Integral Operators Related to Problems of Interface Dynamics

Pa Pa Lin

Abstract—This research work is concerned with the eigenvalue problem for the integral operators which are obtained by linearization of a nonlocal evolution equation. The purpose of section II.A is to describe the nature of the problem and the objective of the project. The problem is related to the “stable solution” of the evolution equation which is the so-called “instanton” that describe the interface between two stable phases. The analysis of the instanton and its asymptotic behavior are described in section II.C by imposing the stability questions it is important to determine the part of the spectrum outside the unit ball. This is not simple in this present context because \( \beta > 1 \), while the rest of the spectrum is strictly less than one. It is possible to investigate the problem only when \( \beta \) is >1, while the rest of the spectrum is strictly inside the unit ball. This kind of problem has been studied by some other authors such as \([1]\)-\([2]\). The results will hopefully provide for many physical applications.

II. Evolution Equation

A. Nature of the Problem

This paper is concerned with the problem which arises from the analysis of the evolution equation in \( C^{\infty}(3;[-1,1]) \):

\[
\frac{\partial m}{\partial t} = -m + \tanh(\beta J \ast m + \beta h)
\]

where \( \beta > 1, h \geq 0, m \) is a real-valued function on 3 for each \( t \in 3 \), \( * \) is the convolution product defined by:

\[
(J \ast m)(x) = \int J(x, y)m(y)dy
\]

for each \( m \in C^{\infty}(3;[-1,1]) \). The operators \( A : C^{\infty}(3;[-1,1]) \to C^{\infty}(3;[-1,1]) \) can be considered. Which are related to the linearization of the right hand side of (1), namely given \( m \in C^{\infty}(3;[-1,1]) \),

\[
p_m(x) = \frac{\beta}{\cosh^2{\beta J \ast m(x)}}
\]

is set and defined \( A_m \), as the function of \( m \) with kernel \( J(x, y) \) by

\[
A_m(x, y) = p_m(x) J(x, y).
\]

Then (when \( h = 0 \)), the operator \( L_m = A_{m,1} \) is the linearization around \( m \) of the right hand side of (1). The operators \( A_m \) for each \( m \in C^{\infty}(3;[-1,1]) \) can be studied. Let \( C_b(3) \) be the space of all bounded real-valued functions on 3. Solutions \( m \) of (1) in \( C_b(\mathbb{X}) \) with sup norm \( ||m||_\infty \leq 1 \) can be observed and that are differentiable with respect to time \( t \). The Cauchy problem in this setup is well-posed with a unique global solution because the right hand side of (1) is uniformly Lipschitz because the set \( \{ m \in C_b(3); ||m||_\infty \leq 1 \} \) is left invariant, since tanh \( z < 1 \) for all \( z \). Equation (1) arises in the study of spin systems with Glauber dynamic and Kac interactions where it is derived in a continuum limit, \([3]\); \( m \) is then interpreted as a magnetization density and \( \beta \) as the product of the absolute temperature and the Boltzmann constant. The analysis of Gibbs measures with Kac interactions, that started in the late sixties \([4]\)-\([5]\) is by now a well established theory. It proves the validity, in equilibrium Statistical Mechanics setting, of the Van der Waals theory by showing that its typical phase diagram is exhibited by systems
with Kac interactions, in a suitable scaling limit. The critical
temperature corresponds to $\beta = 1$, according to our
normalization condition on the interaction, so that $\{ \beta > 1 \}$ is
the phase transition region. For each value of $\beta > 1$ there are
two pure thermodynamic phases with magnetization respectively equal to $\pm m_\beta: m_\beta$ being the positive standard of
the equation
\[ m_\beta = \tanh(\beta m_\beta) \]  
(5)

B. Objective and Motivation

The main objective of this paper is to investigate spectral
property of integral operators in problem of interface
dynamics. Interface dynamics concerns the analysis of the
Cauchy problem for (1) with initial data close to different
phases in various regions of space. This problem has been
extensively studied in the last ten years with special attention
to the multi-dimensional case where it has been proved that on
a suitable space time scaling limit the evolution is ruled by a
to the multi-dimensional case where it has been proved that on
a suitable space time scaling limit the evolution is ruled by a

As proved in [1], the interface described by the instanton
is “stable” and any initial datum “close to an instanton” is
attracted and eventually converges to some translate of the
stable phase at the expense of the metastable one, [7]. When
$h = 0$ there are stationary solutions with two coexisting
phases: they are all identical, modulo translations and reflection, [5], to “the instanton” $m(x)$, which is a $C^\infty$
strictly increasing, antisymmetric function which identically verifies
\[ m(x) = \tanh(\beta J * m(x)) \]  
(6)
where $m(x)$ is the stationary pattern that connects the minus
and plus phases, as
\[ \lim_{x \to \pm \infty} m(x) = \pm m_\beta \]  
(7)

\[ \lim_{x \to \pm \infty} \frac{m(x)}{m_\beta} = \pm 1 \]  
and it has therefore the interpretation of a “diffuse interface”.
However, since the derivative $m'(x)$ of $m(x)$ vanishes
exponentially fast as $x \to \infty$, [3], then loosely speaking, the
fraction of space not occupied by pure phase is vanishingly
small. In this sense, that can be made precisely by introducing
scaling, the interface is sharp and the transition from one
phase to the other one is “instantaneous”. That is why ( $m$ or,
more properly $\tilde{m}$) is called the instanton.

As proved in [1], the interface described by the instanton is
“stable” and any initial datum “close to an instanton” is
attracted and eventually converges to some translate of the
instanton. In the present context, (1) in a neighborhoods of
functions $m$ of the form
\[ m_\xi(x) = m(\xi - |x|), \quad \xi > 0 \]  
(8)
is studied. By linearization leads to the operator $A_\alpha$ with either
$m = m_\xi \lor m$ close to it (and $\xi$ large).This is a first motivation
for studying $A_\alpha$. In the analysis of the operator $A_\alpha$, with $m = m_\xi + h$ is interested or closed to such a function. The existence of
these attractive one dimensional unstable manifolds reflects
the presence of an isolated, simple, maximal eigenvalue $\lambda_0 > 1$
for $A_\alpha$ and of a spectral gap, with the rest of the spectrum
strictly in the unit ball. In this paper, these can be established
and other properties of the operators $A_\alpha$ which are the building
ingredients for the applications to interface dynamics.

C. Asymptotic Behavior of the Instanton

In this section, the asymptotic behavior of the instanton $m(x)$ as $x \to \infty$ is studied. The instanton $m(x)$ is an
antisymmetric, continuous increasing function of $x \in \mathbb{R}$
that solves (6). In Proposition 2.2 of [3] it is proved that there
exists $c > 0$ and $\eta > 0$ such that for all $x > 0$,
\[ |m(x) - m_\beta| \leq ce^{-\eta x} \]  
(9)
and in proposition 2.1 of [6] it is also proved that $\tilde{m} > 0$ and
$m'$ is an eigenvector of $A_\alpha$ with eigenvalue 1. Namely
\[ m' = pJ * m' \]  
(10)
where the shorthand notation
\[ p(x) = p_\alpha(x) = \beta \left[ 1 - m_\alpha(x)^2 \right] \]  
(11)
is used. Equation (10) is obtained by differentiating with
respect to $x$ the instanton equation (6). After integration by
parts the convolution on the right hand side of (10) is deduced
in this section, the asymptotic behavior of the instanton $m(x)$ as $x \to \infty$ is studied. The instanton $m(x)$ is an
antisymmetric, continuous increasing function of $x \in \mathbb{R}$
that solves (6). In Proposition 2.2 of [3] it is proved that there
exists $c > 0$ and $\eta > 0$ such that for all $x > 0$,
\[ |m(x) - m_\beta| \leq ce^{-\eta x} \]  
(9)
and in proposition 2.1 of [6] it is also proved that $\tilde{m} > 0$ and
$m'$ is an eigenvector of $A_\alpha$ with eigenvalue 1. Namely
\[ m' = pJ * m' \]  
(10)
where the shorthand notation
\[ p(x) = p_\alpha(x) = \beta \left[ 1 - m_\alpha(x)^2 \right] \]  
(11)
is used. Equation (10) is obtained by differentiating with
respect to $x$ the instanton equation (6). After integration by
parts the convolution on the right hand side of (10) is deduced
that $\tilde{m}$ is a bounded continuous function and by further
differentiations that all the derivatives of $m$ share such a
property. Since
\[ \lim_{|x| \to \infty} p(x) = p_\alpha (\text{say}) = \beta \left[ 1 - m_\alpha^2(x) \right] < 1 \]  
(12)
the obvious conjecture is that the asymptotic behavior of
$m'(x)$ as $x \to \pm \infty$ is ruled by the equation
\[ v = p_\alpha J * v. \]  
(13)
Looking for a solution of (13) of the form
\[ v(x) = e^{-\alpha x} \]  
(14)
$\alpha$ must be solved
\[ p_\alpha \int J(0,y)e^{-\alpha y} \, dy = 1 \]  
(15)
When the domain of an integral is not specified then it will be
taken the whole 3. Now, basic results for instanton can be
given as follows:

Proposition 1.

There is a strictly positive, decreasing $C^1$ function $\alpha_\alpha$, $p 
\in (0,1)$, such that $\pm \alpha_\alpha$ are the only solutions of the equation
\[ p \int J(0,y)e^{-\lambda y} \, dy = 1, \quad \lambda \in \mathbb{R}. \]  
(16)
$\alpha = \alpha_\alpha, p_\alpha = \beta(1 - m_\alpha^2)$ is written and introduced
the following kernel
\[ K(x,y) = p_\alpha(x,y)e^{-\alpha(x-y)}. \]  
This kernel can be considered as the asymptotic expression for $x$ and $y$ large of
the transition probability $P(x,y)$ given by
\begin{align}
P(x, y) &= A_m(x, y) \frac{m'(y)}{m(x)}, \quad x, y \in \mathbb{R}.
\end{align}

The following is an important result for the instanton.

**Theorem 1.**

There are \( M > 0 \) and \( \delta \in (0, \alpha) \) positive such that
\[
\lim_{x \to -\infty} e^{\alpha x} \bar{m}(x) = M, \quad \lim_{x \to +\infty} e^{\alpha x} \bar{m}'(x) = 0.
\]

An analogous statement holds for \( x \to -\infty \), as \( \bar{m}(x) \) is antisymmetric (and \( \bar{m}'(x) \) symmetric). Without loss of generality, it is restricted to \( x > 0 \). To prove Theorem 1 some useful facts have been developed which will be needed in the proof of the Theorem 1. So the theorem will be proved later.

**D. The Green Function \( \bar{G}_s(x, y) \)**

First, the following convention is introduced. Given a positive integer \( s \) the following equation in \((3)\) is considered.
\[
\begin{cases}
  v(x) = p(x)J \ast v(x) & \text{for } x \geq s \\
  v(x) = \bar{m}'(x) & \text{for } x < s.
\end{cases}
\]

It can be seen that \( \bar{m}' \) solves (18) and it is a Green function \( \bar{G}_s(x, y) \) \([q]\) for (19). An expression for \( \bar{m}'(x) \), \( x > s \), in terms of \( \bar{G}_s(x, y) \) can be obtained and that will eventually lead to the proof of Theorem 1. An identity is satisfied by \( \bar{m}'(x) \).

**Proposition 2.**

For any \( x \geq s \), the following identity holds:
\[
\bar{m}'(x) = \int_{s-1}^x G'_s(x, y) \bar{m}'(y) dy
\]

Where
\[
G_s(x, y) = \sum_{n=1}^\infty R_s^{(n)}(x, y)
\]

and, setting \( x = y_0 \) and \( y = y_n \), \( n \geq 1 \)
\[
R_s^{(n)}(y_0, y_n) = \int_{y_0}^x dy_1 \int_{y_0}^{y_1} \cdots \int_{y_0}^{y_{n-1}} J(y_{n-1}, y) dy_n
\]

**Proposition 3.**

If \( x \geq s \) and \( y \in [s-1, s] \), then
\[
e^{-\alpha(x-y)} g_s(x, y) \leq G_s(x, y) \leq e^{-\alpha(x-y)}[1 + \varepsilon(s)] g_s(x, y)
\]

where
\[
\varepsilon(s) = ce^{-\alpha s}
\]

with \( c \) a suitable, positive constant.

**E. Estimates on \( g_s(x, y) \)**

In this section, the function \( g_s(x, y) \) can be estimated.
\[ |I_0(x) - I_0'| \\
\leq \int_0^x [g_0(x-s,y) - \rho(y)] e^{\alpha(s+y)} \tilde{m}(s+y) dy \\
\leq \sup_{x \to +\infty} e^{\alpha y} \tilde{m}(z) \int_0^x [g_0(x-s,y) - \rho(y)] dy.
\]
Hence by (26), there exists \( c > 0 \) and \( \delta > 0 \) such that
\[ |I_0(x) - I_0'| \leq ce^{-\delta(x-s)} \sup_{x \to +\infty} e^{\alpha y} \tilde{m}(z). \]
This shows that
\[ I_0' = \lim_{x \to +\infty} I_0(x). \]
Let
\[ \bar{M} = \liminf_{x \to +\infty} e^{\alpha y} \tilde{m}(y), \quad \overline{M} = \limsup_{x \to +\infty} e^{\alpha y} \tilde{m}(y). \]
Thus from (31), (34) and (35),
\[ I_0' \leq M \leq \overline{M} \leq I_0' (1 + e(s)) \]
is obtained. This yields
\[ \bar{M} - M \leq I_0' (1 + e(s)) - I_0' = e(s) I_0'. \]
But, by (31) and (32), for \( s > s_0 \) large enough,
\[ I_0' \leq \int_{x_0}^x \rho(y) (1 + e(s_0)) I_0 (s+y) dy \]
and hence (37) become
\[ \bar{M} - M \leq e(s) \int_{x_0}^x \rho(y) (1 + e(s_0)) I_0 (s+y) dy \]
for \( s > s_0 \) and \( s_0 \) large enough.
\[ \delta(s) = ce^{-\eta y} \]
for \( c > 0, \eta > 0 \), and, by (31) and (36),
\[ \limsup_{x \to +\infty} I_0 (s+y) \leq \limsup_{x \to +\infty} e^{\alpha(s+y)} \tilde{m}(s+y) \]
\[ \leq I_0' (1 + e(s_0)) \]
is obtained. Using (40) and (41) it is seen that the right hand side of (39) vanishes as \( x \to +\infty \) and hence \( \bar{M} - M \).
This shows that \( s \to +\infty \) exists.
Therefore there exists \( M > 0 \) so that
\[ \lim_{x \to +\infty} e^{\alpha y} \tilde{m}(x) = M. \]
Thus the first limit in the theorem (as well as (18)) is proved.
The proof of the second one is similar. Indeed both \( M \) and \( e^{\alpha y} \tilde{m}(x) \) are in the interval with extremes min \( \{ I_0(x), I_0' \} \) and \( (1 + e(s)) \) max \( \{ I_0(x), I_0' \} \).
Then by (33) there is \( c > 0 \) such that all \( s \) large enough and all \( x > s \),
\[ |M - e^{\alpha y} \tilde{m}(y)| \leq |I_0(x) - I_0'| + e(s) [I_0(x) + I_0'] \leq c e^{-\delta(x-s)} + e^{-\eta y}. \]
If \( s \) is chosen such that \( (x-s)d_l = s \),
\[ |M - e^{\alpha y} \tilde{m}(y)| \leq 2ce^{-\eta y}. \]
If \( \delta = \eta \delta_1 / \eta + \delta_1 \), then
\[ (M - e^{\alpha y} \tilde{m}(x)) \leq 2ce^{-\delta x}. \]
Thus, Theorem 1 is proved.
The following Theorems are Corollaries of Theorem 1.

**Theorem 2.**
Let \( \alpha = M \alpha^{-1} \) and \( 0 < \delta < \delta \) as in Theorem 1. Then
\[ \lim_{x \to +\infty} e^{(\alpha + \delta^s y)} [\tilde{m}(x) - (m_{\beta} - ae^{-\alpha x})] = 0. \]
Proof: Since, from (7) \( \tilde{m}(y) \to m_{\beta} \) as \( x \to +\infty \), it follows that
\[ m_{\beta} - \tilde{m}(x) = \int_{x}^{+\infty} \tilde{m}(y) dy. \]
Then, recalling that \( M = \alpha \alpha \),
\[ e^{(\alpha + \delta^s y)} \int_{x}^{+\infty} [ae^{-\alpha y} - \tilde{m}(y)] dy \]
\[ \leq \int_{x}^{+\infty} e^{(\alpha + \delta^s y)} [M - e^{\alpha y} \tilde{m}(y)] dy \]
Now, if \( x \to +\infty \), then \( y \to +\infty \) so that \( e^{\delta y} [M - e^{\alpha y} \tilde{m}(y)] \to 0 \).
By Theorem 1. Hence the left hand side of (43) vanishes as \( x \to +\infty \). Thus the Theorem is proved.

**Theorem 3.**
Let \( \delta > 0 \) be as in Theorem 1. Then
\[ \lim_{x \to +\infty} e^{(\alpha + \delta^s y)} [e^{\alpha y} \tilde{m}(x) + \alpha M] = 0. \]
Proof: By differentiating (10) which is stated as
\[ \tilde{m}' = p(x)J * \tilde{m}(x) \]
(see (11))
\[ \tilde{m}'(x) = -2\beta \tilde{m}(x) \tilde{m}'(x)J * \tilde{m}(x) + p(x)J' * \tilde{m}(x) \]
where \( \theta \theta \theta(x, y) = \frac{\partial}{\partial x} J(x, y). \)
By Theorem 1
\[ \lim_{x \to +\infty} e^{(\alpha + \delta^s y)} [-2\beta \tilde{m}(x) \tilde{m}'(x)J * \tilde{m}(x)] = 0. \]
is obtained. So that
\[ \lim_{x \to +\infty} e^{(\alpha + \delta^s y)} [e^{\alpha y} \tilde{m}(x) - p(x)J' * e^{\alpha y} \tilde{m}'(x)] dy \]
\[ = 0. \]
Thus,
\[
\begin{align*}
|e^{\delta x} p(x) \int J'(x, y) e^{-\alpha(y-1)} \left[ e^{\delta y} \overline{m}(y) - M \right] dy & = p(x) \sum_{x=1}^{n x} \int J'(x, y) e^{-(\alpha-1)y} e^{\delta y} \left[ e^{\delta y} \overline{m}(y) - M \right] dy \\
& \leq p(x) \sum_{x=1}^{n x} \left| J'(x, y) \right| e^{-(\alpha-1)y} e^{\delta y} \left[ e^{\delta y} \overline{m}(y) - M \right] dy
\end{align*}
\]

so by Theorem 2, \( e^{\delta x} \left[ e^{\delta y} \overline{m}(y) - M \right] = 0 \) as \( x \to +\infty \). This yield \( \lim_{x \to +\infty} e^{\delta x} p(x) \int J'(x, y) e^{-(\alpha-1)y} e^{\delta y} \overline{m}(y) dy = \lim_{x \to +\infty} e^{\delta x} p(x) \int J'(x, y) e^{-(\alpha-1)y} M dy \).

Thus, from (47)
\[
\lim_{x \to +\infty} e^{\delta x} \left[ e^{\alpha x} \overline{m}(x) - p(x) \int J'(x, y) e^{-(\alpha-1)y} M dy \right] = 0.
\]

Again from (15),
\[
\int J'(x, y) e^{-(\alpha-1)y} dy = -\alpha \int J(x, y) e^{-(\alpha-1)y} dy
\]
(99)

\[
\alpha = \frac{\alpha}{p_{\alpha}}
\]

(recall that \( \alpha = \alpha_{p_{\alpha}} \) is obtained. Then (99) becomes
\[
\lim_{x \to +\infty} e^{\delta x} \left[ e^{\alpha x} \overline{m}(x) + \frac{p(x)}{p_{\alpha}} \alpha M \right] = 0
\]
(100)

Thus,
\[
\frac{p(x)}{p_{\alpha}} - 1 = \frac{1}{1 - \alpha} \left[ m^2 + \overline{m}(x) \right] = 0
\]

is obtained. Since \( 0 < \delta < \alpha \), it follows from Theorem 2 that \( e^{\delta x} \left[ m^2 + \overline{m}(x) \right] = 0 \).

Then, by (100),
\[
\lim_{x \to +\infty} e^{\delta x} \left[ e^{\alpha x} \overline{m}(x) + \alpha M \right] = 0
\]

is got. This completes the proof of the Theorem. From Theorem 1, 2 and 3, the following Theorem is obtained.

**Theorem 4.**

If \( m = \overline{m} \) is an instanton. Then there are \( \alpha \) and a positive \( \alpha_{p_{\alpha}} > \alpha \) and \( c > 0 \) such that for all \( x \geq 0 \),
\[
\left| \overline{m}(x) - \left( \frac{m^2 + \overline{m}(x)}{a(1 - a)} \right) \right| + \left| \overline{m}(x) - aae^{-\alpha x} \right| + \left| \overline{m}(x) + aae^{-\alpha x} \right| \leq ce^{-\alpha x}
\]
(101)

\[m_{\alpha,l} \] is still an integral operator and its kernel is
\[
A_{m_{\alpha,l}}(x, y) = A_{m_{\alpha,l}}(x, y) + A_{m_{\alpha,l}}(x, y, \cdot, \cdot, \cdot) \quad (x, y) \in 3.
\]

This can be interpreted as a reflecting boundary condition called the Neumann condition at 0 and the original problem for \( A_{m_{\alpha,l}} \) on \( C^{\alpha,l}(3) \) is actually the problem on the half line with Neumann conditions at 0. By adding another reflecting at \( \ell > 1 \), a new operator \( A_{m_{\alpha,l}} \) on \( C([0, \ell]) \) can be defined by setting the reflection rule
\[
R_{\ell}(x) = \begin{cases} |x| & \text{for } -\ell \leq x \leq \ell \\ |x| - (x - \ell) & \text{for } \ell \leq x \leq \ell + 1 \end{cases}
\]
(102)

Now, for \( x, y \) in \([0, \ell]\),
\[
A_{m_{\alpha,l}, \ell}(x, y) = \sum_{x \in R_{\ell}(x)} A_{m_{\alpha,l}}(x, z)
\]
(103)

is defined. Then \( A_{m_{\alpha,l}} \) (in fact \( A_{m_{\alpha,l}} \)) is the operator on \([0, \ell]\) with kernel \( A_{m_{\alpha,l}, \ell}(x, y) \). The case \( \ell = +\infty \) is included by setting \( R_{\ell}(x) = -x \), then \( A_{m_{\alpha,l}, \ell} = A_{m_{\alpha,l}, \ell} \). It can be worked in finite volume and by proving estimates uniform in \( \ell \), the original case in the limit \( \ell \to +\infty \). This is not only a technical device, but in fact the analysis in the bounded domains has its own interest. When \( m \) is taken close to a double instanton, the analysis in \([0, \ell]\) with Neumann conditions corresponds to two double instantons, one across 0 and the other one across \( \ell \). The spectral properties in this case reflect the interaction between these two structures. When \( \ell \) is finite, the classical Perron-Frobenius theorem is obtained. It can be stated the Perron-Frobenius Theorem with several other properties of the maximal eigenvalue \( \lambda_{m_{\alpha,l}} \) and the corresponding left and right eigenvectors \( u_{m_{\alpha,l}} \) and \( v_{m_{\alpha,l}} \). It is not yet supposing that \( m \) is close to a double instanton, which statements refer to general \( m \in C^{\alpha,l}(0, \ell) \), but the results are not uniform in \( \ell \). Now, the following Perron - Frobenius Theorem can be stated.

**Theorem 5.**

Let \( l > 1 \) and \( m \in C([0, \ell], [-1,1]) \). Then there are \( \lambda_{m_{\alpha,l}} \) \( > 0 \), \( u_{m_{\alpha,l}} \) and \( v_{m_{\alpha,l}} \) in \([0, \ell], u_{m_{\alpha,l}} \) and \( v_{m_{\alpha,l}} \) strictly positive, such that
\[
A_{m_{\alpha,l}} \cdot v_{m_{\alpha,l}} = \lambda_{m_{\alpha,l}} v_{m_{\alpha,l}} \quad , \quad u_{m_{\alpha,l}} \cdot A_{m_{\alpha,l}} = \lambda_{m_{\alpha,l}} u_{m_{\alpha,l}}
\]
(104)

\[v_{m_{\alpha,l}}(x) = p_{m_{\alpha,l}}(x) u_{m_{\alpha,l}}(x).\]

(105)

Any other point of the spectrum is strictly inside the ball of radius \( \lambda_{m_{\alpha,l}} \). Recall that
\[ p_m(x) = \frac{\beta}{\cosh^2\{\beta \ast m(x)\}}. \]

Since the theorem is too general it cannot say much about the localization of the spectrum and the dependence on \( \ell \) of \( \lambda_{m,\ell} \) and \( v_{m,\ell} \), for that more assumptions on \( m \) is needed. To complete, the proof of Theorem is needed that a Markov chain whose transition probability is conjugated to \( A_{m,\ell} \).

A. Auxiliary Markov Chains

In Theorem 5, it is mentioned that \( \lambda_{m,\ell} \) and \( v_{m,\ell} \) are strictly positive. Hence the function

\[ Q_{m,\ell}(x,y) = A_{m,\ell}(x,y) \frac{v_{m,\ell}(y)}{\lambda_{m,\ell} v_{m,\ell}(x)}, x, y \in [0, \ell] \]

is well-posed and its defines a transition probability on \( [0, \ell] \) conjugated to the operator \( A_{m,\ell} \) : the spectrum of \( A_{m,\ell} \) is obtained from that of \( Q_{m,\ell} \) after multiplication by \( \lambda_{m,\ell} \). In particular the spectral gap in Theorem 5 is related to the mixing properties of the Markov chain with transition probability \( Q_{m,\ell} \). If \( m \) is an instanton, \( m = \bar{m} \), then

\[ \lambda_{\bar{m}} = 1 \] and \( v_{\bar{m}} = \bar{m} \), i.e. \( \lambda_{m} = \bar{m} \), obtained by differentiating the instanton equation (6). The analogue of (28) defines the basic transition probability:

\[ P(x,y) = A_m(x,y) = \frac{\bar{m}}{\bar{m}(x)} \bar{m}(y), x, y \in \mathbb{R}. \]

In the problem with a (reflected) instanton at \( \xi \), i.e. \( \xi - x \), and Neumann conditions, i.e. reflection at \( 0 \) and \( \ell / 2 \), \( \xi > 1 \), an important role will be played by the transition probability:

\[ Q_{\xi,\xi}(x,y) = \sum_{R(\xi,\xi)} P(\xi - x, \xi - z), x \text{ and } y \text{ in } [0, \ell]. \]

The above three Markov chains can be seen as describing similar, discrete time, jump processes of a particle on the line \( \mathbb{R} \).

B. Bounds for Eigen-vectors

The following proposition provides the local bounds for eigenvectors for the operator \( A_{m,\ell} \).

**Proposition 6:** There is \( b > 1 \) so that for any \( |x - y| \leq 1 \)

\[ b^{-1} \leq \frac{v_{m,\ell}(x)}{v_{m,\ell}(y)} \leq b. \]

IV. CONCLUSION

The evolution equation in the space of symmetric bounded function is given by

\[ \frac{\partial m}{\partial t} = -m + \tanh(\beta \ast m + \beta h) \]

with \( \beta > 1, h \geq 0 \) and

\[ (J \ast m)(x) = \int J(x,y)m(y)dy \]

This project is devoted to a study of the eigenvalue problem for the integral operator \( A_m \) related to the given evolution equation. It is noted that spectral properties of the operator \( A_m \) are obtained by showing the existence of a simple positively of the corresponding eigenvector. The analysis of the instanton and of its asymptotic behavior has been discussed.

ACKNOWLEDGMENT

The author is grateful to His Excellency U Thaung, Minister, Ministry of Science and Technology for allowing me conduct this research. She wish to my gratitude to Dr.Khaing Aye, Associate Professor and Head, Department of Engineering Mathematics, Mandalay Technological University for her support of this research.

REFERENCES


