The Banzhaf-Owen value for fuzzy games with a coalition structure

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Abstract—In this paper, a generalized form of the Banzhaf-Owen value for cooperative fuzzy games with a coalition structure is proposed. Its axiomatic system is given by extending crisp case. In order to better understand the Banzhaf-Owen value for fuzzy games with a coalition structure, we briefly introduce the Banzhaf-Owen values for two special kinds of fuzzy games with a coalition structure, and give their explicit forms.

Keywords—cooperative fuzzy game, Banzhaf-Owen value, multilinear extension, Choquet integral

I. INTRODUCTION

As an important kind of games, games with coalition structures proposed by Owen [1] have been researched by many scholars and experts. The payoff indices for games with coalition structures are researched by Owen [1, 2] and Alonso-Mejide and Fiestras-Janeiro [3]. The axiomatic systems of the payoff indices for games with coalition structures are considered in [1-8]. But all above are only introduced for crisp games.

As Aubin [9] pointed, there are some situations where players do not fully participate in a coalition, but take action only according to the rate of participation. Generally speaking, it is difficult to identify a characteristic function of fuzzy games in practice. Hence, the characteristic function of fuzzy games is often constructed based on the characteristic function of the original crisp games. Owen [10] defined a kind of fuzzy games, which is called fuzzy games with multilinear extension form. Tsurumi et al. [11] defined a kind of fuzzy games with weighted function, and gave the expression of the Shapley function on this limited multilinear extension form. In section 2, we recall some notations and basic definitions, which will be used in the following. In section 3, an axiomatic definition of the Banzhaf-Owen value for fuzzy games with coalition structures is offered, and its explicit form is given. In section 4, we briefly discuss the Banzhaf-Owen values for two special kinds of fuzzy games with coalition structures, and give and investigate the explicit forms of the given Banzhaf-Owen values.

II. CRISP GAMES WITH COALITION STRUCTURES

Let \( N = \{1, 2, \ldots, n\} \) be the player set, and \( P(N) \) be the set of all crisp subsets in \( N \). The coalitions in \( P(N) \) are denoted by \( S_0, T_0, \ldots \). For any \( S_0 \in P(N) \), the cardinality of \( S_0 \) is denoted by the corresponding lower case \( s \). A function \( v_0 : P(N) \rightarrow \mathbb{R}_+ \), satisfying \( v_0(\emptyset) = 0 \), is called a crisp game. Let \( G_0(N) \) denote the set of all crisp games in \( N \).

A crisp coalition structure \( \Gamma = \{B_1^0, B_2^0, \ldots, B_n^0\} \) is a partition of \( N \), i.e., \( \bigcup_{1 \leq j \leq n} B_j^0 = N \) and \( B_i^0 \cap B_j^0 = \emptyset \) for \( i \neq j \) in \( M = \{1, 2, \ldots, m\} \). A crisp coalition structure in \( N \) is denoted by \((N, \Gamma)\). For any \( S_0 \), \( S_0 \in P(N, \Gamma) \) is called a feasible coalition, where \( P(N, \Gamma) \) denotes the set of all feasible coalitions in \((N, \Gamma)\).

Example 1: Let \( N = \{1, 2, 3, 4\} \) and \( \Gamma = \{B_1^0, B_2^0\} \), where \( B_1^0 = \{1, 2\} \) and \( B_2^0 = \{3, 4\} \), then \( P(N, \Gamma) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\} \). Owen [10] proposed the Banzhaf-Owen value for crisp games with a coalition structure, which can be expressed by

\[
\varphi_i(N, v_0, \Gamma) = \sum_{R \subseteq M \setminus k} \sum_{T_0 \subseteq B_0^k \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{n-k-1}} \left( v_0(Q_0 \cup T_0 \cup i) - v_0(Q_0 \cup T_0) \right) \quad \forall i \in N, \tag{1}
\]

where \( Q_0 = \bigcup_{q \in R} B_q^k \), \( B_0^k \in \Gamma \) is an union such that \( i \in B_0^k \).

III. THE BANZHAF-OWEN VALUE FOR FUZZY GAMES WITH A COALITION STRUCTURE

Let \( L(N) \) denote the set of all fuzzy coalitions in \( N \). The coalitions in \( L(N) \) are denoted by \( S, T, \ldots \). For a fuzzy coalition \( S \) and player \( i \), \( S(i) \) indicates the membership grade of \( i \) in \( S \), i.e., the rate of the \( i \)-th player’s participation in \( S \). For any \( S \in L(N) \), the support is denoted by \( \text{Supp} S = \{i \in N | S(i) > 0\} \), and the cardinality is denoted by \( |\text{Supp} S| \). We use the notation \( S \subseteq T \) if and only if \( S(i) = T(i) \) or \( S(i) = 0 \) for any \( i \in N \). A function \( v : L(N) \rightarrow \mathbb{R}_+ \),
satisfying \( v(\emptyset) = 0 \), is called a fuzzy game. Let \( G(N) \) denote the set of all fuzzy games in \( L(N) \). For any \( S \in L(N) \), we will use \( S = \{ S(i) \}_{i \in N} \) to denote it, and we will omit braces for singletons, e.g., by writing \( v(S), S \cap \bigwedge T, S \cup U \) instead of \( v(S), \{ S \} \cup \{ T \}, \{ S \} \setminus \{ T \}, \{ U \} \) for any \( v \in G(N) \) and any \( U, V \in L(N) \). In this paper, union and intersection of two fuzzy coalitions are defined as usual, i.e., \( (S \cap \bigwedge T) = S \cap \bigwedge T \) and \( (S \cup \bigwedge T) = S \cup \bigwedge T \).

Similar to crisp case, a fuzzy coalition structure \( \Gamma_F = \{ B_1, B_2, \ldots, B_m \} \) in a partition of \( U \), i.e., \( \bigcup_{i \leq m} \text{Supp} B_i = \text{Supp} U \) and \( \text{Supp} B_i \cap \text{Supp} B_j = \emptyset \) for any \( i \neq j, i, j \in \{ 1, 2, \ldots, m \} \), \( B_i \subseteq U \). The fuzzy coalition structure \( \Gamma_F \) in \( U \) is denoted by \( (U, \Gamma_F) \). For any \( S \subseteq L(U, \Gamma_F) \), \( S \) is considered a fuzzy coalition, where \( L(U, \Gamma_F) \) denotes the set of all feasible fuzzy coalitions in \( (U, \Gamma_F) \).

**Example 2:** Let \( N = \{ 1, 2, 3, 4 \} \), and \( U = \{ U(i) \}_{i \in N} \) be a fuzzy coalition in \( L(N) \) such that \( U(i) > 0 \) for any \( i \in N \). \( \Gamma_F = \{ B_1, B_2 \} \) is a fuzzy coalition structure in \( U \), where \( B_1 = \{ U(1), U(2) \} \) and \( B_2 = \{ U(3), U(4) \} \), then \( L(U, \Gamma_F) = \{ 0, \{ U(i) \}_{i \in N}, \{ U(1), U(2) \}, \{ U(3), U(4) \}, \{ U(1), U(2), U(3), U(4) \}, \{ U(1), U(3), U(4) \}, \{ U(2), U(3), U(4) \}, U \} \).

**Remark 1:** In this paper, if there is no special explanation, \( \Gamma_F = \{ B_1, B_2, \ldots, B_m \} \) and \( M = \{ 1, 2, \ldots, m \} \) always hold. Moreover, we use \( N, v_0, \Gamma \) and \( (U, v, \Gamma_F) \) to denote crisp game \( v_0 \in G_0(N) \) in coalition structure \( (N, \Gamma) \), and fuzzy game \( v \in G(N) \) in coalition structure \( (U, \Gamma_F) \), respectively.

**Definition 1:** Let \( v \in G(N) \), \( v^B \) is called a quotient game on \( \Gamma_F \), if it satisfies \( v^B(R) = v(\bigcup_{i \in R} B_i) \) for all \( R \subseteq M \).

**Theorem 1:** Let \( v \in G(N) \), the function \( \beta : (U, v, \Gamma_F) \to \mathbb{R}_+^L(U) \) is defined by

\[
\beta_i(U, v, \Gamma_F) = \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{ U(i) \}} \frac{1}{2^{m-1} \text{Supp} B_k} \left( v(S \cup \bigwedge U) - v(S \cap \bigwedge U) \right) \quad \forall i \in \text{Supp} U,
\]

where \( Q = \bigvee_{R \subseteq M \setminus k} B_R \), and \( m \) is the number of the players. Then \( \beta \) is the unique Banzhaf-Owen value in \( (U, v, \Gamma_F) \).

**Proof:** Existence. From Eq.(2), we easily get Axioms 1, 2. Axiom 3: From Eq.(2), we have

\[
\beta_i(U, v, \Gamma_F) = \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{ U(i) \}} \frac{1}{2^{m-1} \text{Supp} B_k} \left( v(S \cup \bigwedge U) - v(S \cap \bigwedge U) \right) \quad \forall i \in \text{Supp} U,
\]

where \( Q = \bigvee_{R \subseteq M \setminus k} B_R \), and \( m \) is the number of the players. Then \( \beta \) is the unique Banzhaf-Owen value in \( (U, v, \Gamma_F) \).
Axiom 4: From Eq.(2), we have
\[ \beta(U, v, \Gamma_F) + \beta(U, v, \Gamma_F) = \sum_{R \subseteq M\setminus k, S \subseteq B_k} \sum_{f \in \text{Supp}S} \frac{1}{2^{\text{supp}S}} (v(S \cap U(i)) - v(S \cap \emptyset)) \]
\[ \sum_{R \subseteq M\setminus k, S \subseteq B_k} \sum_{f \in \text{Supp}S} \frac{1}{2^{\text{supp}S}} (v(S \cap U(i)) - v(S \cap \emptyset)) \]
Define the unanimity quotient game as follows:
\[ u_T(R) = \begin{cases} 
1 & T \subseteq R \\
0 & \text{otherwise} 
\end{cases} \]
From Axiom 2, we only need to show the uniqueness of Eq.(2) on the unanimity games. For any $\emptyset \neq T \subseteq L(U, \Gamma_F)$, let
\[ M' = \{ k \in M : \text{Supp}B_k \cap \text{Supp}T \neq \emptyset \} \]
Define the unanimity quotient game as follows:
\[ u_T(R) = \begin{cases} 
1 & M' \subseteq R \\
0 & M' \not\subseteq R 
\end{cases} \]
where $R \subseteq M$.
Let $f$ be a solution in $(U, u_T, \Gamma_F)$ that satisfies the mentioned axioms in Definition 2. From Axiom 1, we have $f_i(U, u_T, \Gamma_F) = 0$ for any $i \notin \text{Supp}T$.
Case (i): When $m' = 1$, where $m'$ denote the cardinality of $M'$. From Axioms 1, 2, 3, 4, we get
\[ f_i(U, u_T, \Gamma_F) = \begin{cases} 
\frac{1}{2^{\text{supp}S}}(u_B(R \cup k) - u_B(R)) & i \in \text{Supp}B_k, k \in M' \\
0 & \text{otherwise} 
\end{cases} \]
From Eq.(2), we have $\beta(U, u_T, \Gamma_F) = f(U, u_T, \Gamma_F)$.
Case (ii): When $m' > 1$, and Supp$B_k = \{1\}$. From Axiom 5, we get
\[ f_i(U, u_T, \Gamma_F) = \begin{cases} 
\frac{1}{2^{\text{supp}S}}(u_B(R \cup k) - u_B(R)) & i \in \text{Supp}B_k, k \in M' \\
0 & \text{otherwise} 
\end{cases} \]
On the other hand, from Eq.(2), we have
\[ \beta(U, u_T, \Gamma_F) = \sum_{R \subseteq M\setminus k, S \subseteq B_k} \sum_{f \in \text{Supp}S} \frac{1}{2^{\text{supp}S}} (v(S \cap U(i)) - v(S \cap \emptyset)) \]
\[ \sum_{R \subseteq M\setminus k, S \subseteq B_k} \sum_{f \in \text{Supp}S} \frac{1}{2^{\text{supp}S}} (v(S \cap U(i)) - v(S \cap \emptyset)) \]
Thus, we have $\beta(U, u_T, \Gamma_F) = f(U, u_T, \Gamma_F)$.
Case (iii): When $m' > 1$, and Supp$B_k > \{1\}$. Without loss of generality, suppose $\text{Supp}B_k = \{ h \}$, where Supp$B_k = \{1, j_1, j_2, \ldots, j_{l_1} \}$. Let $\Gamma_F^{j_1} = \{B_1', B_2', \ldots, B_{m_1+1}'\}$, where $B_{m_1+1}' = \{U(j_1)\}$, $B_k = B_k \cup U(j_1)$ and $B_l = B_l$ for any $i \in \{1, 2, \ldots, k - 1, k + 1, \ldots, m\}$; Let $\Gamma_F^{j_1, j_2} = \{B_1', B_2', \ldots, B_{m_1+1}', B_{m_2+2}'\}$, where $B_{m_2+2}' = \{U(j_2)\}$. $B_k = B_k \cup U(j_2)$ and $B_l = B_l$ for any $i \in \{1, 2, \ldots, k - 1, k + 1, \ldots, m, m + 1\}$, from recursive
method and Axioms 5, 6, we get
\[ f_i(U,u_T,\Gamma_F) = \sum_{R \subseteq \{M \cup H\}^k \cap \mathbb{N}} \frac{1}{2^{m+k-2}} (u^R(R \cup k) - u^R(R)) \]
\[ = \sum_{M \cap k \subseteq R \subseteq \{M \cup H\}^k \cap \mathbb{N}} \frac{1}{2^{m+k-2}} \]
\[ = \frac{1}{2^{m+k-2}} \sum_{M \cap k \subseteq R \subseteq \mathbb{M}} \frac{1}{2^{m+k-2}} \]
\[ = \frac{1}{2^{m+k-2}} \sum_{M \cap k \subseteq R \subseteq \mathbb{M}} \frac{1}{2^{m+k-2}} \]
where $H' = \{1,2,...,h-1\}$. From Eq.(2), we obtain
\[ \beta_i(U,u_T,\Gamma_F) = \frac{1}{2^{m+k-1}} \frac{1}{2^{m+k-1}} = \frac{1}{2^{m+k-2}}. \]

Thus, Eq.(2) is unique on the unanimity games, and the proof is finished.

IV. THE BANZHAF-OWEN VALUE FOR TWO SPECIAL KINDS OF FUZZY GAMES WITH A COALITION STRUCTURE

In this section, we shall discuss the Banzhaf-Owen value for two special kinds of fuzzy games with a coalition structure, which are proposed by Owen [10] and Tsurumi et al. [11]. The fuzzy coalition value for fuzzy games with "multilinear extension form" is expressed by (Owen [10])
\[ v_0(U) = \sum_{T \subseteq U} \Pi_{i \in \text{Supp} T} U(i) \Pi_{i \in \text{Supp} T} \cap \text{Supp} T (1-U(i)) \quad \forall U \in \mathbb{L}(N). \]

The fuzzy coalition value for fuzzy games with "Choquet integral form" is written as (Tsurumi et al. [11])
\[ v_c(U) = \sum_{i=1}^n q(U) v_0([U]_{h_i}) h_i h_i \quad \forall U \in \mathbb{L}(N). \]
where $[U]_{h_i} = \{i \in N | U(i) \geq h_i\}$ and $Q(U) = \{U(i) | U(i) > 0, i \in N\}$, $q(U)$ denotes the cardinality of $Q(U)$. The elements in $Q(U)$ are written in the increasing order as $0 = h_0 \leq h_1 \leq \cdots \leq h_n$. Let $G_0(U)$ and $G_c(U)$ denote the set of all fuzzy games in $U \in \mathbb{L}(N)$ given by Owen [10] and Tsurumi et al. [11], respectively. For $(U,v,\Gamma_F)$, when we restrict $v \in G(N)$ in the setting of $v_0 \in G_0(U)$ and $v_c \in G_c(U)$, then we get the coalition structure $(U,v_0,\Gamma_F)$ and $(U,v_c,\Gamma_F)$, respectively.

Theorem 2: Let $v_0 \in G_0(U)$, the function $\beta^O : (U,v_0,\Gamma_F) \to \mathbb{R}^U$ is defined by
\[ \beta^O(U,v_0,\Gamma_F) = \sum_{R \subseteq \{M \cup H\}^k \cap \mathbb{N}} \frac{1}{2^{m+k-1}} \frac{1}{2^{m+k-2}} \sum_{\text{Supp} R = \{1,2,...,N\}} \]
where $L(R)$ denotes the power set of $R$, $\Gamma^{S\cup V}(U) \cap \text{Supp} B_p$ and $\Gamma^{S\cup V}(U) \cap \text{Supp} B_p$ are the fuzzy coalition structures in $S \cup U(i) \in \text{Supp} B_p$ and $\Gamma^{S\cup V}(U) \cap \text{Supp} B_p$ with respect to $\Gamma_F$. $L(S \cup U(i) \in \text{Supp} B_p, \Gamma_F)$ and $\Gamma^{S\cup V}(U) \cap \text{Supp} B_p$ denote the support of $L(S \cup U(i) \in \text{Supp} B_p, \Gamma_F)$ and $L(S \cup U(i) \in \text{Supp} B_p, \Gamma_F)$, respectively. $Q = \cup R \in B_p$. Then $\beta^O$ is the unique Banzhaf-Owen value in $(U,v_0,\Gamma_F)$.

Proof: The proof of Theorem 2 is similar to that of Theorem 1.

Example 3: (cf. Example 2). Let the player set $N = \{1,2,3,4\}$, and $U = \{U(i)\}_{i \in N}$ be a fuzzy coalition in as defined in Example 3.1, i.e., $U(i) > 0$ for any $i \in N$, and $\Gamma_F = \{B_1, B_2\}$ is a fuzzy coalition structure in $U$ as defined in Example 3.1, i.e., $B_1 = \{U(1), U(2)\}$ and $B_2 = \{U(3), U(4)\}$.
\[ \text{Supp}_L(U,\Gamma_F) = \{\emptyset, \{i\} | i \in N, \{1,2\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, N\}. \]
\[ \text{Supp}_L(S,\Gamma_F) = \{\emptyset, \{i\} | i \in \{1,2,3\}, \{1,2\}, \{1,2,3\}\}. \]
\[ \text{Supp}_L(S,\Gamma_F) = \{\emptyset, \{i\} | i \in \{1,2,3\}, \{1,2\}, \{1,2,3\}\}. \]
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\[ \text{Supp}_L(S,\Gamma_F) = \{\emptyset, \{i\} | i \in \{1,2,3\}, \{1,2\}, \{1,2,3\}\}. \]

Theorem 3: Let $v_c \in G_c(U)$, the function $\beta^C : (U,v_c,\Gamma_F) \to \mathbb{R}^U$ is defined by
\[ \beta^C(U,v_c,\Gamma_F) = \sum_{i=1}^n \beta(U[i],h_i) v_0([U]_{h_i}) h_i \quad \forall i \in \text{Supp} U. \]

\[ \beta^C(U,v_0,\Gamma_F) = 0.84, \beta^C(U,v_0,\Gamma_F) = 0.83, \]
\[ \beta^C(U,v_0,\Gamma_F) = 0.78, \beta^C(U,v_0,\Gamma_F) = 1.63. \]
\[ \beta^0_\varphi(U, v, \Gamma_F) = 1.51. \]

When we restrict \( v \in G(N) \) in domain of \( v_C \in G_C(U) \), from Eq.(6), we get the player Banzhaf-Owen values with respect to \( v_C \) are

\[ \beta^1_\varphi(U, v_C, \Gamma_F) = 1.56, \beta^2_\varphi(U, v_C, \Gamma_F) = 1.69, \]
\[ \beta^3_\varphi(U, v_C, \Gamma_F) = 1.5, \beta^4_\varphi(U, v_C, \Gamma_F) = 2.6, \]
\[ \beta^5_\varphi(U, v_C, \Gamma_F) = 2.2. \]

V. CONCLUSION

We have researched the Banzhaf-Owen value for fuzzy games with coalition structures, which extends the using scope of the Banzhaf-Owen value. Like other payoff indices, there are other axiomatic systems to show the existence and uniqueness of the Banzhaf-Owen value, such as Owen [2], Alonso- Meijide et al. [3]. Moreover, we briefly study two special kinds of fuzzy games with coalition structures.

At present, the research about payoff indices for fuzzy games with coalition structures is not perfect. We shall continue to study in this area. Furthermore, we shall study other kinds of game theory, such as network formation games and playing online games.

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