On Submaximality in Intuitionistic Topological Spaces

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Abstract—In this study, a minimal submaximal element of \( \text{LIT}(X) \) (the lattice of all intuitionistic topologies for \( X \), ordered by inclusion) is determined. Afterwards, a new contractive property, intuitionistic mega-connectedness, is defined. We show that the submaximality and mega-connectedness are not complementary intuitionistic topological invariants by identifying those members of \( \text{LIT}(X) \) which are intuitionistic mega-connected.

Keywords—Intuitionistic set; intuitionistic topology; intuitionistic submaximality and mega-connectedness.

I. INTRODUCTION

The submaximality was defined and characterized by Bourbaki ([2], [11]). It is worth mentioning that the submaximality is a significant condition for maximal topologies of many topological invariants (e.g. connectedness, quasi-H-closure and pseudo-compactness) [4]. Arhangelskii and Collins have carried out a detailed study of how the submaximality affects the structure of familiar topological spaces and groups [16]. Dontchev has presented several characterizations of submaximality [8]. Also Dontchev and Rose have approached submaximality via topological ideals as characterizations of submaximality [9]. Recently, the submaximality has been studied prominently by many researchers [3,10].

Let \( \text{LT}(X) \) be the family of all topologies definable for an infinite set \( X \) forming a complete atomic and complemented lattice (ordered by inclusion). For a given member \( \tau \) of \( \text{LT}(X) \), having property \( P \), if all members of \( \text{LT}(X) \) weaker (stronger) than \( \tau \) have the property \( P \), then a topological invariant property \( P \) is called contractive (expansive). For a given contractive (expansive) property \( P \), a member \( \tau \) of \( \text{LT}(X) \) is called maximal \( P \) (minimal \( P \)) if \( \tau \) has the property \( P \) but no stronger (weaker) member of \( \text{LT}(X) \) has not property \( P \).

An expansive property \( P \) and a contractive property \( Q \) are called complementary when the minimal \( P \) members of \( \text{LIT}(X) \) coincide with the maximal \( Q \) members.

\[ T_1 \text{ and 'all proper closed sets are finite'; door and 'filter-connected'; } T_3 \text{ and nested; disconnected and principal of order two' are some examples of complementary topological invariants ([4], [12], [13], [17]).} \]

The main purpose of this article is to identify those members of \( \text{LIT}(X) \) (the lattice of all intuitionistic topologies for \( X \), ordered by inclusion) which are minimal submaximal by using the definition of intuitionistic submaximality [15] and show that submaximality and mega-connectedness in Intuitionistic topological spaces are not complementary topological invariants.

II. PRELIMINARIES

The idea of “intuitionistic fuzzy set” was suggested by Krassimir T. Atanassov [1]. Later, D. Coker has presented the classical version of this concept [5]. The definitions which are actively used in this paper, are listed below.

Definition 2.1. Let \( X \) be a nonempty set. An intuitionistic set (IS for short) \( A \) is an object having the form \( A=<X, A_1, A_2, \emptyset> \) (or \( A=<A_1, A_2, \emptyset> \)), where \( A_1 \) and \( A_2 \) are disjoint subsets of \( X \). The set \( A_1 \) is called the set of members of \( A \), while \( A_2 \) is called the set of nonmembers of \( A \) [5,1].

Definition 2.2. Let \( X \) be a nonempty set and \( A \) and \( B \) be in the form \( A=<A_1, A_2, B_1, B_2, \emptyset> \) respectively. Furthermore, let \( \{A_i: i \in I\} \) be an arbitrary family of IS’s in \( X \), where \( A_i=<A_i^{(1)}, A_i^{(2)}> \) then;

a) \( A_i^{(1)} \cap A_j^{(1)} = \emptyset \), \( X = X \setminus \emptyset \)

b) \( A_i \supset B_j \) if \( A_1 \subset B_1 \) and \( A_2 \supset B_2 \)

c) \( A_i \supset B_j \) if \( A_1 \supset B_1 \) and \( A_2 \supset B_2 \)

d) \( A_i^{(1)} = A_j^{(1)}, A_i^{(2)} \) here \( A_i^{(2)} \) complementary of \( A_i^{(1)} \)

e) \( \bigcap A_i^{(1)} \cup A_i^{(2)} \) and \( A_i^{(2)} \supset \bigcup A_i^{(1)}, \bigcup A_i^{(2)} \) [5].

Definition 2.3. An intuitionistic topology (IT for short) on a nonempty set \( X \) is a family \( \tau \) of IS’s in \( X \) containing \( \emptyset \) and \( X \) which is closed under arbitrary unions and finite intersections. In this case the pair \( (X, \tau) \) is called an intuitionistic topological space (ITS for short) and any IS in \( \tau \) is known as an intuitionistic open set (IOS for short) in \( X \), the
complement of such an IOS in X is called an intuitionistic closed set (ICS for short) in X [6,7].

**Example 2.4.** Any topological space \((X,\tau)\) is obviously an intuitionistic topological space with the form; 
\[\tau = \{ A \in \tau \mid A \text{ is an ICS} \}\]
whenever we identify a subset A in X with its counter pair \(A = (A^c, A)\).

**Example 2.5.** Let \(X = \{a, b, c, d\}\) and consider the family \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}\). If we let \(T = \{a, b\}\) and \(T^c = \{c, d\}\), then \(T\) is not minimal in \(X\).

**Example 2.6.** Let \(T_1\) and \(T_2\) be two IT on X. Then \(T_1\) is said to be contained in \(T_2\) if \(G \subseteq T_2\) for each \(G \subseteq T_1\). In this case, we also say that \(T_2\) is coarser than \(T_1\), or \(T_1\) is finer than \(T_2\) [7].

**Definition 3.1.** Let \((X, \tau)\) be an ITS and \(\Delta = \{A_1, A_2\}\) be an IS in \(X\). Then the interior and closure of \(A\) are defined by:
\[\text{int}(A) = \{G \subseteq X \mid |G \subseteq A\}\]
\[\text{cl}(A) = \{K \subseteq X \mid K \supseteq A\}\]
\(\Delta\) is an ICS in \(X\) if \(\text{int}(\Delta) = \emptyset\) and \(\text{cl}(\Delta) = X\).

**Example 2.7.** Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}\). If we let \(B = \{a, c\}\), then \(\text{int}(B) = \emptyset\) and \(\text{cl}(B) = X\), Therefore, \(B\) is a dense IS in \(X\).

**Example 2.8.** Let \((X, \tau)\) be an ITS and \(\Delta = \{A_1, A_2\}\) be an IS in \(X\). The set \(\Delta\) is called \(X\)-dense in \(X\) if \(\text{cl}(\Delta) = X\).

**Example 2.9.** Let \(X = \{a, b, c, d, e\}\) and \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}\). If we let \(B = \{a, c, d\}\), then \(\text{int}(B) = \{a, c, d\}\) and \(\text{cl}(B) = X\), Therefore, \(B\) is a dense IS in \(X\).

**III. SUBMAXIMALITY AND MEGACONNECTEDNESS IN INTUITIONISTIC TOPOLOGICAL SPACES**

A topological space \((X, \tau)\) is submaximal if every dense subset of \(X\) is \(\tau\)-open [2]. The concept of submaximal space was defined by Ozcelik and Narli [15]. In this section the definition of submaximality for an ITS is given. In addition, some characterizations of the submaximality on IT are investigated.

**Definition 3.1.** Let \((X, \tau)\) be an ITS. Then \(\tau \subseteq \text{LT}(X)\) is called the submaximal intuitionistic topology (IT-sub for short) if every \(\tau\)-dense subset of \(X\) is an IOS in \(X\) [15].

**Example 3.2.** Let \(X = \{a, b\}\) and the family \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{\emptyset\}, \{a, b\}\}\) is an IT-sub on \(X\).

**Remark 3.3.** As the following example indicates, it is not necessary that the intuitionistic form of a submaximal space is submaximal.

**Example 3.4.** Let \(X = \{1, 2\}\) and \(\tau = \{\emptyset, X, \{1\}\}\), then the topological space \((X, \tau)\) is submaximal. The intuitionistic form of \(\tau\) is \(\tau = \{\emptyset, X, \{1\}, \{2\}\}\) and \(\Delta = \{1\}\) is a \(\tau\)-dense IS but \(A\) is not an IOS. As a result, \(\tau\) is not an IT-sub on \(X\).

**Theorem 3.5.** The submaximality is an expansive intuitionistic topological invariant.

**Proof.** Let \(\tau_1, \tau_2\) in \(\text{LIT}(X)\) such that \(\tau_1 \subseteq \tau_2\) and \(\tau_2\) is IT-sub. Take \(A\) as a \(\tau_2\)-dense IS then \(A\) is a \(\tau_1\)-dense IS. Since \(\tau_2\) is an IT-sub, \(A\) is a \(\tau_2\)-IOS. Then \(A\) is a \(\tau_1\)-IOS so \(\tau_2\) is an IT-sub.

**Theorem 3.6.** Let \(X\) be a nonempty set and \(A\) be an IS. Then the submaximality of \(M(A) = \{G : G \subseteq X, G \subseteq C, \emptyset \subseteq C\}\) is an IT-sub member of \(\text{LIT}(X)\).

**Proof.** Let us take \(\Delta = \{A_1, A_2\}\).

i) If \(B \subseteq B_1, B_2\) is any IS, since \(\emptyset, A_1 \cup B_1\) is an IOS then \(\text{cl}(B) \subseteq \emptyset, A_2 \cup B_2\).

ii) If \(A_1 \subseteq B_1 \subseteq X\) then \(\text{cl}(B) \subseteq X\).

Since \(A\) is a submaximal IT on \(X\), McCARTAN [12] has shown that \(M(A) = \{G : G \subseteq X, G \subseteq C, \emptyset \subseteq C\}\) is a minimal submaximal submember of \(\text{LIT}(X)\). The following example shows that \(M(A)\) is not a minimal IT-sub member of \(\text{LIT}(X)\).

**Example 3.7.** Let \(X = \{1, 2, 3\}\) and \(\Delta = \{1\}\) be an IS. Consider the IT-sub members of \(\text{LIT}(X)\).

\[M(A) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}\]

\[\emptyset \subseteq \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\]

Since \(\tau\) is coarser than \(M(A), \tau\) is not minimal submaximal IT member of \(\text{LIT}(X)\).

**Theorem 3.8.** Let \(X\) be a nonempty set and \(A\) is a subset of \(X\). The family
\[\delta(A) = \{\emptyset, B : A \subseteq B \subseteq X\} \cup \{\emptyset, C : (A) \subseteq C, D \subseteq C\}\]

is a minimal IT-sub member of \(\text{LIT}(X)\).

**Proof.** First of all we show that \(\delta(A)\) is an IT-sub on \(X\). We look at the following cases:

- Let \(E = \{E_1, E_2\}\) be an IS.
  
  i) If \(E_1 \subseteq A\) then \(\emptyset, A \subseteq E_1\) so \(\text{cl}(E) \subseteq \emptyset, A \supseteq X\)
  
  ii) If \(E \subseteq A\) then either \(\emptyset, A \subseteq E, A \subseteq \emptyset\). This implies that \(\text{cl}(E) \subseteq A \supseteq X\).
iii) If $A \subseteq E_1$ then either $E_1 = X$ in which case $E_1 \subseteq X \in \delta(A)$ or $cl(E) \not\subseteq E_1 \cap \emptyset \neq X$

iv) If $A \cap E_1$ and $E \in \alpha(A)$ then either $E \in \delta(A)$ or $cl(E) \not\subseteq \cap \emptyset \neq X$

These cases show that $\delta(A)$ is an IT-sub.

Let $\tau$ be an IT-sub and $\tau \in \delta(A)$. If any IS $E \in$ in the family $\{<C,D>: A \subseteq C \subseteq D \subseteq C\}$ is $\delta(A)$-dense then $E$ is $\tau$-dense. Since $\tau$ is IT-sub, $E$ is a $\tau$-IOS. On the other hand, all the IS

in the form $<\emptyset,X\{x\}>$ where $x \in A'$ must be $\tau$-IOS, because if $<\emptyset,X\{x\}>$ is not an $\tau$-IOS then the IS $<X\{x\},\emptyset> \in \tau$-dense, this contradicts the submaximality of $\tau$. This implies that all the IS in the family $\{<\emptyset,B>: A \subseteq B \subseteq X\}$ are $\tau$-IOS. From this $\delta(A) = \tau$

**Definition 3.9.** $\tau \in LIT(X)$ is called megaconnected if there exist no IS of X which is both IOS and is "sandwiched" (that is there exist non-empty proper ICS's $E_1$, $E_2$ such that $E_1 \subset A \subset E_2$ between non-empty proper $\tau$-ICS of X).

**Corollary 3.10.** The mega-connectedness is a contractive invariant.

**Proof.** This immediately follows from the definition 3.9.

**Corollary 3.11.** $\delta(A)$ is a megaconnected member of $LIT(X)$.

**Proof.** In the family $\{<\emptyset,B>: A \subseteq B \subseteq X\}$, there is no IOS which contains any ICS different from $\emptyset$ and in the family $\{<\emptyset,D>: A' \subseteq C \subseteq D \subseteq C\}$, there is no IOS which is contained by any ICS different from $X$. This means there is not any IOS in $\delta(A)$ which is sandwiched. Therefore, $\delta(A)$ is megaconnected.

Following example shows that $\delta(A)$ is not a maximal megaconnected member of $LIT(X)$.

**Example 3.12.** Let $X = \{1,2,3,4\}$ and $A = \{4\}$ which is a subset of $X$. Consider the following families:

$\delta(A) = \{<\emptyset,\{4\}>,<\emptyset,\{1,4\}>,<\emptyset,\{2,4\}>,<\emptyset,\{3,4\}>,<\emptyset,\{1,2,4\}>,<\emptyset,\{1,3,4\}>,<\emptyset,\{2,3,4\}>,<\emptyset,\{1,2,3\},\emptyset>,<X,\emptyset>,<\{2,3,4\}>,\omega\}$

$\omega = \delta(A) \cup \{<\{1,2,3,4\}>,<\{1,\},\{4\}>,<\{1,\},\{2,4\}>,<\{1,\},\{3,4\}>,\omega\}$

where $\omega$ is the megaconnected member of $LIT(X)$.

**Result 3.12.** From theorem 3.8, corollary 3.11 and example 3.12, the submaximality and megaconnectedness are not complementary intuitionistic topological invariants.

**Theorem 3.13.** Let X be a non-empty set and A a non-empty subset of X. The family $N(A) = \{<B,C>: A \subseteq C \subseteq B \subseteq C \subseteq \emptyset\}$ is megaconnected member of $LIT(X)$.

**Proof.** Let $\tau \in LIT(X)$ such that $N(A) \subseteq \tau$. Then $\tau$ must contain at least one of the following sets:

$B = <F_1,G_1>$, where $A \subseteq F_2 \subseteq G_1$

We should look at the following cases:

i) If $B \in \tau$ then either $B \in N(A)$ or since IOS’s $<A \cup G$, $\emptyset>$ and $<\emptyset, F_1\cup A>$ are N(A)-open, then these IOS’s are $\tau$-open. This implies that $N(A) \subseteq \tau$-open. Since $<A \cup G\subseteq C \subseteq B$ and $B \subseteq <F_1\cup A, \emptyset>$ implies that IOS B is sandwiched between these IOS’s.

ii) If $C \in \tau$ then either $C = X$ or with the above argument C is sandwiched between IOS’s $<\emptyset, A \cup C\subseteq B$ and $C \subseteq <F_2, \emptyset>$. This show that $\tau \in LIT(X)$ is not megaconnected and N(A) is a maximal megaconnected member of LIT(X).

**IV. Conclusion**

In this study, we have shown that in intuitionistic topological spaces the submaximality and mega-connectedness are not complementary. Further studies can define contractive topological invariant property which is complementary with submaximality in intuitionistic topological spaces.

**REFERENCES**


