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S-Fuzzy Left h-Ideal of Hemirings

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***Abstract*—** The notion of S-fuzzy left h-ideals in a hemiring is introduced and it's basic properties are investigated.We also study the homomorphic image and preimage of S-fuzzy left h-ideal of hemirings.Using a collection of left h-ideals of a hemiring, S-fuzzy left h-ideal of hemirings are established.The notion of a finite-valued S-fuzzy left h-ideal is introduced,and its characterization is given.S-fuzzy relations on hemirings are discussed.The notion of direct product and S-product are introduced and some properties of the direct product and S-product of S-fuzzy left h-ideal of hemiring are also discussed.

***Keywords*—** hemiring,left h-ideal,anti fuzzy h-ideal,S-fuzzy left h-ideal,t-conorm , homomorphism.

I. Introduction

THE concept of fuzzy subset was introduced by L.A.Zadeh [8]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. B.Schweizer and A.Sklar [5,6] introduced the notions of Triangular norm (t-norm) and Triangular conorm (t-conorm).Triangular norm (t-norm) and Triangular conorm (t-conorm or s-norm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively.The ideal theory plays an important role in algebraic structure.La Torre [7] studied the notion of h-ideals and k-ideals in hemirings.Then Y.B Jun et. al[4] introduced the notion of fuzzy h-ideal of hemirings and discussed related properties.First,Abu Osman [1] introduced the notion of fuzzy subgroup with respect to t-norm. Following this,J.Zhan [9] introduced the notion of T-fuzzy left h-ideal of hemirings.Then,J.Zhan [10] introduced the notion of fuzzy hyper ideals in hyper near-rings with respect to t-norm.Recently,Y.U Cho et. al[3] introduced the notion of fuzzy subalgebras with respect to t-conorm of BCK-algebras and M.Akram et. al.[2] introduced the notion of sensible fuzzy ideal with respect to t-conorm in BCK-algebras.Using the idea of [2] and [3],In this paper we introduce the notion of S-fuzzy left h-ideal of hemirings and investigate it is related properties.Also,we review several results described in [9] using t-conorm.

II. Preliminaries

An algebra $(R;+,.)$ is said to be a *semiring* if $(R;+)$ and $(R;.)$ are semigroups satisfying $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$ for all $a,b,c \in R$.A semiring R is said to be *additively commutative* if $a + b = b + a$ for all $a,b,c \in S$. A semiring R may have an identity 1, defined by $1.a = a = a.1$ and a zero 0, defined by $0 + a = a = a + 0$ and $a.0 = 0 = 0.a$ for all $a \in R$.A semiring R is said to be a *hemiring* if it is an additively commutative with zero.A non-empty subset I of R is said to be a *left (resp., right ideal)* if $x, y \in I$ and $r \in R$ imply that $x + y \in I$ and $rx \in I$ $(resp., xr \in I)$. If I is both left and right ideal of R , we say I is a *two-sided ideal, or simply ideal*, of R. A left ideal I of a semiring R is said to be a *k-ideal* if $a \in I$ and $x \in R$, and if $x + a \in I$ or $a + x \in I$ then $x \in I$. Right k-ideal is defined dually, and two-sided k-ideal or simply a k-ideal is both a left and a right k-ideal.A left ideal I of a hemiring R is called a *left h-ideal* if $x + a + z = b + z$ implies that $x \in I$ for all $x, y \in R$ and $a, b \in R$.Right h-ideals are defined similarly.

Definition 2.1: Let X be a non-empty set. A fuzzy subset of X is a function $\mu : X \to [0,1]$. Let μ be the fuzzy subset of a set X. For a fixed $0 \leq t \leq 1$, the set

$$L(\mu;t) = \{x \in X : \mu(x) \leq t\}$$

is called a *lower level set* or simply *level set* of μ.

Definition 2.2: A fuzzy subset μ of a hemiring R is said to be *fuzzy left (resp., right) ideal* of R if
$(FI1)\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and
$(FI2)\mu(xy) \geq \mu(y) \quad (resp., \mu(xy) > \mu(x))$
for all $x, y \in R$.
If μ is a *fuzzy ideal* of R if it is both fuzzy left and a fuzzy right ideal of R.

Definition 2.3: A fuzzy subset μ of a hemiring R is said to be an *anti fuzzy left (resp., right) ideal* of R if
$(FI1)\mu(x+y) \leq \max\{\mu(x), \mu(y)\}$ and
$(FI2)\mu(xy) \leq \mu(y) \quad (resp., \mu(xy) \leq \mu(x))$
for all $x, y \in R$.
If μ is an *anti fuzzy ideal* of R if it is both an anti fuzzy left and anti fuzzy right ideal of R.

Definition 2.4: Let R and R' be hemirings. A mapping $f : R \to R'$ is said to be a *homomorphism* if
$f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$
for all $x, y \in R$.

Definition 2.5: A fuzzy subset μ of a hemiring R is said to be a *fuzzy left (resp., right) h-ideal* of R if
$(AFI1)\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and
$(AFI2)\mu(xy) \geq \mu(y) \quad (resp., \mu(xy) \geq \mu(x))$
for all $x, y \in R$.
$(AFI3)$ If $x + a + z = b + z$ implies that
$\mu(x) \geq min\{\mu(a), \mu(b)\}$, for all $a, b, x, z \in S$.
If μ is *fuzzy h-ideal* of R if it is both a fuzzy left and fuzzy right h-ideal of R.

Definition 2.6: A fuzzy subset μ of a hemiring R is said to be an *anti fuzzy left (resp., right) h-ideal* of R if
$(AFI1)\mu(x+y) \leq \max\{\mu(x), \mu(y)\}$ and
$(AFI2)\mu(xy) \leq \mu(y) \quad (resp., \mu(xy) \leq \mu(x))$

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for all $x, y \in R$.
$(AFI3)$ If $x + a + z = b + z$ implies that
$\mu(x) \leq max\{\mu(a), \mu(b)\}$, for all $a, b, x, z \in S$.
If μ is an *anti fuzzy h-ideal* of R it is both an anti fuzzy left h-ideal and anti fuzzy right h-ideal of R.

Definition 2.7: A triangular conorm (t-conorm) is a mapping $S : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ that satisfies the following conditions:
$(S1)\, S(x, 0) = x$,
$(S2)\, S(x, y) = S(y, x)$,
$(S3)\, S(x, S(y, z)) = S(S(x, y), z)$,
$(S4)\, S(x, y) \leq S(x, z)$ whenever $y \leq z$,
for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition S, we obtain the concept of t-norm T.

Proposition 2.8: For a t-conorm S.Then the following statement holds $S(x, y) \geq max(x, y)$, for all $x, y \in [0, 1]$.

Definition 2.9: Let S be a t-conorm. A fuzzy subset μ in a hemiring R is called *sensible* with respect to S
if $Im\mu \subseteq \triangle_S$,where $\triangle_S = \{t \in [0, 1] | S(t, t) = t\}$.

III. S-FUZZY LEFT H-IDEALS IN HEMIRINGS

In what follows,R and S denotes a hemiring and t-conorm respectively, unless otherwise specified.

Definition 3.1: A fuzzy subset μ of R is called a ***S-fuzzy left ideal*** of a hemiring R (briefly, fuzzy left ideal with respect to t-conorm) if it satisfies the following conditions:
$(SFI1)\mu(x + y) \leq S(\mu(x), \mu(y))$,
$(SFI2)\mu(xy) \leq \mu(y)$,for all $x, y \in S$.
S-fuzzy right ideals are defined similarly.

Definition 3.2: A S-fuzzy ideal μ of R is said to be a ***S-fuzzy left h-ideal*** if it satisfies the following condition:
$(SFI3)x+a+z = b+z$ implies that $\mu(x) \leq S(\mu(a), \mu(b))$, for all $a, b, x, z \in S$.
S-fuzzy right h-ideals are defined similarly.

Definition 3.3: A S-fuzzy left h-ideal μ of R is said to be a *sensible* if it satisfies the sensible property.

Example 3.4: Let R be the set of natural numbers including 0, and R is a hemiring with usual addition and multiplication .Define a fuzzy subset $\mu : R \longrightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 0 & if\ x\ \ is\ even\ or\ 0, \\ 1 & otherwise. \end{cases}$$

and let $S_m : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be a function defined by $S_m(\alpha, \beta) = min\{x + y, 1\}$ for all $x, y \in [0, 1]$.Then, S_m is a t-conorm.By routine calculation, we know that μ is a sensible S-fuzzy left h-ideal of R.

Proposition 3.5: Let S be a t-conorm .Then, every sensible S-fuzzy left h-ideal μ of a hemiring R is a anti fuzzy left h-ideal of R.

Proof: The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [9]. ■

Corollary 3.6: If μ is a sensible S-fuzzy left h-ideal of R, then each non-empty level subset $L(\mu; t)$ of μ is a left h-ideal of R.

Proof: Assume that μ is a sensible S-fuzzy left h-ideal of R and $L(\mu; t)$ is a non-empty level subset of μ in R.
(i) Since $L(\mu; t)$ is a non-empty level subset of μ, there exists $x, y \in L(\mu; t)$, $\mu(x + y) \leq S(\mu(x), \mu(y)) = t$.
Thus $x + y \in L(\mu; t)$.
(ii) Let $x, y \in L(\mu; t)$, such that $\mu(xy) \leq \mu(y) \leq t$.
Thus $xy \in L(\mu; t)$.
(iii) Let $a, b, x, z \in L(\mu; t)$, If $x + a + z = b + z$ implies that $\mu(x) \leq S(\mu(a), \mu(b)) = t$.Thus $x \in L(\mu; t)$

Hence,$L(\mu; t)$ is a left h-ideal of R. ■

The following example shows that there exists a t-conorm S such that an anti fuzzy h-ideal of R may not be an sensible S-fuzzy left h-ideal of R.

Example 3.7: Let R be a hemiring in $Example[3.4]$.Define a fuzzy subset $\mu : R \longrightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} \frac{1}{5} & if\ x\ \ is\ even\ or\ 0, \\ \\ \frac{1}{3} & otherwise. \end{cases}$$

is an anti fuzzy h-ideal of R.
Let $\nu = (0, 1)$ and define the binary operation S_ν on $(0, 1)$ as follows

$$S_\nu(\alpha, \beta) = \begin{cases} \max\{\alpha, \beta\} & if\ \ \min\{\alpha, \beta\} =\ 0, \\ 0 & \max\{\alpha, \beta\} > 0,\ \alpha + \beta \geq 1 + \nu \\ \nu & otherwise. \end{cases}$$

Then,S_ν is a t-conorm.It is easy to check that μ is a S-fuzzy left h-ideal of R, but

$$S_\nu\left(\mu(0), \mu(0)\right) = S_\nu\left(\frac{1}{5}, \frac{1}{5}\right) = \nu \neq \mu(0).$$

Hence,μ is not a sensible S-fuzzy left h-ideal R.

Now, we consider the following theorem.

Theorem 3.8: Let S be a t-conorm and let μ be a sensible fuzzy subset in a hemiring R, then μ is a sensible S-fuzzy left h-ideal of R if and only if each non-empty level subset $L(\mu; t)$ of μ is a left h-ideal of R.

Proof: The necessary condition can be given by $corollary[3.6]$.Coversely,assume that each non-empty level subset $L(\mu; t)$ is a left h-ideal of R.
(i) Let $x, y \in R$.Let if possible,$\mu(x + y) > S(\mu(x), \mu(y))$.Set $t_0 := \frac{1}{2}\{\mu(x + y) + S(\mu(x), \mu(y))\}$,we have $x \in L(\mu; t_0)$ and $y \in L(\mu; t_0)$,since $L(\mu; t)$ is a left h-ideal of R.Then $x + y \in L(\mu; t_0)$ and $\mu(x + y) \leq t_0$,a contradiction.Thus $\mu(x + y) \leq S(\mu(x), \mu(y))$.
(ii) If $x, y \in L(\mu; t)$ then $xy \in L(\mu; t)$.Then
$\mu(xy) \leq \mu(y) \leq t$.Thus $\mu(xy) \leq \mu(y)$.
(iii) Let $a, b, x, z \in R$.If $x + a + z = b + z$ implies that $x \in L(\mu; t)\}$.Define $t = min\{\mu(a), \mu(b)\}$.Then $\mu(x) \leq t = min\{\mu(a), \mu(b)\}$. Thus $\mu(x) =\leq max\{\mu(a), \mu(b)\}$.
Hence,μ is a sensible S-fuzzy left h-ideal of R. ■

Definition 3.9: Let R be a hemiring and a family of fuzzy sets $\{\mu_i | i \in I\}$ in R.Then the union $\left(\bigvee_{i \in I} \mu_i\right)$ of $\{\mu_i | i \in I\}$ is defined by

$$\left(\bigvee_{i \in I} \mu_i\right)(x) = \sup\{\mu_i(x) | i \in I\}$$

Theorem 3.10: If $\{\mu_i | i \in I\}$ is a family of S-fuzzy left h-ideal of R,then $(\bigvee_{i \in I} \mu_i)(x)$ is a S-fuzzy left h-ideal of R.

Proof: Let $\{\mu_i | i \in I\}$ be a family of S-fuzzy left h-ideal of R.

(i)For all $x, y \in R$,we have

$$\left(\bigvee_{i\in I} \mu_i\right)(x+y) = \sup\{\mu_i(x+y) | i \in I\}$$

$$\leq \sup\{S(\mu_i(x), \mu_i(y)) | i \in I\}$$

$$= S(\sup(\mu_i(x)|i \in I), \sup(\mu_i(y)|i \in I))$$

$$= S\left(\left(\bigvee_{i\in I} \mu_i\right)(x), \left(\bigvee_{i\in I} \mu_i\right)(y)\right)$$

(ii) For all $x, y \in R$,we have

$$\left(\bigvee_{i\in I} \mu_i\right)(xy) = \sup\{\mu_i(xy) | i \in I\}$$

$$\leq \sup\{S(\mu_i(x)) | i \in I\}$$

$$= S\left(\left(\bigvee_{i\in I} \mu_i\right)(x)\right)$$

(iii) For all $a, b, x, z \in R$ and if $x + a + z = b + z$ then

$$\left(\bigvee_{i\in I} \mu_i\right)(x) = \sup\{\mu_i(x) | i \in I\}$$

$$\leq \sup\{S(\mu_i(a), \mu_i(b)) | i \in I\}$$

$$= S(\sup(\mu_i(a)|i \in I), \sup(\mu_i(b)|i \in I))$$

$$= S\left(\left(\bigvee_{i\in I} \mu_i\right)(a), \left(\bigvee_{i\in I} \mu_i\right)(b)\right)$$

Hence $\left(\bigvee_{i\in I} \mu_i\right)$ is a S-fuzzy left h-ideal of R. ∎

Definition 3.11: Let $f : R \longrightarrow R'$ be a mapping ,where R and R' are non-empty sets and μ is a fuzzy subset of R.The *preimage* of μ under f written μ^f,is a fuzzy subset of R defined by $\mu^f = \mu(f(x))$,for all $x \in R$.

Theorem 3.12: Let $f : R \longrightarrow R'$ be a homomorphism of hemirings. If μ is a S-fuzzy left h-ideal of R',then μ^f is S-fuzzy left h-ideal of R.

Proof: Suppose μ is a S-fuzzy left h-ideal of R',then

(i) For all $x, y \in R$,we have

$$\begin{aligned}\mu^f(x+y) = \mu(f(x+y)) &= \mu(f(x) + f(y)) \\ &\leq S(\mu(f(x)), \mu(f(y))) \\ &= S(\mu^f(x), \mu^f(y))\end{aligned}$$

(ii)For all $x, y \in R$,we have

$$\begin{aligned}\mu^f(xy) = \mu(f(xy)) &= \mu(f(x)f(y)) \\ &\leq \mu(f(y)) = \mu^f(y)\end{aligned}$$

(iii)For all $a, b, x, z \in R$ and if $x + a + z = b + z$ then

$$\begin{aligned}\mu^f(x) &= \mu(f(x)) \\ &\leq S(\mu(f(a)), \mu(f(b))) \\ &= S(\mu^f(a), \mu^f(b))\end{aligned}$$

∎

Hence μ^f is a S-fuzzy left h-ideal of R.

Theorem 3.13: Let $f : R \longrightarrow R'$ be a homomorphism of hemirings. If μ^f is a S-fuzzy left h-ideal of R ,then μ is S-fuzzy left h-ideal of R'.

Proof: Suppose μ is a S-fuzzy left h-ideal of R',then

(i)Let $x', y' \in R'$,there exists $x, y \in R$ such that $f(x) = x'$ and $f(y) = y'$,we have

$$\begin{aligned}\mu(x' + y') &= \mu(f(x) + f(y)) \\ &= \mu(f(x+y)) \\ &= \mu^f(x+y) \\ &\leq S(\mu^f(x), \mu^f(y)) \\ &= S(\mu(f(x)), \mu(f(y))) \\ &= S(\mu(x'), \mu(y'))\end{aligned}$$

(ii)Let $x', y' \in R'$,there exists $x, y \in R$ such that $f(x) = x'$ and $f(y) = y'$,we have

$$\begin{aligned}\mu(x'y') &= \mu(f(x)f(y)) = \mu(f(xy)) \\ &= \mu^f(xy) \\ &\leq \mu^f(y) \\ &= \mu(f(y)) \\ &= \mu(y')\end{aligned}$$

(iii)Let $a', b', x', z' \in R'$,there exists $a, b, x, z \in R$ such that $f(a) = a', f(b) = b', f(x) = x', f(z) = z'$.If $x' + a' + z' = b' + z'$.Then $f(x+a+z) = f(b+z)$ and so $f(x)+f(a)+f(z) = f(b) + f(z)$.It follows that

$$\begin{aligned}\mu(x') &= \mu(f(x)) \\ &= \mu^f(x) \\ &\leq S(\mu^f(a), \mu^f(b)) \\ &= S(\mu(f(a)), \mu(f(b))) \\ &= S(\mu(a'), \mu(b'))\end{aligned}$$

Hence μ is a S-fuzzy left h-ideal of R'. ∎

Definition 3.14: Let f be a mapping defined on R.If ν is a fuzzy subset in $f(R)$,then the fuzzy subset $\mu = \nu \circ f$ in R(i.e., the fuzzy subset defined by $\mu(x) = \nu(f(x))$ for all $x \in R$) is called the *preimage* of ν under f.

Proposition 3.15: An onto homomorphic preimage of a S-fuzzy left h-ideal R is S-fuzzy left h-ideal.

Proof: The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [9 ,Proposition 3.10]. ∎

Let μ be a fuzzy subset in a hemiring R and f be a mapping defined on R.Then the fuzzy subset μ^f in $f(R)$ defined by $\mu^f(y) = \inf\limits_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(R)$ is called the *image* of μ under f.A fuzzy subset μ in R is said to have an *inf property* if for every subset $H \subseteq R$,there exists $h_0 \in H$ such that $\mu(h_0) = \inf\limits_{h \in H} \mu(h)$.

Proposition 3.16: An onto homomorphic image of S-fuzzy left h-ideal with inf property is S-fuzzy left h-ideal.

Proof: Let $f : R \longrightarrow R'$ be an onto homomorphism of semirings and let μ be a S-fuzzy left h-ideal of R with the inf property.

(i)Given $x', y' \in R'$,we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \inf_{h \in f^{-1}(x')} \mu(h), \quad \mu(y_0) = \inf_{h \in f^{-1}(y')} \mu(h)$$

respectively.Then , we have

$$\mu^f(x'+y') = \inf_{z\in f^{-1}(x'+y')} \mu(z) \le \max\{\mu(x_0),\mu(y_0)\}$$

$$\le S(\mu(x_0),\mu(y_0))$$

$$= S\left(\inf_{h\in f^{-1}(x')}\mu(h), \inf_{h\in f^{-1}(y')}\mu(h)\right)$$

$$= S\left(\mu^f(x'),\mu^f(y')\right)$$

(ii)Given $x',y'\in R'$,we let $x_0\in f^{-1}(x')$ and $y_0\in f^{-1}(y')$ be such that

$$\mu(x_0) = \inf_{h\in f^{-1}(x')}\mu(h),\ \ \mu(y_0) = \inf_{h\in f^{-1}(y')}\mu(h)$$

respectively.Then , we have

$$\begin{aligned}\mu^f(x'y') &= \inf_{z\in f^{-1}(x'y')}\mu(z) \le \mu(y_0)\\ &= \inf_{h\in f^{-1}(y')}\mu(h) = \mu^f(y')\end{aligned}$$

(ii) Given $a',b',x',y'\in R'$, we let $a_0\in f^{-1}(a')$, $b_0\in f^{-1}(b')$, $x_0\in f^{-1}(x')$, $z_0\in f^{-1}(z')$ be such that

$$\begin{aligned}\mu(a_0) &= \inf_{h\in f^{-1}(a')}\mu(h),\ \ \mu(b_0) = \inf_{h\in f^{-1}(b')}\mu(h)\\ \mu(x_0) &= \inf_{h\in f^{-1}(x')}\mu(h),\ \ \mu(z_0) = \inf_{h\in f^{-1}(z')}\mu(h)\end{aligned}$$

respectively.If $x'+a'+z'=b'+z'$ then $x_0+a_0+z_0=b_0+z_0$, where $(x_0+a_0+z_0)\in f^{-1}(x'+a'+z')$ and $(b_0+z_0)\in f^{-1}(b'+z')$,we have

$$\mu^f(x') = \inf_{z\in f^{-1}(x')}\mu(z) \le \max\{\mu(a_0),\mu(b_0)\}$$

$$= S\left(\inf_{h\in f^{-1}(a')}\mu(h), \inf_{h\in f^{-1}(b')}\mu(h)\right)$$

$$= S\left(\mu^f(a'),\mu^f(b')\right)$$

■

Hence,μ^f is a S-fuzzy left h-ideal of R'.

Definition 3.17: A $t-$conorm S on $[0,1]$ is called a continuous $t-$conorm if S is a continuous function from $[0,1]\times[0,1]\longrightarrow[0,1]$ with respect to usual topology.

We observe that the function " max " is always a continuous t-conorm

Proposition 3.18: Let S be a continuous $t-$conorm and let f be a homomorphism on a hemiring R.If μ is a S-fuzzy left h-ideal of R,then μ^f is a S-fuzzy left h-ideal of $f(R)$.

Proof: Let $A_1=f^{-1}(y_1), A_2=f^{-1}(y_2)$ and $A_{12}=f^{-1}(y_1+y_2)$,where $y_1+y_2\in f(R)$. Consider the set

$$A_1+A_2 = \{x\in R | x=a_1+a_2\ \mathit{for\ some}\ a_1\in A_1, a_2\in A_2\}$$

If $x\in A_1+A_2$,then $x=x_1+x_2$ for some $x_1\in A_1$ and $x_2\in A_2$ so that we have $f(x)=f(x_1+x_2)=f(x_1)+f(x_2)=y_1+y_2$,that is , $x\in f^{-1}(y_1+y_2)=A_{12}$.Thus , $A_1+A_2\subseteq A_{12}$.It follows that

$$\begin{aligned}\mu^f(y_1+y_2) &= \inf\{\mu(x)|x\in f^{-1}(x_1+x_2)\}\\ &= \inf\{\mu(x)|x\in A_{12}\}\\ &\le \inf\{\mu(x)|x\in A_1+A_2\}\\ &\le \inf\{\mu(x_1+x_2)|x_1\in A_1, x_2\in A_2\}\\ &\le \inf\{S(\mu(x_1),\mu(x_2))\,|x_1\in A_1, x_2\in A_2\}\end{aligned}$$

Since S is continuous for every $\epsilon>0$,we see that if

$$\begin{aligned}&inf\{\mu(x_1)|x_1\in A_1\} - x_1^\star \le \delta \text{ and}\\ &inf\{\mu(x_2)|x_2\in A_2\} - x_2^\star \le \delta\text{,then}\end{aligned}$$

$$S(\inf\{\mu(x_1)|x_1\in A_1\}, \inf\{\mu(x_2)|x_2\in A_2\}) - S(x_1^*,x_2^*) \le \epsilon$$

Choose $a_1\in A_1$ and $a_2\in A_2$,such that

$$\begin{aligned}&inf\{\mu(x_1)|x_1\in A_1\} - \mu(a_1) \le \delta \text{ and}\\ &inf\{\mu(x_2)|x_2\in A_2\} - \mu(a_2) \le \delta\text{,then}\end{aligned}$$

$$\begin{aligned}&S(\inf\{\mu(x_1)|x_1\in A_1\}, \inf\{\mu(x_2)|x_2\in A_2\})\\ &\qquad -S(\mu(a_1),\mu(a_2)) \le \varepsilon\end{aligned}$$

Thus,we have

$$\begin{aligned}(i)\mu^f(y_1+y_2) &\le \inf\{S(\mu(x_1),\mu(x_2))\,|x_1\in A_1, x_2\in A_2\}\\ &= S(\inf\{\mu(x_1)|x_1\in A_1\}, \inf\{\mu(x_2)|x_2\in A_2\})\\ &= S\left(\mu^f(y_1),\mu^f(y_2)\right)\end{aligned}$$

(ii)Similarly, we can prove that

$$\mu^f(y_1y_2) \le \mu^f(y_2)$$

(iii) Now , let $a_1,b_1,x_1,z_1\in f(R)$ be such that $x_1+a_1+z_1=b_1+z_1$.we can prove that

$$\mu^f(x_1) \le S\left(\mu^f(a_1),\mu^f(b_1)\right)$$

Hence,μ^f is a S-fuzzy left h-ideal of $f(R)$.

■

Lemma 3.19: Let T be a t-norm.Then t-conorm S can be defined as

$$S(x,y) = 1 - T(1-x, 1-y).$$

Proof: Straightforward. ■

Theorem 3.20: A fuzzy subset μ of R is a T-fuzzy left h-ideal if and only if its complement μ^c is a S-fuzzy left h-ideal of R.

Proof: Let μ be a T-fuzzy left h-ideal of R.

(i) For all $x,y\in R$,we have

$$\begin{aligned}\mu^c(x+y) &= 1-\mu(x+y)\\ &\le 1-T(\mu(x),\mu(y))\\ &= 1-T(1-\mu^c(x), 1-\mu^c(y))\\ &= S(\mu^c(x),\mu^c(y))\end{aligned}$$

(ii)For all $x,y\in R$,we have

$$\mu^c(xy) = 1-\mu(xy) \le 1-\mu(y) = \mu^c(y)$$

(iii)For all $a,b,x,z\in R$,If $x+a+z=b+z$ such that

$$\begin{aligned}\mu^c(x) &= 1-\mu(x)\\ &\le 1-T(\mu(a),\mu(b))\\ &= 1-T(1-\mu^c(a), 1-\mu^c(b))\\ &= S(\mu^c(a),\mu^c(b))\end{aligned}$$

Hence $\mu^c(x)$ is a S-fuzzy left h-ideal of R.
The converse is proved similarly. ■

IV. Chain conditions

Let μ and ν be a fuzzy subset in a hemiring R.Then the S−h-product of μ and ν is defined by

$$\mu \circ_h \nu(x) = \begin{cases} \inf\left(S\left(\mu(a_i), \mu(b_i)\right) \mid i = 1,2\right) \\ \qquad \text{if } x \text{ can be expressed as } \\ \qquad x + a_1b_1 + z = a_2b_2 + z, \\ 0 \qquad \text{otherwise.} \end{cases}$$

Proposition 4.1: Let μ and ν be a fuzzy subset of R.If they are S-fuzzy left h-ideal of R,then so $\mu \cup \nu$,where $\mu \cup \nu$ is defined by $(\mu \cup \nu)(x) = S(\mu(x), \nu(x))$ for all $x, \in R$.Moreover,If μ and ν are a S-fuzzy right h-ideal and a S-fuzzy left h-ideal respectively,then $\mu \circ_h \nu \subseteq \mu \cup \nu$

Proof: The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [9,proposition 4.2]. ■

Theorem 4.2: Let μ be a fuzzy subset in R and $Im(\mu) = \{\alpha_0, \alpha_1, ..., \alpha_k\}$,where $\alpha_i < \alpha_j$ whenever $i > j$.Let $\{A_n | n = 0, 1, ..., k\}$ be a family of ideals of R such that
(i) $A_0 \subset A_1 \subset ... \subset A_k = R$,
(ii) $\mu(A^*) = \alpha_n$,where $A_n^* = A_n \setminus A_{n-1}, A_{-1} = \phi$ for $n = 0, 1, ..., k$.
Then μ is a S-fuzzy left h-ideal of R.

Proof: Suppose $\{A_n | n = 0, 1, ..., k\}$ be a family of ideals of R.
(i) For all $x, y \in R$,Then we discuss the following cases:If $x + y \in A_n$ and $y \in A_n$ such that $x \in A_n$,since A_n is an ideal of R.thus

$$\mu(x + y) \leq \alpha_n = S(\mu(x), \mu(y)).$$

If $x + y \notin A_n^*$ and $y \notin A_n^*$,then the following four cases arise:

1) $x + y \in R \setminus A_n$ and $y \in R \setminus A_n$
2) $x + y \in A_{n-1}$ and $y \in A_{n-1}$
3) $x + y \in R \setminus A_n$ and $y \in A_{n-1}$
4) $x + y \in A_{n-1}$ and $y \in R \setminus A_n$

But,in either cases,we know that

$$\mu(x + y) \leq S(\mu(x), \mu(y)).$$

If $x + y \in R \setminus A_n^*$ and $y \notin A_n^*$ then either $y \in A_{n-1}$ or $y \in R \setminus A_n$. It follows that either $x \in A_n$ or $x \in R \setminus A_n$.Thus

$$\mu(x + y) \leq S(\mu(x), \mu(y)).$$

If $x + y \notin R \setminus A_n^*$ and $y \in A_n^*$ then by similar process we have

$$\mu(x + y) \leq S(\mu(x), \mu(y)).$$

(ii) Similarly, for $x, y \in R$,we have

$$\mu(xy) \leq \mu(y).$$

(iii)For all $a, b, x, z \in R$,If $x + a + z = b + z$ such that $a \in A_n$ and $b \in A_n$ then $x \in A_n$.By the above process it is easy to show that

$$\mu(x) \leq S(\mu(a), \mu(b)).$$

Hence μ is a S-fuzzy left h-ideal of R. ■

Theorem 4.3: Let $\{A_n | n \in N\}$ be a family of h-ideals of a hemiring R which is nested,that is,$R = A_1 \supset A_2 \supset$ ....Let μ be a fuzzy subset in R defined by

$$\mu(x) = \begin{cases} \frac{1}{n+1} & if\ x \in A_n \setminus A_{n+1}, n = 1, 2, 3..., \\ 0 & if\ x \in \bigcap_{n=1}^{\infty} A_n\,. \end{cases}$$

for all $x \in R$.Then μ is a S-fuzzy left h-ideal of R.

Proof: Let $x, y \in R$.
(i)Suppose that $x \in A_k \setminus A_{k+1}$ and $y \in A_r \setminus A_{r+1}$ for $k = 1, 2, ...; r = 1, 2, ...$ .Without loss of generality,we may assume that $k \leq r$.Then $x + y \in A_k$ and so

$$\mu(x + y) \leq \frac{1}{k+1} = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$ then $x + y \in \bigcap_{n=1}^{\infty} A_n$ and thus

$$\mu(x + y) = 0 = S(\mu(x), \mu(y))$$

If $x \in \bigcap_{n=1}^{\infty} A_n$ then $y \notin \bigcap_{n=1}^{\infty} A_n$,then there exists $i \in N$ such that $y \in A_i \setminus A_{i+1}$.It follows that $x + y \in A_i$ so that

$$\mu(x + y) \leq \frac{1}{i+1} = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$$

Similarly,we know that

$$\mu(x + y) \leq S(\mu(x), \mu(y))$$

for all $x \notin \bigcap_{n=1}^{\infty} A_n$ then $y \in \bigcap_{n=1}^{\infty} A_n$.

(ii)Now,if $y \in A_r \setminus A_{r+1}$ for some $k = 1, 2, ...$,then $xy \in A_k$ for all $x \in R$.Thus

$$\mu(x + y) \leq \frac{1}{k+1} = \mu(y)$$

If $y \in \bigcap_{n=1}^{\infty} A_n$ then $xy \in \bigcap_{n=1}^{\infty} A_n$ for all $x \in R$.Thus

$$\mu(xy) = 0 = \mu(y)$$

(iii) Now,let $a, b, x, z \in R$ be such that $x + a + z = b + z$.If $a, b \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3...$, then $x \in A_r$ as A_r is a left h-ideal of R.Thus

$$\mu(x) \leq \frac{1}{k+1} = \max\{\mu(a), \mu(b)\} \leq S(\mu(a), \mu(b))$$

If $a, b \in \bigcap_{n=1}^{\infty} A_n$ then $x \in \bigcap_{n=1}^{\infty} A_n$ and so

$$\mu(x) = 0 = S(\mu(a), \mu(b))$$

Assume that $a \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3, ...$,and $b \in \bigcap_{n=1}^{\infty} A_n$(or , $a \in \bigcap_{n=1}^{\infty} A_n$ and $b \in A_r \setminus A_{r+1}$ for some $r = 1, 2, 3...$).Then $x \in A_r$ and so

$$\mu(x) \leq \frac{1}{r+1} = \max\{\mu(a), \mu(b)\} \leq S(\mu(a), \mu(b))$$

Hence, μ is a S-fuzzy left h-ideal of R. ■

Let $\mu : R \longrightarrow [0,1]$ be a fuzzy subset of R.The smallest S-fuzzy left h-ideal containing μ is called the S-fuzzy left h-ideal generated by μ, and μ is said to be n-valued if $\mu(R)$ is a finite set of n elements.When no specific n is intended, we call μ a finite-valued fuzzy subset.

Theorem 4.4: A S-fuzzy left h-ideal ν of R is finite valued if and only if a finite-valued fuzzy subset μ of R is generated by ν .

Proof: If $\nu : R \longrightarrow [0,1]$ is a finite-valued S-fuzzy left h-ideal of R,then one may choose $\mu = \nu$.Consequently, assume that $\mu : R \longrightarrow [0,1]$ is a n-valued fuzzy subset with n distinct values $t_1, t_2, ..., t_n$,where $t_1 < t_2 < ... < t_n$.Let G^i be the inverse image of t_i under μ, that is,$G^i = \mu^{-1}(t_i)$.Obviously, $\bigcup_{i=1}^{j} G^i \subseteq \bigcup_{i=1}^{r} G^i$ when $j < r$.We denote by A^j the left h-ideal of R generated by the set $\bigcup_{i=1}^{j} G^i$.Then we have the following chain of left h-ideals:

$$A^1 \supseteq A^2 \supseteq ... \supseteq A^n = R$$

Define a fuzzy $\nu : R \longrightarrow [0,1]$ by

$$\nu(x) = \begin{cases} t_n & if \in A^n, \\ t_j & if \ \in A^j \backslash A^{j-1}; j = 1,2,...,n-1 \end{cases}$$

We claim that ν is a S-fuzzy left h-ideal of R and μ is generated by ν.Let $x, y \in R$ and let i and j be the largest integer such that $x \in A^i$ and $y \in A^j$.we may assume that $i < j$ without loss of generality.Then $x + y \in A^i$ and $xy \in A^i$ and so

$$\begin{aligned} \nu(x+y) &\le t_j = \max\{t_i, t_j\} = \max\{\nu(x), \nu(y)\} \\ &\le S(\nu(x), \nu(y)) \end{aligned}$$

and

$$\nu(xy) \le t_j = \nu(y)$$

Now,let $a, b, x, z \in R$ be such that $x + a + z = b + z$.If $a \in A^i$ and $b \in A^j$ for some $i < j$,then $a, b \in A^i$ and so $x \in A^i$ as A^i is a h-ideal of R.Thus

$$\begin{aligned} \nu(x) &\le t_j = \max\{t_i, t_j\} = \max\{\nu(a), \nu(b)\} \\ &\le S(\nu(a), \nu(b)) \end{aligned}$$

Hence, μ is a S-fuzzy left h-ideal of R.

If $x \in R$ and $\mu(x) = t_j$,then $x \in G^j$ and so $x \in A^j$.But we get $\nu(x) \le t_j = \mu(x)$.Consequently, $\nu \subseteq \mu$.Let γ be any S-fuzzy left h-ideal of R which is a subset of μ.Then, $\bigcup_{i=1}^{j} G^i = L(\gamma; t_j) \subseteq L(\mu; t_j)$, and thus $A^j \subseteq L(\gamma; t_j)$.Hence,$\gamma \subseteq \mu$ and μ is generated by ν.Note that $\mid Im\mu \mid = n = \mid Im\nu \mid$.Thus completing the proof. ∎

A semiring R is a said to be *left h-artinian* (see [9]) if it satisfies the descending chain condition on left h-ideals of R.

Theorem 4.5: If R is a h-artinian hemiring, then every S-fuzzy left h-ideal of R is finite valued.

Proof: Let $\mu : R \longrightarrow [0,1]$ be a S-fuzzy left h-ideal of R which is not finite valued.Then,there exists sequence of distinct numbers $\mu(0) = t_1 > t_2 > ... > t_n$,where $t_1 = \mu(x_i)$ for some $x_i \in R$. This sequence induces an infinite sequence of distinct left h-ideals of R:

$$L(\mu; t_1) \supset L(\mu; t_2) \supset ... \supset L(\mu; t_n) \supset ... \ .$$

This is a contradiction. ∎

Combining Theorem 8 and Theorem 9,we have the following corollary.

Corollary 4.6: If R is a h-artinian hemiring, then every S-fuzzy left h-ideal of R is generated by a finite fuzzy subset in R.

V. S-PRODUCT OF S-FUZZY LEFT h-IDEALS

Definition 5.1: (see [2]) A fuzzy relation on any set R is a fuzzy subset $\mu : R \times R$.

Definition 5.2: Let S be a t-conorm . If μ is a fuzzy relation on a set R and ν is a fuzzy set in R,then μ is a S-fuzzy relation on ν if $\mu_\nu(x,y) \ge S(\nu(x), \nu(y))$, $for \ all \ x, y \in R$

Definition 5.3: Let S be a t-conorm . Let μ and ν be a fuzzy subset of R . Then direct S-product of μ and ν is defined by $(\mu \times \nu)(x,y) = S(\mu(x), \nu(y))$, $for \ all \ x, y \in R$

Lemma 5.4: Let S be a t-conorm .Let μ and ν be a fuzzy subset of R .Then,

(i) $\mu \times \nu$ is a S-fuzzy relation on S.

(ii) $L(\mu \times \nu; t) = L(\mu; t) \times L(\nu; t)$,for all $t \in [0,1]$

Proof: The proof is obvious. ∎

Definition 5.5: Let S be a t-conorm .Let μ be a fuzzy subset of R,then μ is said to be the strongest S-fuzzy relation on R if $\mu_\nu(x,y) \ge S(\nu(x), \nu(y))$, $for \ all \ x, y \in R$

Lemma 5.6: For given fuzzy subset ν in a set R,let μ_ν be the strongest S-fuzzy relation on R.Then

$L(\mu_\nu; t) = L(\mu; t) \times L(\nu; t)$, for all $t \in [0,1]$.

Proof: The proof is obvious. ∎

Proposition 5.7: For given fuzzy subset ν in a set R,let μ_ν be the strongest S-fuzzy relation on R. If μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$,then $\nu(a) \ge \nu(0)$ for $a \in R$.

Proof: If μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$,then $\mu_\nu(a,a) \ge \mu_\nu(0,0)$ for $a \in R$.This means $S(\nu(a), \nu(a)) \ge S(\nu(0), \nu(0))$ for $a \in R$.Since μ is sensible,then $\nu(a) \ge \nu(0)$ for $a \in R$. ∎

The following proposition is an immediate consequence of lemma 5.6.

Proposition 5.8: Let μ and ν be S-fuzzy left h-ideal of R,then the level left h-ideals of μ_ν are given by $L(\mu_\nu; t) = L(\mu; t) \times L(\nu; t)$, for all $t \in R$.

Theorem 5.9: Let S be a t-conorm. Let μ and ν be S-fuzzy left h-ideal of R,then $\mu \times \nu$ is a S-fuzzy left h-idealof $R \times R$.

Proof: Suppose μ and ν be S-fuzzy left h-ideal of R.Let $\mu \times \nu$ is a S-fuzzy left h-idealof $R \times R$.Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any element of $R \times R$.Then,

(i)

$$\begin{aligned} (\mu \times \nu)(x+y) &= (\mu \times \nu)((x_1, x_2) + (y_1, y_2)) \\ &= (\mu \times \nu)((x_1 + y_1, x_2 + y_2)) \\ &= S(\mu(x_1 + y_1), \nu(x_2 + y_2)) \\ &\le S(S(\mu(x_1), \mu(y_1))), S(\nu(x_2), \nu(y_2))) \\ &= S(S(\mu(x_1), \nu(x_2))), S(\mu(y_1), \nu(y_2))) \\ &= S((\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)) \\ &= S((\mu \times \nu)(x), (\mu \times \nu)(y)) \end{aligned}$$

(ii)

$$\begin{aligned}(\mu \times \nu)(xy) &= (\mu \times \nu)((x_1, x_2)(y_1, y_2)) \\ &= (\mu \times \nu)((x_1 y_1, x_2 y_2)) \\ &= S(\mu(y_1), \nu(y_2)) \\ &= (\mu \times \nu)(y_1, y_2) \\ &= (\mu \times \nu)(y)\end{aligned}$$

(iii) Let $x = (x_1, x_2)$,$z = (z_1, z_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$.Then,

$$\begin{aligned}(\mu \times \nu)(x) &= (\mu \times \nu)((x_1, x_2)) \\ &= S(\mu(x_1), \nu(x_2)) \\ &\leq S(S(\mu(a_1), \mu(b_1))), S(\nu(a_2), \nu(b_2))) \\ &= S(S(\mu(a_1), \nu(a_2))), S(\mu(b_1), \nu(b_2))) \\ &= S((\mu \times \nu)(a_1, a_2), (\mu \times \nu)(b_1, b_2)) \\ &= S((\mu \times \nu)(a), (\mu \times \nu)(b))\end{aligned}$$

Thus, $\mu \times \nu$ is a S-fuzzy left h-ideal of $R \times R$. ■

Corollary 5.10: Let S be a t-conorm. Let μ and ν be a sensible S-fuzzy left h-ideal of R,then $\mu \times \nu$ is a sensible S-fuzzy left h-ideal of $R \times R$.

Proof: By Theorem 5.9,we have $\mu \times \nu$ is a S-fuzzy left h-ideal of $R \times R$.Let $x = (x_1, x_2)$ be any element in $R \times R$,then

$$\begin{aligned}(\mu \times \nu)(x) &= (\mu \times \nu)((x_1, x_2)) \\ &= S(\mu(x_1), \nu(x_2)) \\ &= S(S(\mu(x_1), \mu(x_1))), S(\nu(x_2), \nu(x_2))) \\ &= S(S(\mu(x_1), \nu(x_2))), S(\mu(x_1), \nu(x_2))) \\ &= S((\mu \times \nu)(x_1, x_2), (\mu \times \nu)(x_1, x_2)) \\ &= S((\mu \times \nu)(x), (\mu \times \nu)(x))\end{aligned}$$

Hence, $\mu \times \nu$ is a sensible S-fuzzy left h-ideal of $R \times R$. ■

As the converse of Corollary 5.10,we have a following question: If $\mu \times \nu$ is a sensible S-fuzzy left h-ideal of $R \times R$,then are both μ and ν sensible S-fuzzy left h-ideal of R? The following example gives a negative answer.

Example 5.11: Let R be a hemiring with $|R| \geq 2$ and let $t \in [0, 1]$.Define a sensible fuzzy subset μ and ν in R by $\mu(x) = 1$ and

$$\nu(x) = \begin{cases} 1 & if\, x = 0, \\ t & otherwise. \end{cases}$$

for all $x \in R$,respectively.

If $x = 0$,then $\nu(x) = 1$, and thus

$$(\mu \times \nu)(x, x) = S(\mu(x), \nu(x)) = S(1, 1) = 1$$

If $x \neq 0$,then $\nu(x) = t$, and thus

$$(\mu \times \nu)(x, x) = S(\mu(x), \nu(x)) = S(1, t) = 1$$

That is,$\mu \times \nu$ is a constant function,and so $\mu \times \nu$ is a sensible S-fuzzy left h-ideal of $R \times R$.Now,μ is a sensible S-fuzzy left h-ideal of R,but ν is not a sensible S-fuzzy left h-ideal of R,since for $x \neq 0$,we have $\nu(0) = 1 > t = \nu(x)$.

Now,we generalize the product of two S-fuzzy left h-ideal of R to the product of n S-fuzzy left h-ideal.we first need to generalize the domain of t-conorm R to $\prod_{i=1}^{n} [0, 1]$ as follows.

Definition 5.12: The function $S_n : \prod_{i=1}^{n} [0, 1] \to [0, 1]$ is defined by

$S_n(\alpha_1, \alpha_2, \ldots, \alpha_n) =$

$$S(\alpha_i, S_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))$$

for all $1 \leq i \leq n$,where $n \geq 2$ $S_2 = S$ and $S_1 = identity$.

Lemma 5.13: For a t-conorm S and every $\alpha_i, \beta_i \in [0, 1]$, where $1 \leq i \leq n$,$n \geq 2$,we have

$$\begin{aligned}&S_n(S(\alpha_1, \beta_1), S(\alpha_2, \beta_2), \ldots, S(\alpha_n, \beta_n)) \\ &\quad = S(S_n(\alpha_1, \alpha_2, \ldots, \alpha_n), S_n(\beta_1, \beta_2, \ldots, \beta_n)).\end{aligned}$$

Proposition 5.14: Let S be a t-conorm. Let $\{R_i\}_{i=1}^{n}$ be the finite collection of hemirings and $R = \prod_{n=1}^{n} R_i$ the S-product of S_i.Let μ_i be a S-fuzzy left h-ideal of S_i,where $1 \leq i \leq n$. Then, $\mu = \prod_{n=1}^{n} \mu_i$ defined by

$$\begin{aligned}\mu(x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} \mu_i(x_1, x_2, \ldots, x_n) \\ &= S_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in R$ is a S-fuzzy left h-ideal of R.

Proof: The proof is similar to the proof of Theorem 10. ■

Definition 5.15: Let μ and ν be fuzzy subset in R.Then,the S-product of μ and ν,written as

$$[\mu \,.\, \nu]_S(x) = S(\mu(x), \nu(x))$$

for all $x \in R$.

Theorem 5.16: Let μ and ν be S-fuzzy left h-idealof R.If S^* is a t-conorm which dominates S,that is,

$$S^*(S(\alpha, \beta), S(\gamma, \delta)) \geq S(S^*(\alpha, \beta), S^*(\gamma, \delta))$$

for all $\alpha, \beta, \gamma, \delta \in R$.Then S^*-product of μ and ν,$[\mu \,.\, \nu]_{S^*}$ is a S-fuzzy left h-ideal of S.

Proof: Let $x, y \in R$,then we have

(i)

$$\begin{aligned}[\mu \,.\, \nu]_{S^*}(x + y) &= S^*(\mu(x + y), \nu(x + y)) \\ &\leq S^*(S(\mu(x), \mu(y)), S(\nu(x), \nu(y))) \\ &\leq S^*(S(\mu(x), \nu(x)), S(\mu(y), \nu(y))) \\ &= S([\mu \,.\, \nu]_{S^*}(x), [\mu \,.\, \nu]_{S^*}(y))\end{aligned}$$

(ii)

$$\begin{aligned}[\mu \,.\, \nu]_{S^*}(xy) &= S^*(\mu(xy), \nu(xy)) \\ &\leq S^*(\mu(y), \nu(y)) \\ &= [\mu \,.\, \nu]_{S^*}(y)\end{aligned}$$

(iii) Now,let $a, b, x, z \in R$ be such that $x + a + z = b + z$.Then

$$\begin{aligned}[\mu \,.\, \nu]_{S^*}(x) &= S^*(\mu(x), \nu(x)) \\ &\leq S^*(S(\mu(a), \mu(b)), S(\nu(a), \nu(b))) \\ &\leq S^*(S(\mu(a), \nu(a)), S(\mu(b), \nu(b))) \\ &= S([\mu \,.\, \nu]_{S^*}(a), [\mu \,.\, \nu]_{S^*}(b))\end{aligned}$$

Hence,$[\mu \,.\, \nu]_{S^*}$ is a S-fuzzy left h-ideal of R. ■

Theorem 5.17: Let $R \longrightarrow R'$ be an onto homomorphism of hemirings.Let S^* be a t-conorm such that S^* dominates S.Let μ and ν be S-fuzzy left h-ideal of S'.If $[\mu \,.\, \nu]_{S^*}$ is the S^*-product of μ and ν, and $[f^{-1}(\mu) \,.\, f^{-1}(\nu)]_{S^*}$ is the S^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$,then

$$f^{-1}([\mu \,.\, \nu]_{S^*}) = [f^{-1}(\mu) \,.\, f^{-1}(\nu)]_{S^*}$$

Proof: Let $x \in R$,then we have

$$\begin{aligned} f^{-1}\left([\mu\ .\ \nu]_{S^*}\right)(x) &= [\mu\ .\ \nu]_{S^*}(f(x)) \\ &= S^*\left(\mu\left(f(x)\right), \nu\left(f(x)\right)\right) \\ &= S^*\left(f^{-1}\left(\mu(x)\right), f^{-1}\left(\nu(x)\right)\right) \\ &= \left[f^{-1}(\mu)\ .\ f^{-1}\left(\nu\right)\right]_{S^*}(x) \end{aligned}$$

■

Theorem 5.18: Let ν be a sensible fuzzy subset of R. Let μ_ν be the strongest S-fuzzy relation on R.Then ν is a sensible S-fuzzy left h-ideal of R if and only if μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$.

Proof: Suppose that ν is a sensible S-fuzzy left h-ideal of R.Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $R \times R$.Then,

(i)

$$\begin{aligned} \mu_\nu(x+y) &= \mu_\nu\left((x_1, x_2) + (y_1, y_2)\right) \\ &= \mu_\nu\left((x_1 + y_1), (x_2 + y_2)\right) \\ &= S\left(\nu(x_1 + y_1), \nu(x_2 + y_2)\right) \\ &\leq S\left(S\left(\nu(x_1), \nu(y_1)\right), S\left(\nu(x_2), \nu(y_2)\right)\right) \\ &= S\left(S\left(\nu(x_1), \nu(x_2)\right), S\left(\nu(y_1), \nu(y_2)\right)\right) \\ &= S\left(\mu_\nu\left(x_1, x_2\right), S\left(\mu_\nu\left(y_1, y_2\right)\right)\right) \\ &= S\left(\mu_\nu\left(x\right), S\left(\mu_\nu\left(y\right)\right)\right) \end{aligned}$$

(ii)

$$\begin{aligned} \mu_\nu(xy) &= \mu_\nu\left((x_1, x_2)(y_1, y_2)\right) \\ &= \mu_\nu\left((x_1 y_1, x_2 y_2)\right) \\ &= S\left(\nu(x_1 y_1), \nu(x_2 y_2)\right) \\ &\leq \mu_\nu\left((x_1, x_2)(y_1, y_2)\right) \\ &= \mu_\nu\left(y_1, y_2\right) \\ &= \mu_\nu(y) \end{aligned}$$

(iii) Let $x = (x_1, x_2)$, $z = (z_1, z_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$.Then,

$$\begin{aligned} \mu_\nu(x) &= \mu_\nu\left((x_1, x_2)\right) \\ &= S\left(\nu(x_1), \nu(x_2)\right) \\ &\leq S\left(S\left(\nu(a_1), \nu(b_1)\right), S\left(\nu(a_2), \nu(b_2)\right)\right) \\ &= S\left(S\left(\nu(a_1), \nu(a_2)\right), S\left(\nu(b_1), \nu(b_2)\right)\right) \\ &= S\left(\mu_\nu\left(a_1, a_2\right), S\left(\mu_\nu\left(b_1, b_2\right)\right)\right) \\ &= S\left(\mu_\nu\left(a\right), S\left(\mu_\nu\left(b\right)\right)\right) \end{aligned}$$

Thus, μ_ν is a S-fuzzy left h-ideal of $R \times R$.

(iv) For any $x = (x_1, x_2) \in R \times R$,then

$$\begin{aligned} S\left(\mu_\nu(x), \mu_\nu(x)\right) &= S\left(\mu_\nu\left(x_1, x_2\right), \mu_\nu\left(x_1, x_2\right)\right) \\ &= S\left(S\left(\nu(x_1), \nu(x_2)\right), S\left(, \nu(x_1), \nu(x_2)\right)\right) \\ &= S\left(S\left(\nu(x_1), \nu(x_1)\right), S\left(, \nu(x_2), \nu(x_2)\right)\right) \\ &= S\left(\nu(x_1), \nu(x_2)\right) \\ &= \mu_\nu\left(x_1, x_2\right) \\ &= \mu_\nu(x) \end{aligned}$$

Hence, μ_ν is a sensible S-fuzzy left h-ideal of R.

Conversely,suppose that μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$. Let $x, y \in R$, we have

(i)

$$\begin{aligned} \nu(x+y) &= S\left(\nu(x+y), \nu(x+y)\right) \\ &= \mu_\nu(x+y, x+y) \\ &= \mu_\nu\left((x,x) + (y,y)\right) \\ &\leq S\left(\mu_\nu(x,x), \mu_\nu(y,y)\right) \\ &= S\left(S\left(\nu(x), \nu(x)\right), S\left(\nu(y), \nu(y)\right)\right) \\ &= S\left(\nu(x), \nu(y)\right) \end{aligned}$$

(ii)

$$\begin{aligned} \nu(xy) &= S\left(\nu(xy), \nu(xy)\right) \\ &= \mu_\nu(xy, xy) \\ &\leq \mu_\nu(y, y) \\ &= S\left(\nu(y), \nu(y)\right) \\ &= \nu(y) \end{aligned}$$

(iii)Let $a, b, x, z \in R$ be such that $(x,x) + (a,a) + (z,z) = (b,b) + (z,z)$.Since μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$.Then

$$\begin{aligned} \nu(x) &= S\left(\nu(x), \nu(x)\right) \\ &= \mu_\nu(x,x) \\ &= \mu_\nu\left((x,x) + (y,y)\right) \\ &\leq S\left(\mu_\nu(a,a), \mu_\nu(b,b)\right) \\ &= S\left(S\left(\nu(a), \nu(a)\right), S\left(\nu(b), \nu(b)\right)\right) \\ &= S\left(\nu(a), \nu(b)\right) \end{aligned}$$

Consequently,ν is a sensible S-fuzzy left h-ideal of R.

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