On weakly prime and weakly quasi-prime fuzzy left ideals in ordered semigroups

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Abstract—In this paper, we first introduce the concepts of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup $S$. Furthermore, we give some characterizations of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup $S$ by the ordered fuzzy points and fuzzy subsets of $S$.

Keywords—Ordered semigroup; ordered fuzzy point; weakly prime fuzzy left ideal; weakly quasi-prime fuzzy left ideal.

I. INTRODUCTION

There has been a rapid growth worldwide in the interest of fuzzy set theory and its applications from the past several years. Evidence of this can be found in the increasing number of high-quality research articles on fuzzy sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences held every year. It seems that the fuzzy set theory deals with the applications of fuzzy technology in information processing. The information processing is already important and it will certainly increase in importance in the future. The fundamental concept of a fuzzy set, introduced by L. A. Zadeh, provides a natural frame-work for generalizing several basic notions of algebra. Following the terminology given by Zadeh, if $S$ is an ordered semigroup, fuzzy sets in ordered semigroups $S$ were first considered by Kehayopulu and Tsingelis in [1], then they defined “fuzzy” analogous for several notations, which have proven useful in the theory of ordered semigroups. Moreover, they proved that each ordered groupoid can be embedded into an ordered groupoid having the greatest element (pogroupoid) in terms of fuzzy sets [2]. A theory of fuzzy sets on ordered semigroups has been recently developed (see [3-8,12,13]). The concept of ordered fuzzy points of an ordered semigroup $S$ was introduced by Xie and Tang [3], and prime fuzzy ideals of an ordered semigroup $S$ were studied in [4]. Authors also introduced the concepts of weakly prime fuzzy ideals, completely prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of an ordered semigroup $S$, and established the relations among five types ideals. Furthermore, Xie and Tang characterize weakly prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of $S$ by their level ideals [3].

As we know, fuzzy ideals (left, right ideals) with special properties of ordered semigroups always play an important role in the study of ordered semigroups structure. The ordered fuzzy points of an ordered semigroup $S$ are key tools to describe the algebraic subsystems of $S$. Motivated by the study of prime fuzzy ideals in rings, semigroups and ordered semigroups, and also motivated by Kehayopulu and Tsingelis’s works in ordered semigroups in terms of fuzzy subsets, in this paper we attempt to introduce and give a detailed investigation of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup $S$. We characterize weakly prime and weakly quasi-prime fuzzy left ideals of $S$ by ordered fuzzy points of $S$. Furthermore, we prove that in commutative ordered semigroups the weakly prime, quasi-prime and weakly quasi-prime fuzzy left ideals of $S$ are the same. Finally, we describe the largest fuzzy ideal $I(f)$ of $S$ contained in a fuzzy left ideal $f$ of $S$ and the largest fuzzy subsemigroup $I(f)$ of $S$ such that the fuzzy left ideal $f$ is a fuzzy ideal of $I(f)$. The paper illustrates that one can pass from the theory of semigroups or ordered semigroups to the theory of “fuzzy” ordered semigroups. As an application of the results of this paper, the corresponding results of semigroup (without order) are also obtained.

II. PRELIMINARIES AND SOME NOTATIONS

Throughout this paper unless stated otherwise $S$ stands for an ordered semigroup, that is, a semigroup $S$ with an order relation “$\leq$ ” such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$ (for example, see [9]). For convenience we use the notation $S^1 := S \cup \{1\}$, where $1 \cdot a = a \cdot 1 := 1$ for all $a \in S$ and $1 \cdot 1 = 1$. A nonempty subset $I$ of $S$ is called a left (resp. right) ideal of $S$ if

(1) $SI \subseteq I$ (resp. $IS \subseteq I$), and

(2) If $a \in I$, $b \leq a$ with $b \in S$, then $b \in I$.

$I$ is called an ideal of $S$ if $I$ is both a left ideal and a right ideal of $S$ [9]. Let $L$ be a left ideal of $S$. $L$ is called weakly prime if for any two ideals $A, B$ of $S$ such that $AB \subseteq L$, it implies that $A \subseteq L$ or $B \subseteq L$; $L$ is called quasi-prime if for any two ideals $L_1$ and $L_2$ of $S$, $L_1 L_2 \subseteq L$, then $L_1 \subseteq L$ or $L_2 \subseteq L$ and $L$ is called weakly quasi-prime if for any two ideals $L_1$ and $L_2$ of $S$ such that $L \subseteq L_1, L_2$ and $L_1 L_2 \subseteq L$, we have $L_1 = L$ or $L_2 = L$ [10].

For $H \subseteq S$, we define

$[H] := \{t \in S \mid t \leq h$ for some $h \in H\}$.

For $H = \{a\}$, we write $[a]$ instead of $\{[a]\}$. We denote by $L(a)$ (resp. $[a]$) the left ideal (resp. ideal) of $S$ generated by $a \in S$. Then $L(a) = [a \cup Sa] = (S^1 a) \cup (a) = (a \cup Sa \cup a S \cup SaS) = (S^1 a S^1)$ [9].

Lemma 2.1 ([9]): Let $S$ be an ordered semigroup. Then the following statements hold:
A fuzzy left ideal of an ordered semigroup \( S \) is a fuzzy subset \( A \) of \( S \) that satisfies the following properties:

1. \( A \) is a fuzzy subset of \( S \).
2. For all \( x, y \in S \), \( A \) satisfies \( f(x/y) \geq f(y) \) (equivalent to \( f(x \cdot y) \geq f(y) \)).
3. \( A \) is a fuzzy subset of \( S \) which is both a fuzzy left ideal and a fuzzy right ideal of \( S \).
4. A fuzzy left ideal of \( S \) is a fuzzy subset of \( S \) which is both a fuzzy left and a fuzzy right ideal of \( S \). Let \( f \) be a fuzzy left ideal of \( S \). Then \( f \) is called a quasi-prime if for any two fuzzy left ideals \( f_1 \) and \( f_2 \) of \( S \) such that \( f_1 \circ f_2 \subseteq f \), it implies that \( f_1 \subseteq f \) or \( f_2 \subseteq f \).

**Lemma 2.2 ([1], [3]):** Let \( S \) be an ordered semigroup and \( A, B \subseteq S \). Then \( f_A \circ f_B = f_{AB} \).

**Lemma 2.3 ([1]):** Let \( S \) be an ordered semigroup and \( \emptyset \not\subseteq A \subseteq S \). Then \( A \) is a left (resp. right) ideal of \( S \) if and only if the characteristic mapping \( f_A \) of \( A \) is a fuzzy left (resp. right) ideal of \( S \).

**Lemma 2.4 ([3]):** Let \( f \) be a fuzzy subset of an ordered semigroup \( S \). Then \( f \) is a strongly convex fuzzy subset of \( S \) if and only if \( x \leq y \Rightarrow f(x) \geq f(y) \), for all \( x, y \in S \).

**Definition 2.5 ([3]):** Let \( S \) be an ordered semigroup, \( a \in S \) and \( \lambda \in [0,1] \). An ordered fuzzy point \( a_\lambda \) of \( S \) is defined by the rule that

\[
 a_\lambda(x) = \begin{cases} 
 \lambda, & \text{if } x \in [a), \\
 0, & \text{if } x \not\in [a]. 
\end{cases}
\]

Clearly, an ordered fuzzy point \( a_\lambda \) of \( S \) is a fuzzy subset of \( S \). For any fuzzy subset \( f \) of \( S \), we also denote \( a_\lambda \subseteq f \) by \( a_\lambda \in f \) in sequel.

**Lemma 2.6 ([3]):** Let \( a_\lambda \) be an ordered fuzzy point of \( S \). Then

1. The fuzzy left ideal generated by \( a_\lambda \), denoted by \( L(a_\lambda) \), is \( (\forall x \in S)(L(a_\lambda)(x)) = \begin{cases} 
 \lambda, & \text{if } x \in [a), \\
 0, & \text{if } x \not\in [a], 
\end{cases} \)

where \( L(a) \) is a left ideal of \( S \) generated by \( a \).

2. The fuzzy ideal generated by \( a_\lambda \), denoted by \( (a_\lambda) \), is \( (\forall x \in S)( (a_\lambda)(x)) = \begin{cases} 
 \lambda, & \text{if } x \in [a), \\
 0, & \text{if } x \not\in [a], 
\end{cases} \)

where \( (a) \) is an ideal of \( S \) generated by \( a \).

**Lemma 2.7 ([3]):** Let \( a_\lambda, b_\mu \ (\lambda \neq 0, \mu \neq 0) \) be ordered fuzzy points of \( S \), and \( f, g \) fuzzy subsets of \( S \). Then the following statements are true:

1. \( (\forall x \in S)(S \circ a_\lambda \circ x)(x) = \begin{cases} 
 \lambda, & \text{if } x \in [a_\lambda), \\
 0, & \text{if } x \not\in [a_\lambda], 
\end{cases} \)

where \( S \circ a_\lambda \) is a fuzzy left ideal of \( S \).

2. \( (\forall x \in S)(S \circ a_\lambda \circ x)(x) = \begin{cases} 
 \lambda, & \text{if } x \in [a_\lambda), \\
 0, & \text{if } x \not\in [a_\lambda], 
\end{cases} \)

where \( (a_\lambda) \) is a fuzzy left ideal of \( S \).

3. \( a_\lambda \circ b_\mu = (ab)_\lambda \mu_\lambda \). In particular, \( a_\lambda \circ a_\lambda = (a^2)_\lambda \).

4. \( (a_\lambda) \subseteq S \circ a_\lambda \circ S \subseteq S \circ (a_\lambda \cup a_\lambda \circ S) \subseteq S \circ a_\lambda \circ S, \ L(a_\lambda) = a_\lambda \cup S \circ a_\lambda \).

5. \( \lambda \lambda_\lambda \subseteq S \circ a_\lambda \circ S \).

6. If \( f, g \) are fuzzy ideals of \( S \), then \( f \circ g, f \lor g \) are fuzzy ideals of \( S \).

7. If \( f \subseteq g, h \subseteq f \subseteq S \), then \( f \circ h \subseteq f \circ g \).

8. If \( S \) is commutative, then, for every ordered fuzzy point \( a_\lambda \) of \( S \), \( S \circ a_\lambda = a_\lambda \circ S \).

The reader is referred to [3, 14, 15] for notation and terminology not defined in this paper.

**III. WEAKLY PRIME FUZZY LEFT IDEALS OF ORDERED SEMIGROUPS**

**Definition 3.1:** A fuzzy left ideal \( f \) of an ordered semigroup \( S \) is called weakly prime if for any two fuzzy ideals \( f_1 \) and \( f_2 \), \( f_1 \circ f_2 \subseteq f \) implies that \( f_1 \subseteq f \) or \( f_2 \subseteq f \).

**Theorem 3.2:** Let \( S \) be an ordered semigroup. Then a fuzzy left ideal \( f \) of \( S \) is weakly prime if and only if for any two ordered fuzzy points \( x_r, y_t \in S \), \( x_r \circ S = y_t \circ S \subseteq f \) implies that \( x_r \in f \) or \( y_t \in f \).

**Proof:** Let \( x_r \) and \( y_t \) be ordered fuzzy points of \( S \) such that \( x_r \circ S = y_t \circ S \subseteq f \). Then \( S \circ x_r \circ S = S \circ y_t \circ S \subseteq f \).
Since $f$ is weakly prime and, $S \circ x_1 \circ S$ and $S \circ y_1 \circ S$ are fuzzy ideals of $S$, then we have $S \circ x_1 \circ S \subseteq f$ or $S \circ y_1 \circ S \subseteq f$. Say $S \circ x_1 \circ S \subseteq f$, then by Lemma 2.7(5),

$$(x_1)^3 \subseteq S \circ x_1 \circ S \subseteq f.$$ 

Thus $x_1 \in (x_1) \subseteq f.$

Conversely, let $f_1, f_2$ be fuzzy ideals of $S$ such that $f_1 \cap f_2 \subseteq f$. If $f_1 \not\subseteq f_2 \not\subseteq f$, then there exist $x, y \in S$ such that $f_1(x) > f(y), f_2(y) > f(y)$. Let $r = f_1(x), t = f_2(y)$. Then $rt > 0, x_r \in f_1, y_t \in f_2$ and so we have

$$x_r \circ y_t \circ S \subseteq f_1 \circ f_2 \subseteq f.$$ 

By hypothesis, $x_r \in f$ or $y_t \in f$. Say $x_r \in f$, then $f(x) \geq r = f_1(x)$, which is impossible.

**Theorem 3.3:** Let $S$ be an ordered semigroup. Then a left ideal $L$ of $S$ is weakly prime if and only if $f_L$ is a weakly prime fuzzy left ideal of $S$.

**Proof.** Let $L$ be a weakly prime left ideal of $S$. By Lemma 2.3, $f_L$ is a fuzzy left ideal of $S$. For any two fuzzy ideals $f_1$ and $f_2$ of $S$, if $f_1 \subseteq f_2 \subseteq f_L$, then $f_1 \subseteq f_L$, or $f_2 \subseteq f_L$. In fact:

If $f_1 \not\subseteq f_L, f_2 \not\subseteq f_L$. Then there exist $x, y \in S$ such that $f_1(x) > f_L(x), f_2(y) > f_L(y)$. Thus $f_1(x) > 0, f_2(y) > 0$, and $f_1(x) = f_2(y) = 0$. It implies that $x \not\in L$. We now show that there exist $r_1, r_2 \in S$ such that $(x_1 r_2) \not\subseteq L$.

Indeed:

If $(x S y) \not\subseteq L$, then $(x S y) \not\subseteq L$. Since $(x S y)$ and $(S y) \not\subseteq L$ is a weakly prime left ideal of $S$, then we have $(S y) \not\subseteq L$ or $(S y) \not\subseteq L$. Say $(S y) \not\subseteq L$, then $(x)^3 \subseteq S(y) \not\subseteq L$. Thus $x \in (x) \subseteq L$. Impossible. Now let $a \in (x_1 r_2)$ such that $a \not\subseteq L$. Then $f_L(a) = 0$, and

$$(f_1 \circ f_2)(a) = \bigvee_{(p, q) \in A_n} \min(f_1(p), f_2(q))$$

$$\geq \min(f_1(x r_1), f_2(y r_2))$$

$$\geq \min(f_1(x), f_2(y)) > 0,$$

which contradicts the fact that $f_1 \circ f_2 \subseteq f_L$. Therefore $f_L$ is a weakly prime fuzzy left ideal of $S$.

Conversely, let $f_L$ be a weakly prime fuzzy left ideal of $S$ and $A_B$ are ideals of $S$ such that $AB \subseteq L$. Then, by Lemma 2.1, we have $(AB) \subseteq (L) = L$. Thus, by Lemma 2.7, $f_A \circ f_B = f_{AB} \subseteq f_L$. By hypothesis and Lemma 2.3, since $f_L$ is weakly prime, then $A_L \subseteq f_L$ or $f_B \subseteq f_L$, that is, $A \subseteq L$ or $B \subseteq L$.

**IV. WEAKLY QUASI-PRIME FUZZY LEFT IDEALS OF ORDERED SEMIGROUPS**

**Definition 4.1:** Let $S$ be an ordered semigroup. A fuzzy left ideal $f$ of $S$ is called weakly quasi-prime if for any two fuzzy left ideals $f_1$ and $f_2$ of $S$ such that $f \subseteq f_1, f \subseteq f_2$ and $f_1 \cap f_2 \subseteq f$, then $f = f_1$ or $f_2 = f$.

**Theorem 4.2:** Let $S$ be an ordered semigroup. Then a left ideal $L$ of $S$ is weakly quasi-prime if and only if $f_L$ is weakly quasi-prime.

**Proof.** Let $L$ be a weakly quasi-prime left ideal of $S$, then, by Lemma 2.3, $f_L$ is a fuzzy left ideal of $S$. Let $f_1$ and $f_2$ be any two fuzzy left ideals of $S$ such that $f_1 \subseteq f_1, f_2 \subseteq f_2$ and $f_1 \cap f_2 \subseteq f_L$. If $f_1 \subseteq f_1, f_2 \subseteq f_L$, then there exist $x, y \in S$ such that $f_1(x) > f_2(x), f_2(y) > f_1(y)$, so that $x, y \not\subseteq L$ and $f_1(x) > 0, f_2(y) > 0$. Furthermore, there exist $r_1, r_2 \in S$ such that

$$(r_1 x) \cup (r_2 y) \cup (r_1 x) \cup (r_2 y) \subseteq L.$$ 

Indeed: If $(S x \cup L)(S y \cup L) \subseteq L$, since $(S x \cup L)$ and $(S y \cup L)$ are all left ideals of $S$ containing $L$, by hypothesis, we have $(S x \cup L) = L$, $(S y \cup L) = L$. Say $(S x \cup L) = L$, we have $S x \subseteq L$. Thus

$$(L \cup S x)(L \cup S x) \subseteq (L \cup L)(L \cup L x) \subseteq (L \cup L x)(L \cup L x) \subseteq (L \cup L x)(L \cup L x).$$ 

Hence $x \in (L \cup L x) = L$. Impossible.

Since $(r_1 x) \cup (r_2 y) \subseteq L$, we have $(r_1 x \cup r_2 y) \subseteq L$ or $(L x \cup L y) \subseteq L$.

1) If $(r_1 x \cup r_2 y) \subseteq L$, then there exists $z \in (r_1 x \cup r_2 y)$ and $z \not\subseteq L$, thus we have

$$(f_1 \circ f_2)(z) \geq \min(f_1(r_1 x), f_2(r_2 y))$$

$$\geq \min(f_1(x), f_2(y)) > 0.$$ 

But $f_L(z) = 0$. Impossible.

2) If $(L x \cup L y) \subseteq L$, then there exist $l \in L$ and $w \in L$ such that $w \in (L x \cup L y)$ and $w \not\subseteq L$, that is, $f_L(w) = 0$. But

$$(f_1 \circ f_2)(w) \geq \min(f_1(l), f_2(r_2 y))$$

$$\geq \min(f_1(l), f_2(y))$$

$$= f_2(y) > 0,$$

which is impossible. Therefore we have $f_1 \subseteq f_L$ or $f_2 \subseteq f_L$.

Conversely, if $f_1$ is weakly quasi-prime left ideal of $S$, then $L$ is weakly quasi-prime. The proof is a simple modification of the proof of reversed part of Theorem 3.3.

**Theorem 4.3:** Let $S$ be a commutative ordered semigroup and $f$ a fuzzy left ideal of $S$. Then the following statements are equivalent:

1) $f$ is weakly prime;
2) $f$ is quasi-prime;
3) $f$ is weakly quasi-prime.

**Proof.** (1) $\implies$ (2) and (2) $\implies$ (3) are obvious.

(3) $\implies$ (1). Let $f \circ f_2 \subseteq f$ for any two fuzzy ideals $f_1$ and $f_2$ of $S$. Since $S$ is commutative, so we have $f$ is a fuzzy ideal of $S$ and

$$(f_1 \cup f) \circ (f_2 \cup f) \subseteq f \circ f_1 \cup f \circ f_2 \cup f_2 \circ f_2 \subseteq f.$$ 

By (3), $f_1 \cup f = f$ or $f_2 \cup f = f$, that is, $f_1 \subseteq f$ or $f_2 \subseteq f$.

**Theorem 4.4:** Let $f$ be a fuzzy left ideal of an ordered semigroup $S$. Then the following statements are equivalent:

1) $f$ is a weak quasi-prime fuzzy left ideal of $S$.
2) For any two fuzzy ideals $f_1$ and $f_2$ of $S$, if $(f_1 \cup f_1 \circ f_2 \cup f) \subseteq f$, then $f_1 \subseteq f$ or $f_2 \subseteq f$.
3) For any two fuzzy ideals $f_1$ and $f_2$ of $S$, if $f \subseteq f_1$ and $f_1 \circ f_2 \subseteq f$, then $f_1 = f$ or $f_2 \subseteq f$.
4) For any two fuzzy ideals $f_1$ and $f_2$ of $S$, if $(f_1 \cup f_1 \circ f_2 \cup f) \subseteq f$, then $f_1 \subseteq f$ or $f_2 \subseteq f$. 

International Scholarly and Scientific Research & Innovation 6(8) 2012 844
(5) For any two ordered fuzzy points \( a_1, b_1 \in S(\tau > 0) \), if \((a_1 \cup f) \circ S \circ (b_1 \cup f) \subseteq f \), then \( a_1 \leq f \) or \( b_1 \leq f \).

Proof. \((1) \Rightarrow (2)\). Since \( f \cup f \) and \( f \cup f \) are fuzzy left ideals of \( S \) such that \( f \subseteq f \cup f \) and \( (f \cup f) \circ (f \cup f) \subseteq f \), then we have \( f = f \cup f \) or \( f = f \cup f \), that is, \( f \subseteq f \) or \( f \subseteq f \).

(2) \Rightarrow (3). For two fuzzy left ideals \( f_1 \) and \( f_2 \) of \( S \), if \( f \subseteq f_1 \) and \( f \subseteq f_2 \), then
\[
(f \cup f_1) \circ (f \cup f_2) \subseteq f_1 \circ (f \cup f_2) \subseteq (f \circ f_1) \cup (f \circ f_2) \subseteq f.
\]

By (2), we have \( f_1 \subseteq f \) or \( f_2 \subseteq f \), so that \( f_1 = f \) or \( f_2 = f \).

(3) \Rightarrow (4). Since \( f \subseteq f \cup f_1 \) and \( (f \cup f_1) \circ f_2 \subseteq f \), by (3) we have \( f_1 \cup f = f \) or \( f_2 \subseteq f \). It thus follows that \( f \subseteq f \) or \( f \subseteq f \).

(4) \Rightarrow (5). Let \( a_1, b_1(\tau > 0) \) be two ordered fuzzy points of \( S \) such that \((a_1 \cup f) \circ S \circ (b_1 \cup f) \subseteq f \). Then \( a_1 \circ S \circ b_1 \subseteq f \) and \( f \circ S \circ b_1 \subseteq f \). Thus
\[
(a_1 \circ S \circ a_1 \cup f) \circ (S \circ b_1) \subseteq (a_1 \circ S \circ b_1) \cup (f \circ f) \subseteq f.
\]

Since \( a_1 \circ S \circ a_1 = L(a_1) \) and \( S \circ b_1 \) are fuzzy left ideals of \( S \). By (4), we have \( L(a_1) \subseteq f \) or \( S \circ b_1 \subseteq f \). If \( L(a_1) \subseteq f \), then \( a_1 \circ S \circ b_1 \subseteq f \), if \( S \circ b_1 \subseteq f \), then
\[
(L(b_1) \cup f) \circ (L(b_1) \subseteq L(b_1) \cup f \circ L(b_1) \subseteq S \circ b_1 \subseteq f.
\]

By (4), we have \( b_1 \subseteq L(b_1) \subseteq f \).

(5) \Rightarrow (1). Let \( f_1 \) and \( f_2 \) be two fuzzy left ideals of \( S \) such that \( f \subseteq f_1 \) or \( f \subseteq f_2 \) or \( f_1 \cup f_2 \subseteq f \). If \( f_1 \neq f \) or \( f_2 \neq f \), then there exist \( x, y \in S \) such that \( f_1(x) > f(x) \) or \( f_2(y) > f(y) \).

Let \( r = f_1(x), t = f_2(y) \). Then \( r > 0 \) and
\[
(x \cup r) \circ S \circ (y \cup t) \subseteq f_1 \circ S \circ f_2 \subseteq f_1 \circ f_2 \subseteq f.
\]

By (5), we have \( x \in L(a) \), which is impossible.

Theorem 4.5: Let \( f \) be a fuzzy left ideal of an ordered semigroup \( S \) and \( \mu \) a fuzzy subset of \( S \) satisfying that:

(1) \( f \cap \mu = 0 \);

(2) For any \( a, b, \in \mu \), \((a \cup f) \circ S \circ (b \cup f) \cap \mu = 0 \).

If \( g \) is a maximal fuzzy left ideal of \( S \) with respect to containing \( f \) and \( g \cap \mu = 0 \), then \( g \) is a weakly quasi-prime fuzzy left ideal of \( S \).

Proof. Let \( f_1 \) and \( f_2 \) be any two fuzzy left ideals of \( S \) such that \( g \subseteq f_1 \) and \( g \subseteq f_2 \). Then \( g = f_1 = f_2 \).

Indeed: If \( g \subseteq f_1 \) and \( g \subseteq f_2 \), then there exist \( a \in f_1 \setminus g \), \( b \in f_2 \setminus g \), \( r > 0 \). Thus
\[
g \cap g \cup L(a) \subseteq f_1 \cap g \cup L(b) \subseteq f_2.
\]

By hypothesis, we have
\[
(g \cup L(a)) \cap \mu \neq 0, (g \cup L(b)) \cap \mu \neq 0.
\]

Let \( e_k \in (g \cup L(a)) \cap \mu(k > 0), d_{l} \in (g \cup L(b)) \cap \mu(l > 0) \).

Then
\[
(c_k \cup f) \circ S \circ (d_l \cup f) \subseteq (f_k \cup f) \circ S \circ (f_l \cup f) \subseteq (f_k \circ S \circ f_l) \circ f \subseteq f \cup f \subseteq f \subseteq g.
\]

Consequently, \((c_k \cup f) \circ S \circ (d_l \cup f) \cap \mu = 0 \), which contradicts with the fact that
\[
((c_k \cup f) \circ S \circ (d_l \cup f)) \cap \mu \neq 0.
\]

V. FUZZY IDEALS \( i(f) \) AND \( I(f) \) OF ORDERED SEMIGROUPS

Let \( f \) be a fuzzy left ideal of an ordered semigroup \( S \), we define two fuzzy subsets of \( S \), denoted by \( i(f) \) and \( I(f) \) respectively, as follows:

\[
i(f)(x) = \bigvee \{ \tau \in f, x \cup \tau \circ S \subseteq f, \tau \in [0, 1] \}
\]

and
\[
I(f)(x) = \bigvee \{ \tau \in f \circ S \cup f \subseteq f, \tau \in [0, 1] \}
\]

for any \( x \in S \).

Theorem 5.1: Let \( f \) be a left ideal of an ordered semigroup \( S \). Then \( i(f) \) is the largest fuzzy ideal of \( S \) contained \( f \).

Proof. For any \( x, y \in S \), let
\[
A(x) = \{ \tau_1 \in f, x_1 \circ S \subseteq f, \tau_1 \in [0, 1] \},
\]

\[
B(x) = \{ w_\beta \in S \circ f \subseteq f, w_\beta \in [0, 1] \},
\]

\[
C(y) = \{ q_\beta \in f \circ S \subseteq f, q_\beta \in [0, 1] \},
\]

then \( i(f)(x) = \bigvee_{\tau_1 \in A(x)} \tau_1 \), \( i(f)(y) = \bigvee_{w_\beta \in B(x)} w_\beta \), and
\[
i(f)(y) = \bigvee_{q_\beta \in C(y)} q_\beta \).
\]

Thus \( t \in B(x) \). It follows that \( i(f)(x) \geq i(f)(x) \). By the same way, we can prove that \( i(f)(x) \geq i(f)(y) \). Also, let \( x \leq y \). For any \( q_\beta \in [0, 1] \), it is clear that \( x \leq q_\beta \). If \( q_\beta \in C(y) \), then \( y \leq q_\beta \in S \leq f \). So we have \( x \leq y \), \( x \leq f \circ S \subseteq f \). It implies that \( q_\beta \in A(x) \), it thus follows that \( i(f)(x) \geq i(f)(y) \). Therefore, \( i(f) \) is a fuzzy ideal of \( S \).

For any \( x \in S \), let \( t \in A(x) \). Then \( x \subseteq f \), that is, \( t \subseteq f \).

Thus \( i(f)(x) \leq i(f)(x) \). Let \( x \) be a fuzzy ideal of \( S \) such that \( g \subseteq f \), and \( x \subseteq f \). Then \( x \subseteq f \), and so we have
\[
g(x) = \bigvee_{x \in g} t \leq i(f)(x).
\]

Therefore \( g \subseteq i(f) \).

Lemma 5.2 (111): Let \( S \) be an ordered semigroup. Then a fuzzy left ideal \( f \) of \( S \) is quasi-prime if and only if for any two ordered fuzzy points \( x, y \in S(\tau > 0) \), \( x \circ S \cup y \subseteq f \) or \( x \cup S \subseteq f \) implies that \( x \in f \) or \( y \in f \).

Theorem 5.3: Let \( S \) be an ordered semigroup with an identity \( e \) and \( f \) a weakly prime fuzzy left ideal of \( S \). If \( i(f) \neq 0 \), then \( i(f) \) is a quasi-prime fuzzy ideal of \( S \).
Proof. Let \(a_t\) and \(b_t\) \((t \geq 0)\) be any two ordered fuzzy points of \(S\) such that \(a_t \circ S \leq b_t \subseteq i(f)\). Then
\[
(S \circ a_t \circ S) \circ (S \circ b_t \circ S) \subseteq i(f) \subseteq f.
\]
Since \(f\) is weakly prime, we have \(S \circ a_t \circ S \subseteq f\) or \(S \circ b_t \circ S \subseteq f\). Say \(S \circ a_t \circ S \subseteq f\). By Theorem 5.1, we have \(S \circ a_t \circ S \subseteq i(f)\). Since \(S\) has an identity \(e\), so that
\[
a_t = (e \circ a_t) = e_t \circ a_t \subseteq S \circ a_t \subseteq i(f).
\]
By Lemma 5.2, the proof is completed.

Theorem 5.4: Let \(f\) be a fuzzy left ideal of an ordered semigroup \(S\). Then \(I(f)\) is the largest strongly convex fuzzy subsemigroup of \(S\) such that \(f\) is a fuzzy ideal of \(I(f)\).

Proof. For any \(x, y \in S\), let \(f(x) = t\). Then \(x_t \in f\), and let
\[
D(x) = \{t_a \mid f \circ x_t \subseteq f, t_a \in [0, 1]\},
\]
\[
E(y) = \{t_r \mid f \circ y_r \subseteq f, t_r \in [0, 1]\}.
\]
(1) Since \(f \circ x_t \subseteq f \circ f \leq f\), thus \(t \in D(x)\), it follows that
\[
f(x) = t \leq \bigvee_{t_a \in D(x)} t_a = I(f)(x),
\]
that is, \(I(f)\) is a fuzzy subset of \(S\) containing \(f\).

(2) \(I(f)\) is a strongly convex fuzzy subsemigroup of \(S\). In fact:
\[
I(f)(x) \land I(f)(y) = \bigvee_{t_a \in D(x), t_r \in E(y)} (t_a \land t_r) = \bigvee_{t_a \in D(x), t_r \in E(y)} (t_a \land t_r).
\]
Since
\[
f \circ (x_t \circ y_r) = (f \circ x_t) \circ y_r = f \circ (x_t \land y_r) \subseteq f \circ y_r \leq f,
\]
we have \(t_a \land t_r : I(f)(x)\). Thus
\[
I(f)(x) \land I(f)(y) \leq I(f)(x).
\]
Furthermore, let \(x \leq y\). For any \(r \in [0, 1]\), it is clear that \(x_r \subseteq y_r\). If \(r \in E(y)\), then \(f \circ x_r \subseteq f \circ y_r \leq f\), which implies that \(x_r \in D(x)\), it thus follows that \(I(f)(x) \geq I(f)(y)\). Therefore, \(I(f)\) is a strongly convex fuzzy subsemigroup of \(S\).

(3) \(f\) is a fuzzy ideal of \(I(f)\). In fact:
Since \(f\) is a left ideal of \(S\), then we have \(I(f) \circ f \subseteq S \circ f \subseteq f\). Furthermore, for any \(x, y \in S\), if \(x\) can be expressible as \(x \leq y\), let
\[
F(z) = \{q_t \mid f \circ z_q \subseteq f, q_t \in [0, 1]\}.
\]
Then
\[
(f \circ I(f))(x) = \bigvee_{(y, z) \in A_x} (f(y) \land I(f)(z)) = \bigvee_{(y, z) \in A_x} (f(y) \land \bigvee_{q_t \in F(z)} q_t).
\]
Since for any \(q_t \in F(z), f \circ z_q \subseteq f\), we have
\[
f(x) \geq f(y) \land q_t(x) \geq f(y) \land q_t,
\]
Thus we have
\[
f(x) \geq \bigvee_{(y, z) \in A_x} (f(y) \land q_t) = (f \circ I(f))(x).
\]
If \(A_x = 0\), then \(f(x) \geq 0 = f(0)\). Therefore, \(f \circ I(f) \subseteq f\).

(4) Let \(g\) be a strongly convex fuzzy subsemigroup of \(S\) such that \(f\) is a fuzzy ideal of \(g\). For any \(x \in S\), if \(g(x) = t\), then \(x_t \in g\), and \(f \circ x_t \subseteq f \circ g \subseteq f\). Thus \(t \in D(x)\), which implies that \(I(f)(x) \geq t = g(x)\). Therefore \(I(f)\) is the largest fuzzy subsemigroup of \(S\) such that \(f\) is a fuzzy ideal of \(I(f)\).

Theorem 5.5: Let \(S\) be an ordered semigroup with an identity \(e\) and \(f\) a fuzzy left ideal of \(S\) but not a fuzzy ideal of \(S\). Then the following statements are equivalent:

(1) \(f\) is a weakly quasi-prime fuzzy left ideal of \(S\).

(2) If \(f_{1} \subseteq f_{2}\) is a fuzzy left ideal of \(S\) such that \(f \circ f_{1} \subseteq f\), then \(f_{1} \subseteq f_{2}\).

(3) For any ordered fuzzy point \(a_{t} \in S\), if \(f \circ (S \circ a_{t}) \subseteq f\), then \(a_{t} \in f\).

(4) \(f\) is the largest fuzzy left ideal of \(S\) contained in \(I(f)\).

Proof. (1) \(\Rightarrow\) (2). Let \(f_{1} \subseteq f_{2}\) be a fuzzy left ideal of \(S\) such that \(f \circ f_{1} \subseteq f\). Then \(f \circ S \subseteq f \circ f_{1} \subseteq f \circ f_{2} \subseteq f\), where \(f \circ f_{2} \subseteq f\). Since \(f \circ f_{2} \subseteq f\). If \(a_{t} \circ S \in S\), then \(a_{t} \in f\). By Theorem 4.4(3), it follows that \(f_{1} \subseteq f\).

(2) \(\Rightarrow\) (3). For any ordered fuzzy point \(a_{t} \in S\), if \(f \circ (S \circ a_{t}) \subseteq f\), then \(a_{t} \in f\). Since \(S \circ a_{t} \subseteq f\). Since \(S\) has an identity \(e\), so
\[
(S \circ a_{t})(a) = \bigvee_{(y, z) \in A_z} (S(y) \land a_{t}(z)) \geq S(e) \land a_{t}(a) = t.
\]
Thus, \(a_{t} \in S \circ a_{t} \subseteq f\).

(3) \(\Rightarrow\) (4). By Theorem 5.4. \(f \subseteq I(f)\). Let \(g\) be a fuzzy left ideal of \(S\) such that \(g \subseteq I(f)\). Then \(g \subseteq f\). Indeed:
For any \(a_{t} \in g\), since \(f\) is a fuzzy ideal of \(I(f)\), we have
\[
f \circ (S \circ a_{t}) \subseteq f \circ (S \circ g) \subseteq f \circ g \subseteq f \circ I(f) \subseteq f,
\]
by (3), we have \(a_{t} \in f\). Thus \(g = \bigvee a_{t} \subseteq f\).

(4) \(\Rightarrow\) (1). Let \(f_{1}\) and \(f_{2}\) be any two fuzzy left ideals of \(S\) such that \(f \subseteq f_{1} \subseteq f \subseteq f_{2}\) and \(f \circ f_{2} \subseteq f_{1} \circ f_{2} \subseteq f_{2}\). Then for any \(x_{t} \in f_{2}\), \(f \circ x_{t} \subseteq f_{2}\). Thus \(I(f)(x) \geq t = x_{t}(x)\), which implies that \(x_{t} \in I(f)\). Therefore \(f_{2} = \bigvee_{x_{t} \in f_{2}} x_{t} \subseteq I(f)\). Since \(f\) is the largest fuzzy left ideal of \(S\) contained in \(I(f)\), then \(f_{2} \subseteq f\). Thus \(f = f_{2}\).
VI. Conclusion

As we know, fuzzy ideals (left, right ideals) of an ordered semigroup with special properties always play an important role in the study of ordered semigroups structure. The ordered fuzzy points of an ordered semigroup are key tools to describe the algebraic subsystems of ordered semigroups. In this paper we have introduced the concepts of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup. Furthermore, we have given some characterizations of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup by the ordered fuzzy points and fuzzy subsets. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of ordered semigroups. Hopefully, some new results in these topics can be obtained in the forthcoming paper.

ACKNOWLEDGMENT

The work is supported by the National Natural Science Foundation (10961014), the National Specific Subject of the Ministry of Education and the Ministry of Finance of the PRC (TS11496), the Anhui Provincial Excellent Youth Talent Foundation (2012SQRL115ZD), the University Natural Science Project of Anhui Province (KJ2012B133, KJ2012Z311) and the Fuyang Normal College Natural Science Foundation (2007LZ01).

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