Abstract—In this paper, we explore the applicability of the Sinc-Collocation method to a three-dimensional (3D) oceanography model. The model describes a wind-driven current with depth-dependent eddy viscosity in the complex-velocity system. In general, the Sinc-based methods excel over other traditional numerical methods due to their exponentially decaying errors, rapid convergence and handling problems in the presence of singularities in end-points. Together with these advantages, the Sinc-Collocation approach that we utilize exploits first derivative interpolation, whose integration is much less sensitive to numerical errors. We bring up several model problems to prove the accuracy, stability, and computational efficiency of the method. The approximate solutions determined by the Sinc-Collocation technique are compared to exact solutions and those obtained by the Sinc-Galerkin approach in earlier studies. Our findings indicate that the Sinc-Collocation method outperforms other Sinc-based methods in past studies.

Keywords—Boundary Value Problems, Differential Equations, Sinc Numerical Methods, Wind-Driven Currents

I. INTRODUCTION

In many fields of study, modelling the governing phenomena leads to a specific set of differential equations, called boundary value problems (BVPs). In most cases, deriving analytical solutions of BVPs is extremely hard or completely impossible. Therefore, various numerical methods were developed to attack these problems. Some of the well-known numerical approximations to BVPs are finite-difference method [1], finite-element method [2], [3], boundary element method [4], shooting method [5], spline method [6], and Sinc methods.

It is well-known that Sinc-based methods are dominant over other numerical methods, especially in the presence of singularities and semi-infinite domains [7]. They are also characterized by exponentially decaying errors and rapid convergence [8]. Sinc methods reduce the governing differential or integral equations to a system of algebraic equations which makes the solution easier. Sinc-based methods have been applied to diverse scientific and engineering problems comprising heat conduction [9], [10], population growth [11], inverse problems [12], [13], astrophysics problems [14], [15], medical imaging [16], elastoplastic problems [17], and oceanography [18], [19]. Very recently, the application of Sinc-Collocation approach to the telegraph equation [20] and the second type of the Painlevé equations [21] has been studied.

In general, there are two equivalent but distinct Sinc approaches: Sinc-Galerkin and Sinc-Collocation. In earlier studies, it has been evidenced that the Sinc-Collocation approach is superior to the Galerkin one regarding its simple implementation and possible extensions to more general BVPs [22].

In the past century, hydrodynamic models and their numerical solutions obtained many accomplishments. The first wind-driven current models were one-dimensional systems based on the work of Ekman [23]. Eventually two- and three-dimensional models were developed [24], [25]. To derive approximate solutions to 3D models, several numerical methods employing spectral methods [26], B-spline approach [27], Chebyshev and Legendre polynomials [28] and eigenfunction approach [29] were developed. Recently, Sinc-Galerkin approaches have been applied to a 3D wind-driven current model [18], [19].

The intent of this paper is to demonstrate an application of the Sinc-Collocation technique to a steady state 3D model of wind-driven currents with a depth-dependent eddy viscosity in coastal regions and semi-enclosed seas. The model is found in the work of Winter et al. [18]. They formulated the model as a complex-valued ordinary differential equation (ODE) and applied the original Sinc-Galerkin approach to solve it. Later, Koonprasert and Bowers [19], developed a block matrix formulation for the Sinc-Galerkin technique and applied it to the same model. In this paper, we apply the Sinc-Collocation approach to the complex-valued system and compare the results with those in earlier studies and exact solutions when available.

Following the introduction, we provide a brief explanation of the model formulation in section 2. Section 3, is devoted to the Sinc-Collocation treatment that we apply to the model. In Section 4, several model problems have been used to examine the accuracy and stability of the method. Finally, in the last section we discuss the results.

II. PROBLEM FORMULATION

In this section we provide a brief explanation of the model found in the work of Winter et al. [18]. We refer interested readers to [18], [19] and references there in.

To develop this model one needs to construct a right-handed coordinate system with the vertical coordinate $z^*$ directed
The ocean depth, $\tau$, is provided in Figure 1. Assuming $[18]$. For a better understanding, a schematic form of the model $z - z$ following BVP:

$$\rho A z$$

Similarly, the separated BCs at the sea surface, and the seabed are given by

$$-\rho A_v(0) \frac{dU^*(0)}{dz} = \tau_w \cos(\chi),$$

$$-\rho A_v(0) \frac{dV^*(0)}{dz} = \tau_w \sin(\chi)$$

$$\rho A_v(D_0) \frac{dU^*(D_0)}{dz} = k_f \rho U^*(D_0),$$

$$\rho A_v(D_0) \frac{dV^*(D_0)}{dz} = k_f \rho V^*(D_0).$$

With the help of the non-dimensional variables

$$z \equiv \frac{z^*}{D_0}, A_v(z) \equiv \frac{A_v(z)}{A_v(0)},$$

and non-dimensional constants, $\kappa$ (depth ratio) and $\sigma$ (bottom friction parameter)

$$\kappa \equiv \frac{D_0}{D_E} = \sqrt{\frac{f}{2A_0}}, \sigma \equiv \frac{A_0 A_v(1)}{k_f D_0} = \frac{A_v(0)}{A_v(D_0)},$$

where $A_0 \equiv A_v(0), D_E \equiv \sqrt{\frac{f \Omega^2}{\rho A(0)}},$ and $U_0 = \frac{\tau_w D_0}{\rho A(0)} = \frac{\sqrt{\tau_w}}{\sqrt{\Omega \rho A(0)}}$, equations (5) and (6) are transferred to non-dimensional equations

$$-\frac{d}{dz} \left( A_v(z) \frac{dU^*(z)}{dz} \right) = -2\kappa^2 V(z), 0 < z < 1,$$

$$-\frac{d}{dz} \left( A_v(z) \frac{dV^*(z)}{dz} \right) = 2\kappa^2 U(z), 0 < z < 1.$$
\[
\frac{dV(z)}{dz} = \frac{dv(z)}{dz} - \kappa \sin(\chi)
\]

Hence the "reduced velocity" components \(u(z)\) and \(v(z)\) satisfy
\[
-\frac{d}{dz} \left( A_v(z) \frac{du(z)}{dz} \right) + \kappa \cos(\chi) A_v(z) = 0
\]
\[
= -2\kappa^2 u(z) - 2\kappa^3 (1 + \sigma - z) \sin(\chi), \quad 0 < z < 1.
\]
\[
-\frac{d}{dz} \left( A_v(z) \frac{dv(z)}{dz} \right) + \kappa \sin(\chi) A_v(z) = 0.
\]
\[
= 2\kappa^2 u(z) + 2\kappa^3 (1 + \sigma - z) \cos(\chi), \quad 0 < z < 1.
\]

where the BCs at the surface and seabed are respectively given by
\[
\frac{du(0)}{dz} = 0, \quad \frac{dv(0)}{dz} = 0
\]
\[
u(1) + \sigma \frac{dv(1)}{dz} = 0, \quad v(1) + \sigma \frac{dv(1)}{dz} = 0
\]

The system defined by (17)-(20) could be written in the complex-velocity system. To obtain the complex-velocity formulation, we need to multiply equation (18) by the imaginary unit \(i\), and add the result to equation (17). Afterwards by defining a complex velocity \(w(z) = u(z) + iv(z)\), we have
\[
L(z) \equiv Lu(z) + iLv(z)
\]
\[
\equiv -\frac{d}{dz} 27 \left( A_v(z) \frac{du(z)}{dz} \right) - i \frac{d}{dz} \left( A_v(z) \frac{dv(z)}{dz} \right)
\]
\[
\equiv -\frac{d}{dz} \left( A_v(z) \frac{dv(z)}{dz} \right)
\]

Hence the complex velocity formulation is shown by
\[
Lw(z) - i2\kappa^2 w(z) = F(z), \quad 0 < z < 1,
\]
where
\[
F(z) = [-\kappa A_v(z) + i2\kappa^3 (1 + \sigma - z)]e^{ix}.
\]

BCs, evolved by the same procedure, are given by
\[
u'(0) = 0,
\]
\[
u(1) + \sigma \nu'(1) = 0.
\]

### III. THE sinc-COLLOCATION APPROACH

In this section, we briefly describe a Sinc-Collocation approach via first derivative interpolation, that has been recently developed by Abdella [30]. We refer the readers to paper [30] where a comprehensive explanation of the method and Sinc preliminaries are provided.

Assume the general second-order two-point BVP:
\[
a(x) y''(x) + b(x) y'(x) + c(x) y(x) = d(x), \quad x \in (a, b),
\]
\[
\alpha_a y(a) + \beta_a y'(a) = \gamma_a,
\]
\[
\alpha_b y(b) + \beta_b y'(b) = \gamma_b.
\]

where \(\alpha_a, \alpha_b, \beta_a, \beta_b, \gamma_a, \) and \(\gamma_b\) are constants.

The Sinc-Collocation approach introduced by Abdella [30], transforms the BVP as follows such that the BCs become homogeneous [31]:
\[
u(x) = y(x) - \eta(x)
\]
where
\[
\eta(x) = y'(a)H_1 + y(a)H_2 + y(b)H_3 + y'(b)H_4
\]
is the univariate Hermite interpolation with the cardinal functions given by:
\[
H_1 = \frac{(x-a)(x-b)^2}{(b-a)^3}, \quad H_2 = \frac{(x-b)^2(2x-3a+b)}{(b-a)^3},
\]
\[
H_3 = \frac{(x-a)^2(2x-3b+a)}{(b-a)^3}, \quad H_4 = \frac{(x-b)(x-a)^2}{(b-a)^3}.
\]

Employing (27) and considering \(\eta(a) = y(a), \eta'(a) = y'(a), \eta(b) = y(b), \eta'(b) = y'(b),\) leads to a new BVP with homogeneous BCs given by:
\[
a(x) u''(x) + b(x) u'(x) + c(x) u(x) = e(x), \quad x \in (a, b),
\]
\[
u(a) = u(b) = 0,
\]
\[
u'(a) = u'(b) = 0,
\]

where
\[
e(x) = d(x) - a(x) \eta'(x) - b(x) \eta(x) - c(x) \eta'(x).
\]

Here is the point that our method changes its way from the original sinc-collocation technique, i.e. it first approximates \(u'(x)\) at sinc points \(x_k\), by:
\[
u'(x_k) = \sum_{k=-N}^{N} S(k, h)(\phi(x_k)) u'(x_k) = \sum_{k=-N}^{N} \delta_{i,k} u'(x_k)
\]

Then \(u(x)\) is approximated by:
\[
u(x_i) = \sum_{k=-N}^{N} h_k(x_i) u'(x_k) = \sum_{k=-N}^{N} h_{i,k} u'(x_k)
\]

Finally, we can approximate \(u''(x)\) via:
\[
u''(x_k) = \sum_{k=-N}^{N} g_k(x_i) u'(x_k) = \sum_{k=-N}^{N} \delta_{i,k} \phi(x_k) \frac{u'(x_k)}{h}
\]

where \(\phi(x)\) is given by
\[
\xi = \phi(x) = \frac{1}{\pi} \log \left( \frac{x-a}{b-x} \right)
\]

with inverse
\[
x = \psi(\xi) = \frac{b+a}{2} + \frac{b-a}{2} \tanh \left( \frac{\pi}{2} \sinh(\xi) \right)
\]

and \(x_k = \psi(kh)\). As well
\[
\delta_{i,k}^{(0)} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}
\]
conditions obtained by requiring that $u_i$ vanishes at the outside Sinc nodes $(-N - 1)$ and $(N + 1)$:

$$\alpha_d C_{N - 2} + \beta_d C_{N - 1} = \gamma_a,$$

$$\alpha_d C_N + \beta_d C_{N + 1} = \gamma_b,$$

$$\sum_{k=-N}^{N} h_i \delta_{k,1} C_k = 0,$$

$$\sum_{k=-N}^{N} h_i \delta_{k,N+1} C_k = 0.$$ 

The matrix representation of the $(n + 4) \times (n + 4)$ system corresponding to equations (40) and (42)-(45) is given by

$$A C = E$$

where $E$, a $(n + 4) \times 1$ vector, and $A$, a $(n + 4) \times (n + 4)$ matrix are given by

$$E = [\gamma_a, \gamma_b, e(x_{-N}), ..., e(x_{0}), e(x_N), 0, 0]^T,$$

$$A = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix},$$

where $B_1$, $B_2$, $B_3$ and $B_4$ are $1 \times (n + 4)$ matrices given by

$$B_1 = [\alpha_d, \beta_d, 0, ..., 0],$$

$$B_2 = [0, 0, ..., \beta_d, \alpha_d],$$

$$B_3 = [0, 0, h_i \delta_{-N,1-N}, ..., h_i \delta_{N,1-N}, 0, 0].$$

The four conditions required to close the system consists of $n + 4$ unknowns including $y'(a)$, $y(a)$, $y'(b)$, $y(b)$ and $u'(x_i)$, $i = -N, -N + 1, ..., N - 1, N$.

We define the $(n + 1) \times 1$ vector $C$ by:

$$C = [C_{-N-2}, C_{-N-1}, C_{-N}, ..., C_N, C_{N+1}, C_{N+2}]^T = [y(a), y'(a), u'(x_{-N})...u'(x_0), ..., u'(x_N), y'(b), y(b)]^T.$$
where \( d, \gamma, \) and \( \beta \) are equal to \( \frac{7}{2}, 2, \) and \( \frac{7}{2} \) respectively. In order to provide dimensional representation of the velocities we need to multiply the results by the natural scale \( U_0. \)

To demonstrate the accuracy of the method, the maximum absolute errors are defined by

\[
\| E_U \| = \max_{-N < z_i < N + 1} \{ U_0[ U(z_i) - U(z_i)] \},
\]

\[
\| E_V \| = \max_{-N < z_i < N + 1} \{ U_0[ V(z_i) - V(z_i)] \},
\]

and

\[
\| E_W \| = \max\{ \| E_U \|, \| E_V \| \}, \tag{53}
\]

where the units are \( m s^{-1}. \)

Example 1 (seabed linear stress condition)

To keep the parameters and variables identical to those in [18], [19], [32], we choose \( \chi = 45^\circ, \) the linear stress condition at the seabed, \( \sigma = 0.1, D_0 = 100 m, D_E = 20 \) and \( \kappa = 5. \) In this example we solve a discrete system of size \( (m \times m) \) given by (46), where \( m = 2N + 5. \) To demonstrate the numerical convergence of the method we repeat the process for \( N=4, 8, 16, \ldots, 64. \) The errors are listed in Table I and exhibit a very high degree of accuracy.

In Figure 2, we depict the exponential convergence of the solutions by the horizontal projection of the Ekman spiral (HPES). Our solution for \( N=64, \) has a high degree of accuracy which makes it hard for us to distinguish it from the exact solution.

In Table II, we exhibit the comparison we made between our findings and those in papers [18], [32]. \( E_U, E_V \) and \( E_W \) convey the maximum errors of our method, the one in [18] and [32] respectively.

Example 2 (No-slip condition at the seabed)

In this example, we assume \( \sigma = 0 \) and all other parameters similar to Example 1. The absolute errors of our solutions are listed in Table III and a very close similarity to those in Example 1 is explored. The HPES for different values of \( N, \) against the exact solution are portrayed in Figure 3. Likewise, Table IV provides the maximum errors of our method, and those in [18], [32] respectively.

### Table I

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case of time-dependent eddy viscosity in which when \( t \to \infty \), it can be considered as a constant.

In seas of shallow to intermediate depth, the eddy viscosity has the maximum values of \( A^*_{v_c}(z^*) \) at the intermediate depths and the minimum values near the surface and seabed. But in deeper seas, it is expected that \( A^*_{v_c}(z^*) \) has the maximum values near the surface and its value decreases going towards the seabed. The latter case is illustrated by

\[
A^*_{v_c}(z^*) = 0.02[1 - (0.0075)z^*]^2, \quad 0 < z^* < D_0. 
\]  

(54)

which decreases quadratically from the value of 0.02 \( \text{m}^2 \text{s}^{-1} \) to the minimum value of 0.00125 \( \text{m}^2 \text{s}^{-1} \). The eddy viscosity in the first case, follows a quadratic model given by

\[
A^*_{v_c}(z^*) = 0.02[1 + (0.12)z^*(1 - (0.01)z^*)], \quad 0 < z^* < D_0. \]

(55)

increasing from the initial value of 0.02 \( \text{m}^2 \text{s}^{-1} \) to the peak value of 0.08 and then decreasing to 0.02 \( \text{m}^2 \text{s}^{-1} \).

Example 3. (The decreasing eddy viscosity)

In this example, we find the approximate solutions \( U_c(z) \) and \( V_c(z) \) via the complex velocity discrete system while the variable eddy viscosity is given by (54). The parameters are chosen identical to those in Example 1. Since there is not any closed form solution of this case, we depict the HPES of decreasing eddy viscosity against that of constant eddy viscosity for different values of \( N \) in Figure 4.

![Figure 4](image-url)

Fig. 4. The Sinc-Collocation Ekman Spiral projection of Example 3 for different values of \( N \) against the exact solution of the constant eddy viscosity case while \( \sigma = 0.11, \chi = 45, \kappa = 5, D_0 = 100 \text{ m}, D_E = 20 \text{ m} \).

Example 4. (The quadratic eddy viscosity)

This example is similar to Example 3, but the eddy viscosity is given by (55). Since no exact solution for this case is reported, we portray the HPES of quadratic eddy viscosity for different values of \( N \), against that of constant eddy viscosity in Figure 5.

![Figure 5](image-url)

Fig. 5. The Sinc-Collocation Ekman Spiral projection of Example 4 for different values of \( N \) against the exact solution of the constant eddy viscosity case while \( \sigma = 0.11, \chi = 45, \kappa = 5, D_0 = 100 \text{ m}, D_E = 20 \text{ m} \).

**TABLE V**

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<td>3.94 \times 10^{-11}</td>
<td>8.03 \times 10^{-11}</td>
<td>8.03 \times 10^{-11}</td>
</tr>
</tbody>
</table>

will be equivalent to \( A_\infty \equiv 4 \). This example is similar to Example 2, in which the eddy viscosity is constant. Consider the steady-state boundary value problem

\[
A_\infty \frac{d^2 w(z)}{dz^2} + 2\kappa^2 i w(z) = -2\kappa^3 i \left( \frac{1 - z}{A_\infty} \right) e^{i\chi} 
\]  

(56)

with time-independent BCs

\[
\frac{dw(0)}{dz} = 0, \quad w(1) = 0.
\]  

(57)

(58)

and the no-slip boundary condition \( \sigma = 0 \).

The exact solution of this problem is \( W(z) = U_0(U(z) + iV(z)) \) where \( U(z) \) and \( V(z) \) are given by

\[
U(z) = R(W_c(z)) \cos(\chi) - I(W_c(z)) \sin(\chi), \]

\[
V(z) = R(W_c(z)) \sin(\chi) - I(W_c(z)) \cos(\chi),
\]

and

\[
W_c(z) = \left( \frac{1 + i}{2} \right) \frac{\sinh \left( (1 - i)\kappa(1 - z) \sqrt{\frac{z}{A_\infty}} \right)}{\sqrt{A_\infty} \cosh \left( (1 - i)\kappa(1 - z) \sqrt{\frac{z}{A_\infty}} \right)}.
\]  

(59)

The results comparing to the exact solution is depicted in Table V. Figure 6, displays the HPES of the current problem for \( N = 4,8,\ldots,64 \) against the exact solution. In Table VI, we compare our results to those in [32].

V. CONCLUSION

In this paper, we applied a Sinc-Collocation approach developed by Abdella [30] to numerically approximate the solution of a 3D oceanography model observed in [18]. The validity, stability and accuracy of our approach is examined by solving...
several examples found in [18], [32] and comparing the results with the exact solutions and those in prior studies. Our results show that the presented Sinc-Collocation approach is very promising in oceanographic problems. In Particular, we would claim that the Sinc-Collocation method is superior to the Galerkin version due to its simple implementation and higher accuracy. As expected, the errors of our method exponentially converges to zero depending on the values of N. In closing, we would claim that the current method is eligible to be a proper alternative to other methods which have been used thus far.

Future research may include an investigation of the model with the time-dependent eddy viscosity which leads to solving partial differential equations using the approach discussed in this paper.

REFERENCES


