A Novel System of Two Coupled Equations for the Longitudinal Components of the Electromagnetic Field in a Waveguide

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Abstract—In this paper, a novel wave equation for electromagnetic waves in a medium having anisotropic permittivity has been derived with the help of Maxwell’s curl equations. The $x$ and $y$ components of the Maxwell’s equations are written with the permittivity ($\epsilon$) being a $3 \times 3$ symmetric matrix. These equations are solved for $E_x$, $E_y$, $H_x$, and $H_y$ in terms of $E_z$, $H_z$, and the partial derivatives. The $Z$ components of the Maxwell’s curl are then used to arrive at the generalized Helmholtz equations for $E_z$ and $H_z$.

Keywords—Electromagnetism, Maxwell’s Equations, Anisotropic permittivity, Wave equation, Matrix Equation, Permittivity tensor.

I. INTRODUCTION

OVER the several decades, anisotropic materials have been extensively analysed and studied as they can successfully describe the real-world configurations. In addition to this, devices incorporating these materials acquire interesting directionally dependent properties which are useful in their operations [1]. In recent years, there have been increasing interests in the interactions between electromagnetic fields and anisotropic media [2]-[20]. Anisotropic medium has wide applications in the design and analysis of various novel antenna and microwave devices of high performance and utility [21]-[25]. With the advancement of the technology, the reconstruction of anisotropic constructive parameters and principal axis in three dimensional problem is of practical importance in microwave range. In the studies of Fedorov, and Borzov, the direction of the principal axis of the anisotropic medium was obtained [24]-[27]. The idea of excluding the transverse components in favour of the longitudinal components of the electromagnetic field in a anisotropic medium has been applied, number of times, in connection with the anisotropic dielectric waveguide or optic fibres [28]-[38]. This work extends this idea for a more general model of a medium. In particular, a novel system of two coupled equations has been presented for the longitudinal components of the electromagnetic field in a waveguide filled with a homogeneous medium characterized by a general symmetric permittivity tensor.

II. WAVE EQUATION WITH ANISOTROPIC PERMITTIVITY

The vector wave equation which describes the wave propagation along a rectangular waveguide with a nonhomogeneous cross section and anisotropic dielectric material may be obtained from the Maxwell equations. The fields are assumed to propagate in the $Z$ direction and have a harmonic time dependence of the form $e^{j\omega t - \gamma z}$ where $\gamma = \alpha + j\beta$, with $\omega$, $\alpha$ and $\beta$ being the angular frequency, the attenuation constant and the phase constant, respectively. The magnetic permeability $\mu$ is assumed to be constant and equal to the free space permeability, i.e., $\mu = \mu_0$. The Maxwell equations are given as follow:

$$\nabla \times E = -j\omega \mu H$$

(1)

$$\nabla \times H = j\omega E$$

(2)

where the electric and magnetic field vectors $E$ and $H$ are given by

$$E = E_x i + E_y j + E_z k,$$

(3)

$$H = H_x i + H_y j + H_z k.$$  

(4)

Here $i$, $j$, and $k$ are the unit vectors along the $x$, $y$ and $z$ directions. The tensor permittivity $\epsilon$ is given by,

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}$$

(5)

Now expanding equation (1) and comparing the coefficients of $i$, $j$ and $k$,

$$\left( \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} \right) \left( \begin{array}{c} E_x \\ E_y \\ E_z \end{array} \right) = -j\omega \mu (H_x i + H_y j + H_z k)$$

(6)

It can be written as follow:

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega \mu H_x \quad \text{or} \quad E_{z,y} + \gamma E_y = -j\omega \mu H_z$$

(7)

$$\frac{\partial E_z}{\partial x} = j\omega \mu H_y \quad \text{or} \quad E_{x,z} + \gamma E_z = j\omega \mu H_y$$

(8)

$$\frac{\partial E_y}{\partial x} = -\frac{\partial E_x}{\partial y} = -j\omega \mu H_z \quad \text{or} \quad E_{y,x} - E_{x,y} = -j\omega \mu H_z$$

(9)

where

$$\frac{\partial}{\partial z} = -\gamma$$

and

$$E_{i,j} = \frac{\partial E_j}{\partial z}, \quad i, j = x, y, z$$

Similarly, from equation (2), we have

$$H_{z,y} + \gamma H_y = j\omega (\epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z)$$

(10)
\[ H_{z,x} + \gamma H_{x} = -j\omega(\epsilon_{zx}E_{x} + \epsilon_{zy}E_{y} + \epsilon_{yz}E_{z}) \]  
\[ H_{y,x} - H_{x,y} = j\omega(\epsilon_{xy}E_{x} + \epsilon_{yx}E_{y} + \epsilon_{zx}E_{z}) \]  
(11)  
(12)

After rearranging equations (7), (8), (10) and (11), we have:

\[ \gamma E_{y} + j\omega H_{x} = -E_{z,y} \]  
\[ \gamma E_{x} - j\omega H_{y} = -E_{z,x} \]  
(13)  
(14)

\[ j\omega_{xx}E_{x} + j\omega_{xy}E_{y} - \gamma H_{x} = H_{z,y} - j\omega_{zx}E_{z} \]  
(15)

\[ j\omega_{yx}E_{x} + j\omega_{yy}E_{y} + \gamma H_{x} = -j\omega_{xy}E_{y} - H_{z,x} \]  
(16)

The above four equations can be written as a matrix equation for the variables \( E_{x}, E_{y}, H_{x} \) and \( H_{y} \) in terms of \( E_{z} \) and \( H_{z} \) and their partial derivatives as follow:

\[
\begin{pmatrix}
0 & \gamma & j\omega \mu & 0 \\
\gamma & 0 & 0 & -j\omega \mu \\
-j\omega_{yx} & j\omega_{xy} & \gamma & 0 \\
-j\omega_{yxy} & j\omega_{xy} & \gamma & 0 \\
\end{pmatrix}
\begin{pmatrix}
E_{x} \\
E_{y} \\
H_{x} \\
H_{y} \\
\end{pmatrix}
= \begin{pmatrix}
-\gamma E_{y} \\
\gamma E_{x} \\
-E_{z,x} \\
-E_{z,y} \\
\end{pmatrix}
\]  
(17)

Or

\[
\begin{pmatrix}
E_{x} \\
E_{y} \\
H_{x} \\
H_{y} \\
\end{pmatrix}
= (A^{-1})^{-1}
\begin{pmatrix}
0 & \gamma & j\omega \mu & 0 \\
\gamma & 0 & 0 & -j\omega \mu \\
-j\omega_{yx} & j\omega_{xy} & \gamma & 0 \\
-j\omega_{yxy} & j\omega_{xy} & \gamma & 0 \\
\end{pmatrix}
\begin{pmatrix}
E_{x} \\
E_{y} \\
H_{x} \\
H_{y} \\
\end{pmatrix}
\]  
(18)

Now, with the help of above components and the equations 9 and 12, we get the following set of equations

\[
\gamma(\gamma^2 + \omega^2 \mu_{xx}) \frac{\partial^2 E_{x}}{\partial y^2} + \omega^2 \mu_{xx} \frac{\partial^2 E_{x}}{\partial x^2} + j\omega^2 \mu_{xy} \frac{\partial E_{x}}{\partial y} - j\omega \mu \gamma_{xy} \frac{\partial E_{x}}{\partial x} = 0 
\]  
(19)

\[
\frac{\partial^2 E_{x}}{\partial y^2} + \omega^2 \mu_{xx} \frac{\partial^2 E_{x}}{\partial x^2} + \gamma(\gamma^2 + \omega^2 \mu_{yy}) \frac{\partial E_{x}}{\partial y} - j\omega \mu \gamma_{xy} \frac{\partial E_{x}}{\partial x} = 0 
\]  
(20)

We write down explicit expression for all the matrix elements of \( B \)

\[
b_{ij} = \frac{(-1)^{i+j}D_{ij}}{\Delta} \]  
(21)

Here,

\[
\Delta = -\omega^4 \left[ \frac{1}{c^2} - \mu(\epsilon_{xx} + \epsilon_{yy}) - \omega^2 (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \right] \\
= -\omega^4 k 
\]  
(22)

\[
k = \frac{1}{c^2} - \frac{\mu(\epsilon_{xx} + \epsilon_{yy})}{c^2} - \omega^2 (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \]  
(23)

Here \( c \) is the speed of light and is equal to \( 3 \times 10^8 \) meter/second and \( k \) is an arbitrary constant. The solution to the transverse components of the electric and magnetic fields is given by

\[
\begin{pmatrix}
E_{x} \\
E_{y} \\
H_{x} \\
H_{y} \\
\end{pmatrix} = B \begin{pmatrix}
-\omega \epsilon_{yx} E_{z} - H_{z,x} - E_{z,y} \\
-\omega \epsilon_{yx} E_{z} + H_{z,x} - E_{z,y} \\
-\omega \epsilon_{yx} E_{z} - H_{z,x} + E_{z,y} \\
-\omega \epsilon_{yx} E_{z} + H_{z,x} + E_{z,y} \\
\end{pmatrix} \]  
(24)

Now the determinant of matrix \( B \) and \( b_{ij} \) are given by

\[
b_{[i=1-4,j=1]} = \begin{pmatrix}
\omega^2 \mu \gamma_{xy} \\
\omega^2 \mu \gamma_{xy} \\
-j\omega \gamma \epsilon_{yy} + j\omega \gamma \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \\
0 \\
\end{pmatrix} \]  
(25)

\[
b_{[i=1-4,j=2]} = \begin{pmatrix}
\omega^2 \mu \gamma_{xy} \\
-j\omega \gamma \epsilon_{yy} + j\omega \gamma \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \\
\omega^2 \mu \gamma_{xy} \\
-j\omega \gamma \epsilon_{yy} + j\omega \gamma \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \\
\end{pmatrix} \]  
(26)

\[
b_{[i=1-4,j=3]} = \begin{pmatrix}
\omega^2 \mu \gamma_{xy} \\
-j\omega \gamma \epsilon_{yy} + j\omega \gamma \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \\
\omega^2 \mu \gamma_{xy} \\
-j\omega \gamma \epsilon_{yy} + j\omega \gamma \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \\
\end{pmatrix} \]  
(27)

\[
b_{[i=1-4,j=4]} = \begin{pmatrix}
\omega^2 \mu \gamma_{xy} \\
-j\omega \gamma \epsilon_{yy} + j\omega \gamma \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \\
\omega^2 \mu \gamma_{xy} \\
-j\omega \gamma \epsilon_{yy} + j\omega \gamma \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \\
\end{pmatrix} \]  
(28)
\[
\begin{align*}
+ j\omega \left[ \gamma^2 \epsilon_{yy} + \omega^2 \mu (\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy} \epsilon_{yx}) \right] \frac{\partial^2 E_z}{\partial y^2} + \\
j \omega \gamma^2 \epsilon_{yz} \frac{\partial^2 E_z}{\partial y \partial z} - \omega^2 \mu \epsilon_{yz} \frac{\partial H_z}{\partial y^2} - j \omega \gamma^2 (\gamma^2 + \omega^2 \mu_{xx}) \frac{\partial^2 H_x}{\partial y^2} - j \omega \gamma^2 (\gamma^2 + \omega^2 \mu_{xx}) \frac{\partial^2 H_y}{\partial y^2} - \gamma (\gamma^2 + \omega^2 \mu_{xx}) \frac{\partial^2 E_z}{\partial y^2} - j \omega \gamma^2 (\gamma^2 + \omega^2 \mu_{xx}) \frac{\partial^2 H_x}{\partial y^2} - j \omega \gamma^2 (\gamma^2 + \omega^2 \mu_{xx}) \frac{\partial^2 H_y}{\partial y^2}
\end{align*}
\]

These are the two wave equations for a rectangular waveguide with anisotropic permittivity.

III. CONCLUSION

We write down the \( x \) and \( y \) components of the Maxwell’s equations

\[
\nabla \times E = -j \omega \mu_0 \mathbf{H}
\]

And

\[
\nabla \times H = j \omega \epsilon_0 \mathbf{E}
\]

Where \( \epsilon \) is a \( 3 \times 3 \) symmetric matrix, given by,

\[
\epsilon = \begin{pmatrix}
\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\
\epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\
\epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}
\end{pmatrix}
\]

We impose the condition that the fields depend on \( z \) as \( e^{j \omega t - \gamma z} \) where \( \gamma = \alpha + j \beta \), \( \omega \), \( \alpha \) and \( \beta \) being the angular frequency, the attenuation constant and the phase constant respectively. Then result is a set of four linear algebraic equations for \( E_z \), \( E_y \), \( H_x \), \( H_y \), \( H_z \), \( H_y \), \( H_x \), \( H_z \), \( H_y \). These are solved yielding \( E_z \), \( E_y \), \( H_x \), \( H_y \) in terms of \( E_z \), \( E_y \), \( H_x \), \( H_y \), \( H_z \), \( H_y \). These expressions are then substituted into the \( z \) components of the \( \nabla \times E \) and \( \nabla \times H \) equations giving thereby a pair of second order coupled linear partial differential equations for \( E_z \), \( H_z \). It is to be noted that \( E_z \) or \( H_z \) can not be set to 0 because one of the pair of equations so obtained may not be satisfied. This is in contrast with the isotropic case.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge Prof. Raj Senani for his constant encouragement and Ar. Rajesh Ayodyhawasi for his support. Authors also gratefully acknowledge the authorities of Netaji Subhash Institute of Technology, New Delhi and Manav Rachna International University, Faridabad, Haryana for the provision of facilities for this research work.

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