
- \( B \) is magnetic flux density,
- \( \varepsilon \) and \( \mu \) are electric and magnetic permeability, \( \xi \) and \( \zeta \) are the value which defines the biaxial isotropic properties.

The constitutive parameters \( \varepsilon, \mu, \xi, \) and \( \zeta \) are arbitrary random tensors, potentially fully populated, and homogeneous within each layer. The electrical permittivity, \( \varepsilon \), and the magnetic permeability, \( \mu \), are second-order tensors, while the chirality parameters \( \xi \) and \( \zeta \) are pseudo-tensors. For bi-isotropic media, the material parameters are scalars and pseudo-scalars.

Maxwell’s equations are form-invariant; however, constitutive relations are only form-invariant when they are written in the biaxial form. Classical theories for homogenization of local biaxial composites, based on the Maxwell Garnett and the Bruggeman formalism, are relevant about some "biaxialotropic effects in left-handed materials".
The Maxwell-Garnett approach [11] has been a very successful theory in describing the effective dielectric properties of composite dielectric media.

A theory of electromagnetic processes in media is called as macroscopic electrodynamics or electrodynamics of continuous media. Relations between \( E, D, H \) and \( B \) acquire a special meaning in view of a strong interest in recent propositions and intensive studies of different artificial electromagnetic structures called as "metamaterials", "left-handed materials", "chiral materials", "bianisotropic materials", etc [12] - [13].

"Metamaterials" are new artificially created structures, able to show material properties unknown in nature. Dot near-fields are originated from "microscopic" electric and magnetic charges.

The term "Metamaterial" has been introduced into the electromagnetic lexicon in recent years to describe new artificial materials with electromagnetic properties that are not found in naturally occurring materials. Metamaterial media and surfaces (with unconventional electromagnetic properties) have attracted a great deal of attention and interest in recent years. Various propositions involving double-negative (DNG) media, single-negative (SN) materials, electromagnetic band gap (EBG) structures, and artificial thin magnetic conductors (AMC) have been explored by many researchers over the past few years. New EEM/BEM hybrid method [14]-[24] can be applied for metamaterial structures determination.

The time-harmonic constitutive relations of a general bianisotropic medium are [10]:

\[
D = \varepsilon_0 (\varepsilon E + \eta \chi \mathbf{H});
\]

\[
B = \frac{1}{c_0} (\sigma E + \eta \mu \mathbf{H}),
\]

where \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability of free space respectively, and \( c_0 \) is the speed of light in free space.

For chiral media the constitutive relations are given by:

\[
D = \varepsilon E - \chi \frac{\partial H}{\partial t};
\]

\[
B = \chi \frac{\partial E}{\partial t} + \mu H, \tag{3}
\]

where \( \chi \) is called the chiral parameter.

When the medium is lossless, \( \varepsilon, \mu \) and \( \chi \) are real.

Tellegen had imagined a new medium for which the constitutive relations were given by:

\[
D = \varepsilon E + \chi H; \tag{4}
\]

\[
B = \chi E + \mu H,
\]

where \( \frac{\chi^2}{\varepsilon \mu} = 1 \).

A chiral medium is characterized by the property that it is not identical with its own mirror image. The microstructure of a chiral medium has a left-handed or right-handed nature. The asymmetric geometry of the coils creates an overall chiral response. The electric and magnetic field vectors rotate as a wave propagates through such a medium.

The general theory of plan wave propagation in the bianisotropic media, including corresponding wave equations and the potential functions [6], is very well developed.

In the present paper solenoid coils [17] with bianisotropic cores are considered.

II. THEORETICAL BACKGROUND

It is considered a long solenoid of circular cross section (radius \( a \)) wound with \( N \) turns of wire per unit length, the wire currying current, having magnitude \( I \) and angular frequency \( \omega \). The solenoid core is bianisotropic, having constitutive relations (4) and conductivity \( \sigma \). Outside of the solenoid is free space.

Inside the solenoid coil, Maxwell’s equations, governing distribution of electromagnetic fields, have the following form:

\[
\begin{align*}
\rot \textbf{H} &= \sigma \textbf{E} + j \omega \textbf{D} ; \quad \rot \textbf{E} = -j \omega \textbf{B} \; ; \quad \text{div} \textbf{D} = 0 \; ; \quad \text{div} \textbf{B} = 0. \tag{5}
\end{align*}
\]

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\end{align*}
\]

By using constitutive relation (4), it is obtained:

\[
\rot \textbf{E} = \left[ \mu \sigma + j \omega (\varepsilon - \chi^2) \right] \textbf{E} ; \quad \text{div} \textbf{E} = 0 \; , \quad \text{and} \quad \text{div} \textbf{H} = 0 . \tag{7}
\]

Outside the solenoid coil is:

\[
\begin{align*}
\rot \textbf{H} &= j \omega \varepsilon \textbf{E} \; , \quad \rot \textbf{E} = -j \omega \mu \textbf{H} \; , \quad \text{div} \textbf{E} = 0 \; , \quad \text{div} \textbf{H} = 0 \; , \quad \textbf{D} = \varepsilon \textbf{E} \; , \quad \text{and} \quad \textbf{B} = \mu \textbf{H} \; . \tag{8}
\end{align*}
\]

The obtained Maxwell’s equation can be solved analytically in cylindrical coordinates, \( r, \theta, z \), where \( z \)-axis coincides with solenoid axis.

Because of the problem axial symmetry, the field components depend only on the radial distance, \( r \). So, \( \partial / \partial \theta = 0 \) and \( \partial / \partial z = 0 \) and angular, \( E_{\theta} \), and axial, \( E_{z} \), electric field component satisfies the following Bessel’s differential equations:

\[
\frac{d^2 E_{\theta}}{dw^2} + \frac{1}{w} \frac{dE_{\theta}}{dw} + \left( \frac{1}{w^2} - \frac{1}{w^2} \right) = 0 \tag{9}
\]

and
\[
\frac{d^2 E}{dw^2} + \frac{1}{w} \frac{dE}{dw} + E_z = 0
\]  \quad (10)

where:

\[ w = kr, \quad r < a \]
\[ w = k_0 r, \quad r > a \]  \quad (11)

\[ k^2 = \omega^2 (\mu - \chi^2) - j/\omega \mu \sigma \]  \quad (12)

and

\[ k_i^2 = \omega^2 \epsilon_0 \mu_0. \]  \quad (13)

The value \( k \) is complex, \( k = k_1 + jk_2 \), where

\[ k_1 = \omega \sqrt{\epsilon_0 \mu - \chi^2} \sqrt{\frac{1 + p^2 + 1}{2}}, \]  \quad (14)

\[ k_2 = \omega \sqrt{\epsilon_0 \mu - \chi^2} \sqrt{\frac{1 + p^2 - 1}{2}}, \]  \quad (15)

\[ p = \frac{\mu \sigma}{\omega (\epsilon_0 \mu - \chi^2)}. \]  \quad (16)

By using boundary conditions:

\[ E_a(r = a - 0) = E_a(r = a + 0), \]
\[ E_z(r = a - 0) = E_z(r = a + 0), \]
\[ H_a(r = a - 0) = H_a(r = a + 0) \quad \text{and} \]
\[ H_z(r = a - 0) - H_z(r = a + 0) = N T, \]  \quad (17)

and by respecting radiation condition for large radial distance, the following field distribution can be obtained:

\[ E_0 = \frac{\omega \chi \mu_0 N T}{kk_0} \left( \frac{J_1(k_0)}{J_2(k_0)} H_2^{(1)}(k_0 a) + J_1(k_0) H_0^{(1)}(k_0 a) - \frac{k_0}{k} \left( \frac{\mu_0}{\mu} + \frac{k_0^2 + \omega^2 \chi^2}{k^2} \right) J_1(k_0) J_0(k_0) \right) \]  \quad (27)

and

\[ H_0 = N T \left( \frac{J_0(k_0) H_2^{(1)}(k_0 a)}{H_1^{(1)}(k_0 a)} + \frac{k_0^2 + \omega^2 \chi^2}{k^2} J_1(k_0) \right) \]  \quad (28)

III. CALCULATION OF SOLENOID COIL IMPEDANCE

By using Pointing’s theorem the power per unit solenoid length, \( P' \), can be evaluated as Pointing’s vector flux through complex surface condensing cylindrical surfaces \( r = a + 0 \) and \( r = a - 0 \),

\[ P' = Z H' = \left[ \Gamma(r = a + 0) - \Gamma(r = a - 0) \right] 2 a \pi. \]  \quad (29)

where:

\[ \Gamma = E_a H_z' - E_z H_a'. \]  \quad (30)
is radial Pointing’s vector component and \( Z' = R' + jX' \) is solenoid coil impedance per unit length.

\[
E_0 = \frac{\omega^2 \chi \mu N^2 I}{kk_0} J_0(k_0)H_0^{(2)}(k_0) + J_1(k_0)H_1^{(2)}(k_0) - \frac{k_0}{k} \left( \frac{\mu_0 + \mu_s}{\mu} \right) \left( \frac{k^2 + \omega^2 \chi^2}{k^2} \right) J_0(k_0)J_0(k_0), \tag{31}
\]

where:

\[
Z'_0 = j\omega L'_0, \quad L'_0 = \mu N^2 a^2 \pi \tag{32}
\]

are static values for input impedance and coil inductance per unit length.

The coil inductance per unit length, for arbitrary frequency, can be determined as:

\[
L' = \frac{X'}{\omega}, \tag{33}
\]

and coil Q-factor is:

\[
Q = \frac{X'}{R'} = \frac{\omega L'}{R'}. \tag{34}
\]

The impedance ratio is obtained:

\[
\frac{Z'}{Z'_0} = \frac{L'_0}{L'_0} \left( 1 - \frac{1}{Q} \right). \tag{35}
\]

Similar procedure can be applied on arbitrary shaped perfect conducting electrodes, where electrodes are replaced by finite system of Equivalent Electrodes [25] - [26].

In contrast to Charge Simulation Method, where the fictitious sources are placed inside the electrodes volume, the EEs are located on the body surface. The radius of EEs is equal to equivalent radius of electrode part, which is substituted.

This consideration can be applied on magnetic materials [15]. Boundary between two magnetic materials can be replaced by equivalent currents (ECU), where ECU are located on the boundary surface of the magnetic layers, having different magnetic permeability. A system of linear equations can be formed again, and surface density of Ampere’s microscopic currents, \( J_s \), are now unknown.

IV. NUMERICAL RESULTS

Computer-aided analysis of field distribution for evaluating electromagnetic device or component performance has become the most effective way of design. Analytical methods have limited uses and experimental methods are expensive and time consuming. Finite difference method, finite element method and boundary element method are used for the numerical solution of the field equations. The boundary element method employs an integral equation formulation. Since the unknowns are placed only on the boundary the dimensionality of the problem is reduced by one.

In order to investigate the influence of the core bianisotropic properties, first solenoid, having isotropic cores, will be considered.

These results are contained in general proposed formulas as special cases for \( \chi = 0 \).

It is clear that the electromagnetic field distribution is simpler in the case of the isotropic core, when only axial magnetic field and angular electric field exists. Low frequency magnetic field is practically homogeneous inside the solenoid and vanishes in the exterior region. In the interior region electric field increases (from zero to value on the solenoid axes) from axes to the windings. Outside the solenoid the electric field strength is negligible and decreases as \( 1/r \). The LF coil impedance has an inductive character. \( L'/L_0' \) dependence on the ratio \( a/\lambda_0 \) (\( \lambda_0 \) is free space wavelength), for solenoid having air core, is presented in Fig.1.

It is noticeable that the maximal value, \( L'_{max} = 1.08808L_0' \), of inductance per unit length corresponds to the frequency as \( a = 0.076\lambda_0 \). With frequency increasing, because of very significant radiation effects, solenoid changes reactance properties, and shows inductive, capacitive and resonant qualities. Q-factor dependence on the ratio \( a/\lambda_0 \), for solenoid having air core, is presented in the Fig.2.

The \( L'/L_0' \) and Q-factor dependencies on the ratio \( a/\lambda_0 \) and for different ratios \( \sigma/a\epsilon \), for solenoid with semiconducting isotropic core (\( \epsilon_r = 1, \mu_r = 4, \chi = 0 \)) are presented in the Fig.3 and Fig.4.
Differently from above mentioned, in the solenoid with bianisotropic core exist both components of electric and magnetic field (axial and angular). Low frequency electric and magnetic field is practically homogeneous inside the solenoid and vanishes in the exterior region. In the interior region electric and magnetic field increases (from zero to value on the solenoid axes) from axes to the windings. Outside the solenoid the electric and magnetic field strength is negligible and decreases as \(1/r\).

The \(L'/L'_0\) and Q-factor dependencies on the ratio \(a/\lambda_0\) and for different ratios \(\chi/\sqrt{\epsilon\mu}\), for solenoid with bianisotropic core are presented in the Fig.5 and Fig.6 (\(\epsilon_r = 1\), \(\mu_r = 7\), \(\sigma = 0\)).

It can be concluded that the influence of the core bianisotropic properties increases together with core conductivity.
Fig. 7 The ratio $L'/L_0$ as a function of $a/\lambda_0$, for solenoid coil with $\varepsilon_r = 1$, $\mu_r = 7$, $\sigma/\omega\varepsilon = 10$, where $\chi'_{\sqrt{\varepsilon\mu}}$ is parameter.

Fig. 8 $Q$-factor as a function of $a/\lambda_0$, for solenoid coil with $\varepsilon_r = 1$, $\mu_r = 7$, $\sigma/\omega\varepsilon = 10$, where $\chi'_{\sqrt{\varepsilon\mu}}$ is parameter.

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