A neighborhood condition for fractional $k$-deleted graphs

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Abstract—Let $k \geq 3$ be an integer, and let $G$ be a graph of order $n$ with $n \geq 9k + 3 - 4\sqrt{2(k-1)^2 + 2}$. Then a spanning subgraph $F$ of $G$ is called a $k$-factor if $d_F(x) = k$ for each $x \in V(G)$. A fractional $k$-factor is a way of assigning weights to the edges of a graph $G$ (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is $k$. A graph $G$ is a fractional $k$-deleted graph if there exists a fractional $k$-factor after deleting any edge of $G$. In this paper, it is proved that $G$ is a fractional $k$-deleted graph if $G$ satisfies $\delta(G) \geq k + 1$ and $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices $x, y \in V(G)$.

Keywords—graph, minimum degree, neighborhood union, fractional $k$-factor, fractional $k$-deleted graph.

I. INTRODUCTION

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let $G$ be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of $x$ in $G$ and by $N_G(x)$ the set of vertices adjacent to $x$ in $G$, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, $N_G(S) = \cup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of $G$ induced by $S$, and $G - S = G[V(G) \setminus S]$. We say that $S$ is independent if $N_G(S) \cap S = \emptyset$. Let $S$ and $T$ be disjoint subsets of $V(G)$. We use $e_G(S, T)$ to denote the number of edges joining $S$ and $T$ in $G$. The minimum vertex degree of $G$ is denoted by $\delta(G)$.

Let $k$ be a positive integer. Then a spanning subgraph $F$ of $G$ is called a $k$-factor if $d_F(x) = k$ for each $x \in V(G)$. If $k = 1$, then a $k$-factor is simply called a 1-factor. A fractional $k$-factor is a way of assigning weights to the edges of a graph $G$ (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is $k$. If $k = 1$, then a fractional $k$-factor is a fractional 1-factor. A graph $G$ is a fractional $k$-deleted graph if there exists a fractional $k$-factor after deleting any edge of $G$. If $k = 1$, then a fractional $k$-deleted graph is a fractional 1-deleted graph. Some other terminologies and notations can be found in [1,2].

Many authors have studied graph factors [3-8]. Many authors have investigated fractional $k$-factors [9–12] and fractional $k$-deleted graphs [13,14]. The following results on $k$-factors, fractional $k$-factors and fractional $k$-deleted graphs are known.

Theorem 1 [15] Let $k$ be an integer such that $k \geq 2$, and let $G$ be a connected graph of order $n$ such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, $kn$ is even, and the minimum degree is at least $k$. If $G$ satisfies $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then $G$ has a $k$-factor.

Theorem 2 [14] Let $k$ be an integer such that $k \geq 2$, and let $G$ be a connected graph of order $n$ such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, and the minimum degree $\delta(G) \geq k$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then $G$ has a fractional $k$-factor.

Theorem 3 [16] Let $k \geq 2$ be an integer. Let $G$ be a connected graph of order $n$ with $n \geq 13k + 1 - 4\sqrt{2(k-1)^2 + 2}$, $\delta(G) \geq k + 2$, and $\delta(G) \geq k$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then $G$ is a fractional $k$-deleted graph.

The purpose of this paper is to weaken the conditions on the order, minimum degree and connectivity of $G$ in Theorem 3. The main result is the following theorem.

Theorem 4 Let $k \geq 3$ be an integer. Let $G$ be a graph of order $n$ with $n \geq 9k + 3 - 4\sqrt{2(k-1)^2 + 2}$, $\delta(G) \geq k + 1$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then $G$ is a fractional $k$-deleted graph.

II. THE PROOF OF THEOREM 4

The following result is essential to the proof of our main theorem.

Lemma 2.1 [17] A graph $G$ is a fractional $k$-deleted graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus (S, d_{G-S}(x) \leq k)\}$

$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S, T)$,

where $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $\varepsilon(S, T)$ is defined as follows,

$\varepsilon(S, T) = \begin{cases} 2, & \text{if} \ T \text{ is not independent,} \\ 1, & \text{if} \ T \text{ is independent, and} \\ e_G(T, V(G) \setminus (S \cup T)) \geq 1, \\ 0, & \text{otherwise.} \end{cases}$

Proof of Theorem 4. Let $G$ be a graph satisfying the hypothesis of Theorem 4, we prove the theorem by contradiction.
Suppose that $G$ is not a fractional $k$-deleted graph. Then by Lemma 2.1, there exists a subset $S$ of $V(G)$ such that
\[
\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S, T) - 1 , \tag{1}
\]
where $T = \{ x : x \in V(G) \setminus S, d_{G-S}(x) \leq k \}$. Firstly, we prove the following claims.

**Claim 1.** $S \neq \emptyset$.

**Proof.** Note that $\varepsilon(S, T) \leq |T|$. If $S = \emptyset$, then by (1) we have
\[
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\
= d_G(T) - k|T| \geq (\delta(G) - k)|T| \\
\geq |T| \geq \varepsilon(S, T) .
\]
It is a contradiction. This completes the proof of Claim 1.

**Claim 2.** $|T| \geq k + 1$.

**Proof.** Assume that $|T| \leq k$. Then from (1) and $|S| + d_{G-S}(x) - k \geq d_G(x) - k \geq \delta(G) - k \geq 1$, we get
\[
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\
\geq |T||S| + d_{G-S}(T) - k|T| \\
= \sum_{x \in T}(|S| + d_{G-S}(x) - k) \\
\geq |T| \geq \varepsilon(S, T) .
\]
That is a contradiction. This completes the proof of Claim 2.

**Claim 3.** $|T| \geq |S| + 1$.

**Proof.** Let $|T| \leq |S|$. Then by (1), we obtain
\[
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq d_{G-S}(T) . \tag{2}
\]
On the other hand, according to the definition of $\varepsilon(S, T)$, we have
\[
d_{G-S}(T) \geq \varepsilon(S, T) ,
\]
which contradicts (2). The proof of Claim 3 is complete.

**Claim 4.** $|S| \leq n - 1$.

**Proof.** In terms of Claim 3 and $|S| + |T| \leq n$, we have
\[
n \geq |S| + |T| \geq 2|S| + 1 ,
\]
that is,
\[
|S| \leq \frac{n - 1}{2} .
\]
The proof of Claim 4 is complete.

In terms of Claim 2, $T \neq \emptyset$. Now we define
\[
h_1 = \min \{ d_{G-S}(x) : x \in T \}
\]
and choose $x_1 \in T$ such that $d_{G-S}(x_1) = h_1$. Clearly, we have $0 \leq h_1 \leq k$. In the following, we consider two cases.

**Case 1.** $T = N_T(x_1)$.

Using Claim 2, $T = N_T(x_1)$ and $0 \leq h_1 \leq k$, we obtain
\[
k \geq h_1 = d_{G-S}(x_1) \geq |T| - 1 \geq k ,
\]
which implies
\[
h_1 = k . \tag{3}
\]
In terms of (3) and Claim 1, we get
\[
\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\
\geq k|S| + h_1|T| - k|T| = k|S| \\
\geq k > 2 \geq \varepsilon(S, T) .
\]
That contradicts (1).

**Case 2.** $T \setminus N_T[x_1] \neq \emptyset$.

Define
\[
h_2 = \min \{ d_{G-S}(x) : x \in T \setminus N_T[x_1] \} .
\]
We choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{G-S}(x_2) = h_2$. Obviously, $0 \leq h_1 \leq h_2 \leq k$ and $x_1 x_2 \notin E(G)$. According to the hypothesis of Theorem 4, we have
\[
\frac{n + k - 2}{2} \leq |N_G(x_1) \cup N_G(x_2)| \\
\leq d_{G-S}(x_1) + d_{G-S}(x_2) + |S| \\
= h_1 + h_2 + |S|,
\]
which implies
\[
|S| \geq \frac{n + k - 2}{2} - h_1 - h_2 . \tag{4}
\]
By (4) and Claim 4, we obtain
\[
\frac{n - 1}{2} \geq \frac{n + k - 2}{2} - h_1 - h_2 ,
\]
that is,
\[
h_1 + h_2 \geq \frac{k - 1}{2} . \tag{5}
\]
In terms of (5), $k \geq 3$, $0 \leq h_1 \leq h_2 \leq k$ and the integrity of $h_2$, we get
\[
h_2 \geq 1 . \tag{6}
\]

**Claim 5.** $0 \leq h_1 \leq k - 1$.

**Proof.** If $h_1 = k$, then by (1) and Claim 1 we get
\[
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\
\geq k|S| + h_1|T| - k|T| = k|S| \geq k \\
\geq 2 \geq \varepsilon(S, T) ,
\]
which is a contradiction. This completes the proof of Claim 5.

Note that
\[
|N_T[x_1]| \leq d_{G-S}(x_1) + 1 = h_1 + 1 . \tag{7}
\]
From (4), (7), $0 \leq h_1 \leq h_2 \leq k$ and $|S| + |T| \leq n$, we have
\[
\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\
\geq k|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) \\
- k|T| \\
= k|S| - (h_2 - h_1)|N_T[x_1]| - (k - h_2)|T| \\
\geq k|S| - (h_2 - h_1)(h_1 + 1) \\
- (k - h_2)n \\
= (2k - h_2)|S| - (h_2 - h_1)(h_1 + 1) \\
- (k - h_2)n \\
\geq (2k - h_2)(\frac{n + k - 2}{2} - h_1 - h_2) \\
-(h_2 - h_1)(h_1 + 1) - (k - h_2)n ,
\]
Thus, we obtain
\[ 0 \geq k - h_1 - h_2 \]
which implies
\[ G \]
is a contradiction.

**Proof of Theorem 4.**

According to (6), (11), \( \delta_G(S, T) \leq 2 \), we get
\[ 1 \geq \epsilon(S, T) - 1 \geq \delta_G(S, T) \geq (2k - h_2)(n + k - 2)/2 - 2h_2 - (k - h_2)n \]
\[ = \frac{1}{2}(4h_2^2 + (n - 9k + 2)h_2 + 2k^2 - 4k) \]
which implies
\[ 4h_2^2 + (n - 9k + 2)h_2 + 2k^2 - 4k - 2 \leq 0. \]  

**Claim 6.** For \( k \geq 3 \), we have
\[ \sqrt{\frac{(k-1)^2+1}{2}} - 1 > \frac{1}{2}. \]

**Proof.** Since \( k \geq 3 \), we have
\[ \frac{(k-1)^2+1}{2} \geq \frac{5}{2} > \frac{9}{4}, \]
that is,
\[ \sqrt{\frac{(k-1)^2+1}{2}} > \frac{3}{2}. \]

Thus, we obtain
\[ \sqrt{\frac{(k-1)^2+1}{2}} - 1 > \frac{1}{2}. \]

The proof of Claim 6 is complete.

According to (6), (11), \( n \geq 9k + 3 - 4\sqrt{2(k-1)^2+2}, \)
\( k \geq 3 \) and Claim 6, we obtain
\[ 0 \geq 4h_2^2 + (n - 9k + 2)h_2 + 2k^2 - 4k - 2 \]
\[ \geq 4h_2^2 + (4\sqrt{2(k-1)^2+2} + 5)h_2 + 2k^2 - 4k - 2 \]
\[ \geq 4h_2^2 - 8\sqrt{\frac{(k-1)^2+1}{2}}h_2 + 2(k-1)^2 + 2 + 5h_2 - 6 \]
\[ = 4\left(\sqrt{\frac{(k-1)^2+1}{2}} - h_2\right)^2 + 2h_2 - 6 \]
\[ \geq 4\left(\sqrt{\frac{k-1}{2}} - 1\right)^2 - 1 \]
\[ > 4\left(\frac{1}{2}\right)^2 - 1 \geq 0, \]
which is a contradiction.

From all the cases above, we deduce the contradictions. Hence, \( G \) is a fractional \( k \)-deleted graph. This completes the proof of Theorem 4.

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