I-Vague Groups
Zelalem Teshome Wale

Abstract—The notions of I-vague groups with membership and non-membership functions taking values in an involutary dually residuated lattice ordered semigroup are introduced which generalize the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. Moreover, various operations and properties are established.

Keywords—Involutary dually residuated lattice ordered semigroup, I-vague set and I-vague group.

I. INTRODUCTION
THE notion of fuzzy groups defined by A. Rosenfeld[13] is the first application of fuzzy set theory in Algebra. Since then a number of works have been done in the area of fuzzy algebra.


K. L. N. Swamy[14], [15], [16] introduced the concept of dually residuated lattice ordered semigroups(DRL-semigroup) which is a common abstraction of Boolean algebras and lattice ordered groups. The subclass of DRL-semigroups which are bounded and involutary(e i.e having 0 as least, 1 as greatest and satisfying 1-(1-x) = x) is which is categorically equivalent to the class of MV-algebras of C. C. Chang[4] and well studied offer a natural generalization of the closed unit interval [0, 1] of real numbers as well as Boolean algebras. Thus, the study of vague sets (\( t_A, f_A \)) with values in an involutary DRL-semigroup promises a unified study of real valued vague sets and also those Boolean valued vague sets[11].

In his thesis T. Zelalem[19] studied the concept of I-vague sets. In this paper using the definition of I-vague sets, we defined and studied I-vague groups where I is an involutary DRL-semigroup. In this paper we shall recall some basic results in [14], [15], [19] without proof. Moreover, notation, terminology and results of [19] are used in this paper. Throughout this paper, we shall denote the identity element of a group (\( G_x \)) by e and the order of an element x of G by O(x). Moreover, for \( x \in G, < x > \) denotes the cyclic group generated by x.

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II. PRELIMINARIES

Definition 2.1: [14] A system \( A = (A, +, \leq, \sim) \) is called a dually residuated lattice ordered semigroup(in short DRL-semigroup) if and only if
i) \( A = (A, +) \) is a commutative semigroup with zero’0’;
ii) \( A = (A, \leq) \) is a lattice such that \( a + (b \cup c) = (a + b) \cup (a + c) \) and \( a + (b \cap c) = (a + b) \cap (a + c) \) for all \( a, b, c \in A \);
iii) Given a, b \in A, there exists a least x in A such that \( b + x \geq a \), and we denote this x by a - b (for a given a, b this x is uniquely determined);
iv) \( (a-b) \cup 0 \leq a \cup b \) for all a, b \in A;
v) \( a - a \geq 0 \) for all \( a \in A \).

Theorem 2.2: [14] Any DRL-semigroup is a distributive lattice.

Definition 2.3: [19] A DRL-semigroup A is said to be involutary if there is an element 1(\( \neq 0 \))(0 is the identity w.r.t. +) such that
i) \( a + (1-a) = 1 + 1 \);
ii) \( 1 - (1-a) = a \) for all \( a \in A \).

Theorem 2.4: [15] In a DRL-semigroup with 1, 1 is unique.

Theorem 2.5: [15] If a DRL-semigroup contains a least element x, then x = 0. Dually, if a DRL-semigroup with 1 contains a largest element \( a \), then \( a = 1 \).

Throughout this paper let \( I = (I, +, -, \vee, \wedge, 0, 1) \) be a dually residuated lattice ordered semigroup satisfying \( 1 - (1-a) = a \) for all \( a \in I \).

Lemma 2.6: [19] Let \( I \) be the largest element of I. Then for \( a, b \in I \)
(i) \( a + (1-a) = 1 \).
(ii) \( 1 - a = 1 - b \iff a = b \).
(iii) \( (a \cup b) = 1 - (a) \cap (1-b) \).

Lemma 2.7: [19] Let I be complete. If \( a_\alpha \in I \) for every \( \alpha \in \Delta \), then
(i) \( 1 - \bigwedge_{\alpha \in \Delta} a_\alpha = \bigwedge_{\alpha \in \Delta} (1 - a_\alpha) \).
(ii) \( 1 - \bigwedge_{\alpha \in \Delta} a_\alpha = \bigvee_{\alpha \in \Delta} (1 - a_\alpha) \).

Definition 2.8: [19] An I-vague set A of a non-empty set G is a pair \( (t_A, f_A) \) where \( t_A : G \rightarrow I \) and \( f_A : G \rightarrow I \) with \( t_A(x) \leq 1 - f_A(x) \) for all \( x \in G \).

Definition 2.9: [19] The interval \( [t_A(x), 1 - f_A(x)] \) is called the I-vague value of \( x \in G \) and is denoted by \( V_A(x) \).

Definition 2.10: [19] Let \( B_1 = [a_1, b_1] \) and \( B_2 = [a_2, b_2] \) be two I-vague values. We say \( B_1 \geq B_2 \) if and only if \( a_1 \geq a_2 \) and \( b_1 \geq b_2 \).

Definition 2.11: [19] An I-vague set \( A = (t_A, f_A) \) of G is said to be contained in an I-vague set \( B = (t_B, f_B) \) of G written as \( A \subseteq B \) if and only if \( t_A(x) \leq t_B(x) \) and \( f_A(x) \geq f_B(x) \) for all \( x \in G \). A is said to be equal to B written as \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).
Definition 2.12: [19] An I-vague set A of G with $V_A(x) = V_A(y)$ for all $x, y \in G$ is called a constant I-vague set of G.

Definition 2.13: [19] Let A be an I-vague set of a non empty set G. Let $A(\alpha, \beta) = \{x \in G: V_A(x) \geq [\alpha, \beta]\}$ where $\alpha, \beta \in I$ and $\alpha \leq \beta$. Then $A(\alpha, \beta)$ is called the $(\alpha, \beta)$ cut of the I-vague set A.

Definition 2.14: [19] Let $S \subseteq G$. The characteristic function of S denoted as $x_S = (t_{x_S}, f_{x_S})$, which takes values in I is defined as follows:

\[ t_{x_S}(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases} \]

and

\[ f_{x_S}(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise}. \end{cases} \]

$x_S$ is called the I-vague characteristic set of S in I. Thus $V_{x_S}(x) = \begin{cases} [1,1] & \text{if } x \in S \\ [0,0] & \text{otherwise}. \end{cases}$

Definition 2.15: [19] Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be I-vague sets of a set G. Then $A \cup B$ and $A \cap B$ are also I-vague sets of G. Let $x \in G$. Then $A \cup B$ and $A \cap B$ have

(i) $V_{A \cup B}(x) = \sup\{V_A(x), V_B(x)\}$;

(ii) $V_{A \cap B}(x) = \inf\{V_A(x), V_B(x)\}$.  

Definition 2.16: [19] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be I-vague values. Then

(i) $\sup\{B_1, B_2\} = [\sup\{a_1, a_2\}, \sup\{b_1, b_2\}]$;

(ii) $\inf\{B_1, B_2\} = [\inf\{a_1, a_2\}, \inf\{b_1, b_2\}]$.

Lemma 2.17: [19] Let A and B be I-vague sets of a set G. Then $A \cup B$ and $A \cap B$ are also I-vague sets of G. Let $x \in G$. Then $A \cup B$ and $A \cap B$ have

(i) $V_{A \cup B}(x) = \inf\{V_A(x), V_B(x)\}$;

(ii) $V_{A \cap B}(x) = \inf\{V_A(x), V_B(x)\}$.  

Definition 2.18: [19] Let I be complete and $\{A_i : i \in \Delta\}$ be a non empty family of I-vague sets of G where $A_i = (t_{A_i}, f_{A_i})$. Then

(i) $\bigcap_{i \in \Delta} A_i = (\bigwedge_{i \in \Delta} t_{A_i}, \bigwedge_{i \in \Delta} f_{A_i})$

(ii) $\bigvee_{i \in \Delta} A_i = (\bigvee_{i \in \Delta} t_{A_i}, \bigvee_{i \in \Delta} f_{A_i})$.

Lemma 2.19: [19] Let I be complete. If $\{A_i : i \in \Delta\}$ is a non empty family of I-vague sets of G, then $\bigcap_{i \in \Delta} A_i$ and $\bigvee_{i \in \Delta} A_i$ are I-vague sets of G.

Definition 2.20: [19] Let I be complete and $\{A_i : i \in \Delta\}$ be a non empty family of I-vague sets of G. Then for each $x \in G$,

(i) $\inf\{V_{A_i}(x) : i \in \Delta\} = \bigwedge_{i \in \Delta} t_{A_i}(x), \bigwedge_{i \in \Delta} (1 - f_{A_i})(x)$.

(ii) $\sup\{V_{A_i}(x) : i \in \Delta\} = \bigwedge_{i \in \Delta} t_{A_i}(x), \bigwedge_{i \in \Delta} (1 - f_{A_i})(x)$.

III. I-Vague Groups

Definition 3.1: Let G be a group. An I-vague set A of a group G is called an I-vague group of G if

(i) $V_A(xy) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$ and

(ii) $V_A(xy^{-1}) \geq V_A(x)$ for all $x \in G$.

Lemma 3.2: If A is an I-vague group of a group G, then $V_A(x) = V_A(x^{-1})$ for all $x \in G$.

Proof: Since A is an I-vague group of G, $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$. Therefore $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 3.3: If A is an I-vague group of a group G, then $V_A(e) \geq V_A(x)$ for all $x \in G$.

Proof: Let $x \in G$.

$V_A(e) = V_A(x^{-1}) \geq \inf\{V_A(x), V_A(x^{-1})\} = V_A(x)$ for all $x \in G$. Therefore $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 3.4: Let $m \in Z$. If A is an I-vague group of a group G, then $V_A(x^m) \geq V_A(x)$ for all $x \in G$.

Proof: Let $m \in Z$. We prove that $V_A(x^m) \geq V_A(x)$ for all $x \in G$. Since $V_A(e) \geq V_A(x)$ for all $x \in G$ by lemma 3.3, the statement is true for $m = 0$.

First we prove that the lemma is true for positive integers by induction.

Since $V_A(x) \geq V_A(x)$, it is true for $m = 1$.

Assume it is true for $m$. Then $V_A(x^{m+1}) \geq \inf\{V_A(x^m), V_A(x)\} = V_A(x^m)$.  

Thus $V_A(x^m+1) = V_A(x)$. Hence the statement is true for non-negative integers.

Suppose that $m$ is a negative integer.

$V_A(x^m) = V_A((x^{-1})^{-m}) \geq V_A(x^{-1}) = V_A(x)$. We have $V_A(x^m) \geq V_A(x)$.

Consequently, $V_A(x^m) \geq V_A(x)$ for all $x \in G$ and for every integer $m$. Hence the lemma follows.

Lemma 3.5: A necessary and sufficient condition for an I-vague set A of a group G to be an I-vague group of G is that $V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

Proof: Let A be an I-vague set of G. Suppose that $V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$. Let $x \in G$.

Then $V_A(e) = V_A(x^{-1}) \geq \inf\{V_A(x), V_A(x^{-1})\} = V_A(x)$.

Thus $V_A(e) \geq V_A(x)$ for all $x \in G$.

$V_A(x^{-1}) \geq \inf\{V_A(e), V_A(x^{-1})\} = V_A(x)$.

Thus $V_A(x) \geq V_A(x)$ for each $x \in G$.

Let $x, y \in G$. Then

$V_A(xy) = V_A(x(y^{-1})^{-1}) \geq \inf\{V_A(x), V_A(y^{-1})\}$

$\geq \inf\{V_A(x), V_A(y)\}$.  

Hence

$V_A(xy) \geq \inf\{V_A(x), V_A(y)\}$ for each $x, y, \in G$, so A is an I-vague group of G.

Conversely, suppose that A is an I-vague group of G. Let $x, y \in G$. Then

$V_A(xy) \geq \inf\{V_A(x), V_A(y)\}$

Therefore $V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

Hence the theorem follows.

Lemma 3.6: Let H be a subgroup of G and $[\gamma, \delta] \leq [\alpha, \beta]$ with $\alpha, \beta, \gamma, \delta \in I$ where $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$V_A(x) = \begin{cases} [\alpha, \beta] & \text{if } x \in H \\ [\gamma, \delta] & \text{otherwise} \end{cases}$

is an I-vague group of G.

Proof: Let H be a subgroup of G. We prove that the I-vague set A defined as above is an I-vague group of G.
Let $x, y \in G$. If $xy^{-1} \in H$, then $V_A(xy^{-1}) = [\alpha, \beta]$.

Hence $V_A(xy^{-1}) \supseteq \text{inf}\{V_A(x), V_A(y)\}$.

If $xy^{-1} \notin H$, then either $x \notin H$ or $y \notin H$.

Thus, $\text{inf}\{V_A(x), V_A(y)\} = \{0, \delta\}$. It follows that $V_A(xy^{-1}) \supseteq \text{inf}\{V_A(x), V_A(y)\}$. Hence $V_A(xy^{-1}) \supseteq \text{inf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$.

Therefore $A$ is an I-vague group of $G$.

**Example:** Consider the group $(Z, +)$. Let $I$ be the unit interval $[0, 1]$ of real numbers. Let $a \oplus b = \min\{1, a + b\}$.

With the usual ordering $(I, \oplus, \leq, -)$ is an involutory DRL-semigroup.

Define the I-vague set $A$ of $G$ as follows:

$$V_A(x) = \begin{cases} [a_1, b_1] & \text{if } x \in 4Z; \\ [a_2, b_2] & \text{if } x \in 2Z - 4Z; \\ [a_3, b_3] & \text{otherwise} \end{cases}$$

where $[a_1, b_1] \leq [a_2, b_2] \leq [a_1, b_1]$ and $a_1, b_1 \in [0, 1]$ for $i = 1, 2, 3$. Then $A$ is an I-vague group of $G$.

We prove that $V_A(xy^{-1}) \supseteq \text{inf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

(i) If $xy^{-1} \in 4Z$, then $V_A(xy^{-1}) = [a_1, b_1] \supseteq \text{inf}\{V_A(x), V_A(y)\}$.

(ii) If $xy^{-1} \in 2Z - 4Z$, then there exist $x, y \in Z$ such that $x \in 4Z$ or $y \notin 4Z$. This implies $\text{inf}\{V_A(x), V_A(y)\} \leq [a_2, b_2] = V_A(xy^{-1})$.

(iii) If $xy^{-1} = x - y$ is odd, then one of them must be odd. Hence $\text{inf}\{V_A(x), V_A(y)\} = [a_3, b_3] \subseteq V_A(xy^{-1})$.

Therefore $A$ is an I-vague group of $G$.

**Lemma 3.7:** Let $H \neq \emptyset$ and $H \subseteq G$. The I-vague characteristic set of $H$, $\chi_H$ is an I-vague group of $G$ iff $H$ is a subgroup of $G$.

**Proof:** Suppose that $H$ is a subgroup of $G$. By Lemma 3.6, $\chi_H$ is an I-vague group of $G$.

Conversely, suppose that $\chi_H$ is an I-vague group of $G$.

We show that $H$ is a subgroup of $G$. Let $x, y \in H$. Then $V_{\chi_H}(xy^{-1}) \supseteq \text{inf}\{V_{\chi_H}(x), V_{\chi_H}(y)\} = [1, 1]$. Hence $V_{\chi_H}(xy^{-1}) = [1, 1]$, so $xy^{-1} \in H$. Therefore $H$ is a subgroup of $G$. Hence the lemma follows.

**Lemma 3.8:** If $A$ and $B$ are I-vague groups of a group $G$, then $A \cap B$ is also an I-vague group of $G$.

**Proof:** Let $A$ and $B$ be I-vague groups of $G$. Then $A \cap B$ is an I-vague set of $G$ by lemma 2.17. Now we show that $V_{A \cap B}(xy^{-1}) \supseteq \text{inf}\{V_{A \cap B}(x), V_{A \cap B}(y)\}$ for each $x, y \in G$.

Let $x, y \in G$. Then

$$V_{A \cap B}(xy^{-1}) = \text{inf}\{V_A(xy^{-1}), V_B(xy^{-1})\} \supseteq \text{inf}\{V_A(x), V_B(x), V_A(y), V_B(y)\} = \text{inf}\{V_A(x), V_A(y), V_A(x), V_B(y)\}$$

Thus $V_{A \cap B}(xy^{-1}) \supseteq \text{inf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$. Therefore $A \cap B$ is an I-vague group of $G$.

**Lemma 3.9:** Let $I$ be complete. If $\{A_i : i \in \Delta\}$ is a nonempty family of I-vague groups of $G$, then $\bigcap_{i \in \Delta} A_i$ is an I-vague group of $G$.

**Proof:** Let $A = \bigcap_{i \in \Delta} A_i$. Then $A$ is an I-vague set of $G$ by lemma 2.19.

Now we prove that $V_A(xy^{-1}) \supseteq \text{inf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$. Let $x, y \in G$. Then

$$V_A(xy^{-1}) = \bigcap_{i \in \Delta} V_{A_i}(xy^{-1}) \supseteq \bigcap_{i \in \Delta} \text{inf}\{V_{A_i}(x), V_{A_i}(y)\} = \text{inf}\{V_A(x), V_A(y)\}$$

Thus $V_A(xy^{-1}) \supseteq \text{inf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$. Hence $A$ is an I-vague group of $G$.

**Theorem 3.11:** An I-vague set $A$ of a group $G$ is an I-vague group of $G$ if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a subgroup of $G$ whenever it is nonempty.

**Proof:** Let $A$ be an I-vague set of $G$.

Suppose that $A$ is an I-vague group of $G$. We prove that $A_{(\alpha, \beta)}$ is a subgroup of $G$ whenever it is nonempty.

Let $x, y \in A_{(\alpha, \beta)}$. Then $V_A(x) \supseteq [\alpha, \beta]$ and...
Let $A$ be an $I$-vague group of $G$. Since $A$ is an $I$-vague group of $G$, $V_A(x^{-1}) \geq \inf\{V_A(x), V_A(y)\} \geq [\alpha, \beta]$. Hence $x^{-1} \in A_{[\alpha, \beta]}$, so $A_{[\alpha, \beta]}$ is a subgroup of $G$.

Conversely, suppose that for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the non-empty set $A_{[\alpha, \beta]}$ is a subgroup of $G$. We prove that $A$ is an $I$-vague group of $G$.

Let $x, y \in G$. Suppose that $V_A(x) = [\alpha, \beta]$ and $V_A(y) = [\gamma, \delta]$. Then $x \in A_{[\alpha, \beta]}$ and $y \in A_{[\gamma, \delta]}$.

Let $\inf\{V_A(x), V_A(y)\} = [\alpha \wedge \gamma, \beta \wedge \delta] = [\eta, \zeta]$. It follows that $x, y \in A_{[\alpha, \zeta]}$. Since $A_{[\alpha, \zeta]}$ is a subgroup of $G$, $x^{-1} \in A_{[\alpha, \zeta]}$. Thus $V_A(x^{-1}) \geq [\eta, \zeta]$. As a result we have $V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\}$. Therefore $A$ is an $I$-vague group of $G$. Hence the theorem follows.

**Theorem 3.12:** Let $A$ be an $I$-vague group of a group $G$.

If $V_A(xy^{-1}) = V_A(x)$ for $x, y \in G$, then $V_A(x) = V_A(y)$.

**Proof:** Suppose that $V_A(xy^{-1}) = V_A(x)$ for $x, y \in G$. Let $V_A(x) = V_A(xy^{-1}) = \inf\{V_A(x)^{-1}, V_A(x)\} = \inf\{V_A(x), V_A(y)\}$. Thus $V_A(x) = V_A(y)$. Therefore $V_A(x) = V_A(y)$. Hence the theorem follows.

The following example shows that the converse of the preceding theorem is not true.

**Example:** Let $I$ be the unit interval $[0, 1]$ of real numbers. Define $[0, 0] + [0, 1] = [1, 1]$. With the usual ordering $(+, \leq, -)$, the interval $[0, 0] + [0, 1]$ is an involutary DRL-seminigroup.

Consider $G = (Z, +)$ and $H = \{3Z, +\}$. Let $A$ be the $I$-vague group of $G$ defined by

$$V_A(x) = \begin{cases} \[1, 1] & \text{if } x \in H \\ \[0, \frac{1}{3}] & \text{otherwise.} \end{cases}$$

Let $x = 2$ and $y = 1$. $V_A(x) = V_A(y) = [0, \frac{2}{3}]$, but $V_A(xy^{-1}) = V_A(2y^{-1}) = V_A(1) = [0, \frac{1}{3}] \neq V_A(0)$.

**Theorem 3.13:** Let $A$ be an $I$-vague group of a group $G$ and $x \in G$. Then $V_A(xy) = V_A(x) = V_A(y)$ for all $y \in G$ iff $V_A(x) = V_A(x')$.

**Proof:** Let $A$ be an $I$-vague group of a group $G$ and $x \in G$. Suppose that $V_A(xy) = V_A(x) = V_A(y)$ for all $y \in G$. Take $y = e$. It follows that $V_A(x) = V_A(e)$.

Conversely, suppose that $V_A(x) = V_A(e)$. We prove that $V_A(xy) = V_A(x) = V_A(y)$ for all $y \in G$.

For any $y \in G$, $V_A(y) \leq V_A(x') = V_A(x)$.

$V_A(xy) \geq \inf\{V_A(x), V_A(y)\} = V_A(y)$.

Hence $V_A(xy) \geq V_A(y)$.

$V_A(y) = V_A(xy) = V_A(x^{-1}xy) \geq \inf\{V_A(x^{-1}), V_A(xy)\} = \inf\{V_A(x), V_A(xy)\} = \inf\{V_A(x), V_A(y)\} = V_A(xy)$.

Thus $V_A(y) \geq V_A(xy)$. Hence we have $V_A(xy) = V_A(y)$.

Similarly, $V_A(xy) = V_A(y)$. Therefore $V_A(xy) = V_A(xy) = V_A(y)$.

Hence the theorem follows.

**Lemma 3.14:** Let $A$ be an $I$-vague group of a group $G$.

Then $G \sqsubseteq A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of $G$.

**Proof:** Let $A$ be an $I$-vague group of a group $G$. Since $e \in G \sqsubseteq A$, $G \sqsubseteq A \neq \emptyset$ and $G \sqsubseteq A \subseteq G$. Let $x, y \in G \sqsubseteq A$. We prove that $xy^{-1} \in G \sqsubseteq A$.

$V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\} = V_A(e)$. Since $V_A(e) \geq V_A(xy^{-1})$ for all $x, y \in G \sqsubseteq A$, $V_A(xy^{-1}) = V_A(e)$. Thus $xy^{-1} \in G \sqsubseteq A$. Therefore $G \sqsubseteq A$ is a subgroup of $G$.

**Lemma 3.15:** Let $A$ be an $I$-vague group of a group $G$.

If $< x > \subseteq < y >$ then $V_A(y) \subseteq V_A(x)$.

**Proof:** Suppose that $< x > \subseteq < y >$. Then $x \in < y >$. It follows that $x = y^m$ for some $m \in Z$.

$V_A(x) \geq V_A(y^m) \geq V_A(y)$. Therefore $V_A(x) \geq V_A(y)$. The following example shows that the converse of lemma 3.15 is not true.

**Example:** Let $I$ be the unit interval $[0, 1]$ of real numbers. Let $(+b, +b) = [1, 1]$. With the usual ordering $(+, \leq, -)$, the interval $(+b, +b)$ is an involutary DRL-seminigroup. Let $G = \{e, a, b, c\}$.

Define the $I$-vague set $A$ of $G$ by

$$V_A(x) = \begin{cases} \[\frac{1}{2}, 1] & \text{if } x \in < a > \\ [0, \frac{1}{2}] & \text{otherwise.} \end{cases}$$

Then $V_A(c) = [\frac{1}{2}, 1] \leq [\frac{1}{2}, 1] = V_A(a)$ but $< a >$ is not a subset of $< c >$.

**Definition 3.16:** Let $A$ be an $I$-vague group of a group $G$. Image of $A$ is defined as $ImA = \{V_A(x) : x \in G\}$.

Since $V_A(e) \geq V_A(x)$ for all $x \in G$, $V_A(e)$ is the greatest element of $ImA$.

**Theorem 3.17:** Let $A$ be an $I$-vague group of a group $G$.

(i) If $G$ is cyclic then $ImA$ has a least element.

(ii) If $V_A(x) \leq V_A(y)$ then $< x > \subseteq < y >$ and $ImA$ has a least element then $G$ is cyclic.

**Proof:** Let $A$ be an $I$-vague group of $G$.

(i) Suppose that $G$ is cyclic. Then $G = < x >$ for some $x \in G$. We prove that $V_A(x)$ is the least element of $ImA$.

Let $y \in G$. Then $y = x^m$ for some $m \in Z$. $V_A(y) = V_A(x^m) \geq V_A(x)$. We have $V_A(x) \leq V_A(y)$ for every $y \in G$. Thus $V_A(x)$ is the least element of $A$.

Hence $ImA$ has a least element.

(ii) Suppose that $ImA$ has a least element say $V_A(x)$ for some $x \in G$. Let $y \in G$. Thus $V_A(y) \geq V_A(x)$ for all $y \in G$. By our condition we have $< y > \subseteq < x >$. Since $y \in < y >$, $y \in < x >$. Hence $G \subseteq < x >$. Consequently, we have $G = < x >$. Therefore $G$ is cyclic.

**Lemma 3.18:** Let $A$ be an $I$-vague group of $G$. Let $x, y \in G$.

The two conditions

i) $V_A(x) = V_A(y) \Rightarrow < x > = < y >$

ii) $V_A(x) > V_A(y) \Rightarrow < x > \subset < y >$

are equivalent to the condition $V_A(x) \geq V_A(y) \Rightarrow < x > \subset < y >$.

**Proof:** Assume that the two conditions are given. We prove that $V_A(x) \geq V_A(y) \Rightarrow < x > \subset < y >$.

If $V_A(x) > V_A(y)$, then $< x > \subset < y >$ by (ii).

If $V_A(x) = V_A(y)$, then $< x > = < y >$ by (i).

We have $< x > \subset < y >$.

Conversely, assume that $V_A(x) \geq V_A(y) \Rightarrow < x > \subset < y >$.

(i) Suppose that $V_A(x) = V_A(y)$.

$V_A(x) = V_A(y) \Rightarrow V_A(x) \geq V_A(y)$ and $V_A(y) \geq V_A(x)$.

$\Rightarrow < x > \subset < y >$ and $< y > \subset < x >$.

$\Rightarrow < x > = < y >$.

Therefore $V_A(x) \geq V_A(y) \Rightarrow < x > \subset < y >$.

Thus $V_A(x) \geq V_A(y) \Rightarrow < x > < y >$. Therefore $V_A(x) \geq V_A(y) \iff < x > < y >$.

(i) and (ii) are equivalent to $V_A(x) \geq V_A(y) \iff < x > < y >$.
Let $A$ be an I-vague group of a group $G$ such that the image set of $A$ is given by $\text{Im}A = \{I_0 \supset I_1 \supset ... \supset I_n\}$ and such that

(i) $V_A(x) = V_A(y) \Rightarrow \langle x \rangle \leq \langle y \rangle$,

(ii) $V_A(x) < V_A(y) \Rightarrow \langle x \rangle > \langle y \rangle$.

Then $G$ is a cyclic group of prime power order.

**Proof:** Let $A$ be an I-vague group of a group $G$. Since $\text{Im}A = \{I_0 \supset I_1 \supset ... \supset I_n\}$, $\text{Im}A$ has a least element. By theorem 3.17, $G$ is cyclic. It follows that $G \cong Z$ or $G \cong Z_m$ for some $m \in N$. Suppose that $G \cong Z$. Consider $V_A(2)$ and $V_A(3)$. If $V_A(2) = V_A(3)$, then $2 \leq \langle 3 \rangle$ by (i). But this is not true since $2 \not\leq 3$. So either $V_A(2) > V_A(3)$ or $V_A(3) > V_A(2)$.

If $V_A(2) > V_A(3)$, then $2 > \langle 3 \rangle$ by (ii). But this is not true since $2 \not\leq 3$.

If $V_A(3) > V_A(2)$, then $3 > \langle 2 \rangle$ by (ii). But this is not true since $3 \not\leq 2$. Therefore $G$ is not isomorphic to $Z_m$.

Thus $G \cong Z_m$ for some $m \in N$.

Suppose that $m$ is not a prime power. Then there exist prime numbers $p$ and $q$ such that $p \neq q$ which are factors of $m$.

Consider $V_A(p)$ and $V_A(q)$.

Since $\text{Im}A = \{I_0 \supset I_1 \supset ... \supset I_n\}$, either $V_A(p) \supset V_A(q)$ or $V_A(q) \supset V_A(p)$. It follows that $p \leq \langle q \rangle$ or $q \leq \langle p \rangle$, a contradiction.

Thus our supposition is false. Therefore $m$ is prime power. Hence the theorem follows.

**Theorem 3.20:** Let $G$ be a cyclic group of prime power order then there is an I and an I-vague group $A$ of $G$ such that for all $x, y \in G$

(i) $V_A(x) = V_A(y) \Rightarrow \langle x \rangle = \langle y \rangle$,

(ii) $V_A(x) > V_A(y) \Rightarrow \langle x \rangle \leq \langle y \rangle$.

**Proof:** Suppose that $G$ is a cyclic group of order $p^n$ where $p$ is prime and $n \in N \cup \{0\}$. We find an I and an I-vague group $A$ of $G$ satisfying (i) and (ii).

Step(1) We construct an I and an I-vague set of $G$.

Let $I$ be the unit interval $[0, 1]$ of real numbers. Define $a \oplus b = \min(1, a + b)$

With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup.

Now we construct our I-vague set of $G$.

Let $x \in G$. Then $O(x) = p^i$ where $i = 0, 1, 2, ..., n$.

Define $A = \{I_0, I_1, ... \}$ where $I_0 : G \rightarrow I$ and $I_1 : G \rightarrow I$ such that $I_0(x) = a_i, I_1(x) = b_i$ where $a_i, b_i \in \{0, 1\}$ satisfying $a_i \leq 1 - b_i$ for $i = 0, 1, 2, ..., n$. Choose the intervals $I_0, I_1, ...$ in such a way that $I_0 > I_1 > ... > I_n$ where $I_0 = [a_0, 1 - b_0]$. Then $V_A(x) = I_0$. Hence $A$ is an I-vague set of $G$. We have $V_A(x) = I_0$.

Step(2) We show that $A$ is an I-vague group of $G$.

Let $x \in G$. $O(x) = O(x^{-1})$ implies $V_A(x) = V_A(x^{-1})$.

To show $A$ is an I-vague group of $G$ it remains to prove that $V_A(x) \supset \text{Im}V_A(x)$ for every $x, y \in G$.

Let $x, y \in G$. Since $G$ is a cyclic group of order $p^n$ and the order of the subgroup divides the order of the group, $O(\langle x \rangle) = p^j, O(\langle y \rangle) = p^k$ and $O(\langle xy \rangle) = p^m$ for some $j, k, m \in \{0, 1, ..., n\}$.

Therefore $V_A(x) = I_j, V_A(y) = I_k$ and $V_A(xy) = I_m$.

Moreover, since $G$ is a cyclic group of prime power order, $\langle x \rangle \leq \langle y \rangle$ or $\langle y \rangle \leq \langle x \rangle$.

If $x \leq y$, then $x, y \in G$. Hence $\langle x \rangle \leq \langle y \rangle$.

If $y \leq x$, then $x, y \in G$. Hence $\langle y \rangle \leq \langle x \rangle$.

Therefore $\langle x \rangle \leq \langle y \rangle$ or $\langle y \rangle \leq \langle x \rangle$.

Assume that $\langle x \rangle \leq \langle y \rangle$. It follows that $O(\langle x \rangle) \leq O(\langle y \rangle)$ or $O(\langle x \rangle) = O(\langle y \rangle)$. If $O(\langle x \rangle) < O(\langle y \rangle)$ then $m < j$. It follows that $I_m > I_j$.

Hence $V_A(xy) = I_m \geq \text{inf}\{I_j, I_k\} = \text{inf}\{V_A(x), V_A(y)\}$.

Thus $V_A(xy) \supset \text{Im}V_A(x)$, $V_A(y)$.

Therefore $V_A(xy) > V_A(x)$ or $V_A(xy) > V_A(y)$.

Thus $V_A(xy) > V_A(x)$ or $V_A(xy) > V_A(y)$.

Hence $G$ is cyclic of prime power order.

**References:**


