Equal sharing solutions for bicooperative games

Fan-Yong Meng and Yan Wang

Abstract—In this paper, we discuss the egalitarianism solution (ES) and center-of-gravity of the imputation-set value (CIV) for bicooperative games, which can be seen as the extensions of the solutions for traditional games given by Dutta and Ray [1] and Driessen and Funaki [2]. Furthermore, axiomatic systems for the given values are proposed. Finally, a numerical example is offered to illustrate the player ES and CIV.

Keywords—bicooperative games, egalitarianism solution, center-of-gravity of the imputation-set value

I. INTRODUCTION


In 2008, Bibao et al. [15] introduced bicooperative games, and researched the Shapley value. Later, Bilbao et al. [16] studied the Banzhaf value for this kind of games. Grabisch and Labreuche [17] proposed another Shapley value for bicooperative games. Following Grabisch and Labreuche [17], Yu and Zhang [18] gave another characterization for the Shapley value given by Grabisch and Labreuche [17], and further studied a special kind of fuzzy games introduced by Tsummi et al. [19].

In this paper, we shall research the ES and CIV for bicooperative games. Based on the existence axiomatic systems, we study characterizations of the given values.

The rest part of this paper is organized as follows. In section 2, some concepts for traditional games are introduced. In section 3, we mainly research the ES and CIV for bicooperative games. Axiomatic systems for the given values are studied. Meantime, some properties are researched. In section 4, a numerical example is given.

II. PRELIMINARIES

Let $N = \{1, 2, ..., n\}$ be a finite set, and $P(N)$ denote the class of all subsets in $N$. The coalitions $P(N)$ in are denoted by $S,T,...$. For any $S \in P(N)$, the cardinality of $S$ is denoted by the corresponding lower case $s$. We will omit braces for singletons, e.g. by writing $S, S \cup \{i\}, S \setminus T, i$ instead of $\{S\}, \{S\} \cup \{i\}, \{S\} \setminus \{T\}, \{i\}$ for any $\{S\}, \{T\}, \{i\} \in P(N)$. A function $v : P(N) \rightarrow \mathbb{R}$, such that $v(\emptyset) = 0$, is called a set function. The set of all set functions in $N$ is denoted by $G(N)$.

Dutta and Ray [1] first introduced egalitarianism for traditional games, which is expressed by

$$\phi_i(N,v) = \frac{v(N)}{n} \quad \forall i \in N. \quad (1)$$

Later, Driessen and Funaki [2] proposed the center-of-gravity of the imputation-set value, which is written as

$$\gamma_i(N,v) = v(i) + \frac{1}{n} \left( v(N) - \sum_{j \in N} v(j) \right) \quad \forall i \in N. \quad (2)$$

III. THE ES AND CIV FOR BICOOPERATIVE GAMES

Let $N = \{1, 2, ..., n\}$ denote the set of players, and define $3^N = \{(S,T) : S,T \in N, S \cap T = \emptyset\}$. A bicooperative game is a pair $(N,v)$ with the player set $N$ and a function $v : 3^N \rightarrow \mathbb{R}$ satisfying $v(\emptyset, \emptyset) = 0$. Let $BG(N)$ denote the set of all bicooperative games on $N$. Grabisch and Lebreuche [17] proposed a relation in $3^N$ as follows:

$$(A,B) \subseteq (C,D) \Leftrightarrow A \subseteq C, B \supseteq D$$

Similar to Dutta and Ray [1], we give the ES for bicooperative games as follows:

$$\beta_i(N,v) = \frac{v(N,\emptyset) - v(\emptyset,N)}{n} \quad \forall i \in N. \quad (3)$$

Fan-Yong Meng is with School of Management, Qingdao Technological University, Qingdao 266033 P.R. China, e-mail: (mengfan_yongtjie@163.com).

Yan Wang is with Business School, Central South University, Changsha, 410083, P.R. China, e-mail: (kmust-wy@sina.com).
Similar to Driessen and Funaki [2], we give the CIV for bicooperative games as follows:
\[
\varphi_i(N, v) = v(i, \emptyset) - v(\emptyset, i) + \left(1 - \frac{1}{n}\right)(v(N, \emptyset) - v(\emptyset, N))
\]
\[
- \left(\sum_{j \in N} (v(j, \emptyset) - v(\emptyset, j))\right) \forall i \in N. \quad (4)
\]

The upper unanimity game \(\bar{u}_{(S,T)}: 3^N \to \mathbb{R}\) is given by
\[
\bar{u}_{(S,T)}(A, B) = \begin{cases} 
1 & (S, T) \subseteq (A, B) \\
0 & \text{otherwise}
\end{cases}
\]
where \((S, T) \in 3^N \setminus \emptyset, \emptyset)\).

The lower unanimity game \(u_{(S,T)}: 3^N \to \mathbb{R}\) is expressed by
\[
u_{(S,T)}(A, B) = \begin{cases} 
1 & (A, B) \subseteq (S, T) \\
0 & \text{otherwise}
\end{cases}
\]
where \((S, T) \in 3^N \setminus \emptyset, \emptyset)\).

Let \(f\) be a solution on bicooperative games. Inspired by van den Brink and Funaki [3], we give the following properties for bicooperative games.

**Efficiency** Let \(v \in BG(N)\), we have
\[
\sum_{i \in N} f_i(N, v) = v(N, \emptyset) - v(\emptyset, N);
\]

**Symmetry** Let \(v \in BG(N)\) and \(i, j \in N\), if we have \(v(S \cup i, T \cup j) = v(S \cup j, T \cup i)\), \(v(S \cup i, T \cup j) = v(S, T \cup j)\) and \(v(S \cup j, T) = v(S, T \cup j)\) for any \((S, T) \in 3^N\) with \(i, j \notin S, T\), then \(f_i(N, v) = f_j(N, v)\);

**Individual Rationality** Let \(v \in BG(N)\), if we have \(\sum_{i \in N} (v(i, \emptyset) - v(\emptyset, i)) \leq v(N, \emptyset) - v(\emptyset, N)\), then
\[
f_i(N, v) \geq v(i, \emptyset) - v(\emptyset, i);
\]

**Additivity** Let \(v \in BG(N)\), if we have \((v + w)(S, T) = v(S, T) + w(S, T)\) for any \((S, T) \in 3^N\), then
\[
f(N, v + w) = f(N, v) + f(N, w);
\]

**Non-Negativity** Let \(v \in BG(N)\) with \(v(S, T) \geq 0\) for any \((S, T) \in 3^N\), then we have \(f_i(N, v) \geq 0\) for any \(i \in N\).

**Theorem 1:** Solution \(f\) satisfies efficiency, symmetry, additivity and non-negativity if and only if \(f = \beta\).

**Proof:** Sufficiency: It is obvious that \(\beta\) satisfies these properties.

Necessity: Let \(f\) be a solution that satisfies the above mentioned properties. Since every bicooperative game \(v \in BG(N)\) can be expressed by
\[
v = \sum_{(S,T) \in 3^N \setminus \emptyset, \emptyset} \delta_{(S,T)} v(S, T).
\]
From additivity, we only need to show \(f = \beta\) for any identity game \(\delta_{(S,T)}\).

Consider identity game \(((N, \emptyset), \beta_{(S,T)})\), where \(S \neq N\). From efficiency, we get \(\sum_{i \in N} f_i((N, \emptyset), \beta_{(S,T)}) = 0\).

From non-negativity, we know
\[
f_i((N, \emptyset), \beta_{(S,T)}) = 0 = \beta_i((N, \emptyset), \beta_{(S,T)}) \quad \forall i \in N.
\]

Consider identity game \((i, (N, \emptyset), \beta_{(S,T)})\). From efficiency, we get
\[
\sum_{i \in N} f_i((N, \emptyset), \delta_{(N, \emptyset)}) = 1.
\]

From symmetry, we have
\[
f_i((N, \emptyset), \delta_{(N, \emptyset)}) = \frac{1}{n} = \delta_i((N, \emptyset), \delta_{(N, \emptyset)}) \quad \forall i \in N.
\]

Similarly, we have
\[
f_i((\emptyset, N), \delta_{(\emptyset, N)}) = -\frac{1}{n} = \delta_i((\emptyset, N), \delta_{(\emptyset, N)}) \quad \forall i \in N
\]
and
\[
f_i((\emptyset, N), \delta_{(S,T)}) = 0 = \beta_i((\emptyset, N), \delta_{(S,T)}) \quad \forall i \in N
\]
where \(T \neq N\).

**Theorem 2:** Solution \(f\) satisfies efficiency, symmetry, additivity and individual rationality if and only if \(f = \varphi\).

**Proof:** Sufficiency: It is obvious that \(\varphi\) satisfies these properties.

Necessity: Let \(f\) be a solution that satisfies the above mentioned properties. Consider the upper unanimity game \((N, \emptyset), \bar{u}_{(S,T)}\), where \(s = 1\).

From individual rationality, we get \(f_i((N, \emptyset), \bar{u}_{(S,T)}) \geq 0\) for any \(i \in N \setminus S\), and \(f_i((\emptyset, N), \bar{u}_{(S,T)}) \geq 1\) for any \(i \in S\).

Form efficiency, we get
\[
f_i((N, \emptyset), \bar{u}_{(S,T)}) = 0 = \varphi_i((N, \emptyset), \bar{u}_{(S,T)})
\]
for any \(i \in N \setminus S\), and
\[
f_i((\emptyset, N), \bar{u}_{(S,T)}) = 1 = \varphi_i((\emptyset, N), \bar{u}_{(S,T)})
\]
for any \(i \in S\).

Similarly, when \(t = 1\), we have
\[
f_i((\emptyset, N), \bar{u}_{(0,T)}) = 0 = \varphi_i((\emptyset, N), \bar{u}_{(0,T)}) \quad \forall i \in N \setminus T
\]
and
\[
f_i((\emptyset, N), \bar{u}_{(0,T)}) = -1 = \varphi_i((\emptyset, N), \bar{u}_{(0,T)}) \quad \forall i \in T.
\]

Next, consider the identity game \((N, \emptyset), \delta_{(S,T)}\) with \(2 \leq s + t \leq n - 1\). From individual rationality, we get \(f_i((N, \emptyset), \delta_{(S,T)}) \geq 0\) for all \(i \in N\).

From efficiency, we obtain \(\sum_{i \in N} f_i((N, \emptyset), \delta_{(S,T)}) = 0\).

Thus, we have
\[
f_i((N, \emptyset), \delta_{(S,T)}) = 0 = \varphi_i((N, \emptyset), \delta_{(S,T)})
\]
for all \(i \in N\).

Similarly, we have
\[
f_i((\emptyset, N), \delta_{(S,T)}) = 0 = \varphi_i((\emptyset, N), \delta_{(S,T)})
\]
for all \(i \in N\), where \(2 \leq s + t \leq n - 1\).
For the identity games \((N, \emptyset), \delta_{(N, \emptyset)}\) and \(((\emptyset, N), \delta_{(\emptyset, N)})\),
from efficiency and symmetry, we get
\[
f_i((N, \emptyset), \delta_{(N, \emptyset)}) = \frac{1}{n} = \varphi_i((N, \emptyset), \delta_{(N, \emptyset)}) \quad \forall i \in N
\]
and
\[
f_i((\emptyset, N), \delta_{(\emptyset, N)}) = \frac{-1}{n} = \varphi_i((\emptyset, N), \delta_{(\emptyset, N)}) \quad \forall i \in N.
\]
Since every bicooperative game \(v \in BG(N)\) can be expressed by
\[
v = \sum_{i \in N} (v(i, \emptyset) u_i - v(\emptyset, i) u_{i(0,i)}) + \sum_{(S, \emptyset) \in N^s, s \geq 2} \left( v(S, \emptyset) - \sum_{i \in S} v(i, \emptyset) \right) \delta_{(S, \emptyset)} + \sum_{(\emptyset, T) \in N^t, t \geq 2} \left( v(\emptyset, T) + \sum_{i \in T} v(\emptyset, i) \right) \delta_{(\emptyset, T)} + \sum_{(S, T) \in N^{s+t}, s+t \geq 2, s, t \geq 1} v(S, T) \delta_{(S, T)}.
\]
From additivity, we get the conclusion.

**Definition 1:** Let \(v \in BG(N)\), if we have \(v(S \cup T, K) \geq v(S, K) + v(T, K)\) and \(v(K, S \cup T) \leq v(K, S) + v(K, T)\) for any \(S, T, K \subseteq N\), where \(S \cap T, S \cap K, T \cap K = \emptyset\). Then, \(v \in BG(N)\) is said to be a superadditive bicooperative game.

**Definition 2:** Let \(v \in BG(N)\), if we have \(\frac{v(S, K)}{s} \geq \frac{v(K, T)}{k}\) and \(\frac{v(T, S)}{s} \leq \frac{v(T, K)}{k}\) for any \(S, T, K \subseteq N\), where \(S \cap T, S \cap K, T \cap K = \emptyset\) and \(s \geq k\). Then, \(v \in BG(N)\) is said to be an average monotonic bicooperative game.

**Definition 3:** Let \(v \in BG(N)\). The vector \(x = (x_1, x_2, ..., x_n)\) is called an imputation for \(v \in BG(N)\) if it satisfies
1. \(x_i \geq v(i, \emptyset) - v(\emptyset, i) \quad \forall i \in N\);
2. \(\sum_{i \in N} x_i = v(N, \emptyset) - v(\emptyset, N)\).

**Theorem 3:** Let \(v \in BG(N)\). If \(v\) is average monotonic, then \((\varphi_i(N, v))_{i \in N}\) is an imputation.

**Proof:** From Definition 2, we get
\[
\frac{v(N, \emptyset) - v(\emptyset, N)}{n} \geq v(i, \emptyset) - v(\emptyset, i).
\]
From Eq.(3), we have \((\varphi_i(N, v))_{i \in N}\) is an imputation.

**Theorem 4:** Let \(v \in BG(N)\). If \(v\) is superadditive, then \((\varphi_i(N, v))_{i \in N}\) is an imputation.

**Proof:** From Definition 3, we have
\[
v(N, \emptyset) - v(\emptyset, N) \geq \sum_{i \in N} (v(i, \emptyset) - v(\emptyset, i)).
\]
From Eq.(4), we get
\[
\varphi_i(N, v) = v(i, \emptyset) - v(\emptyset, i) + \frac{1}{n} (v(N, \emptyset) - v(\emptyset, N)) \geq v(i, \emptyset) - v(\emptyset, i) \quad \forall i \in N.
\]
Furthermore, we have
\[
\sum_{i \in N} \varphi_i(N, v) = \sum_{i \in N} v(i, \emptyset) - v(\emptyset, i) + \frac{1}{n} (v(N, \emptyset) - v(\emptyset, N)) - \left( \sum_{j \in N} v(j, \emptyset) - v(\emptyset, j) \right) = \sum_{i \in N} v(i, \emptyset) - v(\emptyset, i) + v(N, \emptyset) - \left( \sum_{j \in N} v(j, \emptyset) - v(\emptyset, j) \right) = v(N, \emptyset) - v(\emptyset, N).
\]

**IV. Numerical Example**

There are three companies who produce the same products. Denoted by the player \(1, 2, 3\), respectively. Namely, the player set is \(N = \{1, 2, 3\}\). If they compete, their incomes will reduce. If they cooperate, all of them are not willing to get the incomes by cooperation. There exist both competition and cooperation among them. The payoffs got by cooperation are \(v(1, \emptyset) = 2, v(2, \emptyset) = 3, v(3, \emptyset) = 2, v(1, 2, 3, \emptyset) = 12\), and the payoffs obtained by competition are \(v(\emptyset, 1) = 1, v(\emptyset, 2) = 1, v(\emptyset, 3) = 1, v(\emptyset, 1, 2, 3) = 3\).

From Eq.(3), we obtain the player ES are \(\beta_1(N, v) = 3\) \((i = 1, 2, 3)\).

From Eq.(4), we get the player CIV are \(\varphi_1(N, v) = \varphi_2(N, v) = 8/3\) and \(\varphi_3(N, v) = 11/3\).

**V. Conclusion**

We have studied the ES and CTV for bicooperative games, which can be seen as the extensions for the ES and CIS-value for traditional games. Like other payoff induces, there exist other axiomatic systems for the ES and CTV for bicooperative games.

Furthermore, similar to the ENSC-value given by Driessen and Funaki [2], we can get the ENSC-value for bicooperative games, which can be expressed by
\[
\alpha_i(N, v) = v(\emptyset, N\setminus i) - v(N\setminus i, \emptyset) + \frac{1}{n} (v(N, \emptyset) - v(\emptyset, N) + \left( \sum_{j \in N} v(N\setminus j, \emptyset) - v(\emptyset, N\setminus j) \right) \quad \forall i \in N.
\]
By establishing axiomatic system, we can show the above given value is unique.

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