Some Clopen sets in the Uniform Topology on BCI-algebras

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Abstract—In this paper some properties of the uniformity topology on a BCI-algebras are discussed.

Keywords—(Fuzzy) ideal, (Fuzzy) subalgebra, Uniformity, clopen sets.

I. INTRODUCTION

In 1966, K. Iseki introduced the concept of BCI-algebra [4]. In 1965, L.A. Zadeh [6] defined the concept of a fuzzy set, as a function from a non-empty set to [0,1]. In [1], B. Ahmad, apply this notion to BCI-algebra.

In this paper we will discuss some properties of the uniform topology on a BCI-algebra.

II. PRELIMINARIES

Definition 2.1. By a BCI-algebra we mean an algebra \((X;\ast, 0)\) of type \((2,0)\) satisfying the axioms:

BCI-1) \((x \ast y) \ast (x \ast z) = 0\),

BCI-2) \((x \ast (x \ast y)) \ast y = 0\),

BCI-3) \(x \ast x = 0\),

BCI-4) \(x \ast y = y \ast x = 0\) implies \(x = y\),

BCI-5) \(x \ast 0 = 0\) implies \(x = 0\).

For all \(x, y\) and \(z\) in \(X\).

From now on \(X = (X;\ast, 0)\) is a BCI-algebra.

Definition 2.2 [3]. A subset \(B\) of \(X\) is called:

i) an ideal if for any \(x, y\) in \(X\).

\(1)\ 0 \in B\)

\(2)\ x \ast y, y \in B\) imply \(x \in B\).

ii) a subalgebra if for any \(x, y\) in \(B\), \(x \ast y \in B\).

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Definition 2.3 [1]. A fuzzy subset \(\mu\) of \(X\) is called:

i) a fuzzy ideal of \(X\) if for any \(x, y \in X\), we have

\(1)\ \mu(0) \geq \mu(x),\) for all \(x\) in \(X\),

\(2)\ \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}\).

ii) a fuzzy subalgebra of \(X\) if for any \(x, y \in X\)

\(\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}\).

Notation 2.4. The set of all (non-zero fuzzy) ideal of \(X\) is denoted by \((FI(X))\). Note that by non-zero fuzzy set of \(X\) we mean, there is \(x \in X\) such that \(\mu(x) > 0\).

Lemma 2.5. (i) \(A \in I(X)\) iff \(\chi_A \in FI(X)\), where \(\chi_A\) is the characteristic function of \(A\).

(ii) If \(\mu, \eta \in FI(X)\), then \(\mu \cap \eta \in FI(X)\), where \(\mu \cap \eta\) is a fuzzy subset of \(X\) which is defined by

\(\mu \cap \eta(x) = \min\{\mu(x), \eta(x)\},\) for all \(x \in X\).

(iii) If \(\mu \in FI(X)\), then

\(\mu(x \ast y) \geq \min\{\mu(x \ast z), \mu(z \ast y)\}, \forall x, y, z \in X\).

(iv) If \(\mu \in FI(X)\), then \(\mu(0) > 0\).

(v) \(A\) is a subalgebra of \(X\) if and only if \(\chi_A\) is a fuzzy subalgebra.

Proof. The proofs of (i), (ii), (iv) and (v), are easy, and the proof of (iii) follows from BCI-1.

Remark 2.6. (i). By BCI-5, \(\{0\} \in I(X)\) and hence \(\chi_{\{0\}} \in FI(X)\).

(ii) For all \(x \in X\), \(A \in I(X)\), \(\chi_A(x \ast x) = 1\), by BCI-3.

Definition 2.7 [2]. A BCI-algebra \(X\) is called medial if

\((x \ast y) \ast (z \ast u) = (x \ast z) \ast (y \ast u),\) \(\forall x, y, z, u \in X\).

Definition 2.8 [1]. A BCI-algebra \(X\) is called quasi right alternate if

\(x \ast (y \ast y) = (x \ast y) \ast y,\) \(\forall x, y \in X\).

Definition 2.8. Let \(\mu \in FI(X)\). We define the relation

\(\sim_\mu\) on \(X\) as follows:

\(\sim_\mu\).
$x \sim \mu y$ if and only if $\min \{ \mu(x * y), \mu(y * x) \} > 0$.

**Proposition 2.9.** The relation $\sim_\mu$ is an equivalence relation on $X$.

**Notations.** Let $X$ be a non-empty set and $U$, $V$ be subsets of $X \times X$. We let

i) $U \circ V = \{(x,y) \in X \times X \mid \exists z \in X \text{ such that } (x,z) \in V \text{ and } (z,y) \in U\}$;

ii) $U^{-1} = \{(x,y) \in X \times X \mid (y,x) \in U\}$;

iii) $\Delta = \{(x,x) \in X \times X \mid x \in X\}$.

**Definition 2.10** [5]. By a uniformity on $X$ we shall mean a non-empty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

(U1) $\Delta \subseteq U$, for any $U \in \mathcal{K}$;

(U2) If $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$;

(U3) If $U \in \mathcal{K}$, then there exist a $V \in \mathcal{K}$, such that $V \circ V \subseteq U$;

(U4) If $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$;

(U5) If $U \in \mathcal{K}$, and $U \subseteq \mathcal{K}$, then $V \in \mathcal{K}$.

**Theorem 2.11.** Let $\mu \in FI(X)$ and $U_{\mu} = \{(x,y) \in X \times X \mid x \sim_\mu y\}$.

If

$$K^\mu = \{U_{\mu} \mid \mu \in FI(X)\},$$

then $K^\mu$ satisfies the conditions (U1)-(U4).

**Theorem 2.12.** Let $\mathcal{K} = \{U \subseteq X \times X \mid U_{\mu} \subseteq U \text{ for some } \mu \in FI(X)\}$.

Then $\mathcal{K}$ satisfies a uniformly on $X$ and the pair $(X,K)$ is a uniform structure.

**Notation.** Let $x \in X$, and $U \in \mathcal{K}$, we define $U[x] : = \{y \in X \mid (x,y) \in U\}$.

**Theorem 2.13.** Let $\mu = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$. The $\mu$ is a topology on $X$.

**Remark 2.14.** Note that for any $x$ in $X$, $U[x]$ is an open neighborhood of $x$.

**Definition 2.15.** Let $(X,K)$ be a uniform space. Then the topology $\mu$ is called the uniform topology on $X$ induced by $K$.

**III. MAIN RESULTS**

**Proposition 3.1.** Every ideal $I$ of $X$ is a clopen set in $(X,\mu)$.

**Proof.** Let $I$ be an ideal of $X$. To prove that $I$ is closed, we shall show that $I^C = \bigcup_{\mu \in I} [x]$. Indeed, assume $y \in I^C$, then from $y \in U_{\mu}[x]$ it follows that $y \in U_{\mu}[x]$.

$$I^C \subseteq \bigcup_{\mu \in I} U_{\mu}[x].$$

Conversely, let $y \in \bigcup_{\mu \in I} U_{\mu}[x]$. Then there is $z \in I^C$ such that $y \in U_{\mu}[z]$. Hence $y * z$ and $z * y \in I$. Now we show that $y \notin I$. On the contrary, let $y \in I$. Then from $z * y \in I$, we get that $z \notin I$, which is contradiction. Therefore

$$\bigcup_{\mu \in I} U_{\mu}[x] \subseteq I^C$$

consequently from (1) and (2) we obtain that $I$ is closed. To prove that $I$ is open we show that

$$I = \bigcup_{\mu \in I} U_{\mu}[x].$$

Clearly $y \in U_{\mu}[x]$, $\forall y \in X$. Hence, $I \subseteq \bigcup_{\mu \in I} U_{\mu}[x]$.

On the other hand, let $y \in \bigcup_{\mu \in I} U_{\mu}[x]$, then there is $z \in I$ such that $y \in U_{\mu}[z]$. Thus $y * z \in I$ and $z * y \in I$. Now by BCI-2 we have

$$(y * (y * z)) * z = 0 \in I.$$

Since $z \in I$ and $z * y \in I$ we get that $y \in I$. Thus

$$\bigcup_{\mu \in I} U_{\mu}[x] \subseteq I.$$

Therefore (3) holds, and hence $I$ is open.

**Theorem 3.2.** Each $U_{\mu}[x]$ is a clopen subset for all $\mu \in FI(X)$.

**Proof.** Let $\mu \in FI(X)$, $x \in X$. We want to show that $U_{\mu}[x]$ is a closed subset of $X$. Let $y \in (U_{\mu}[x])^c$. We claim that for the given element $y$ we have

$$U_{\mu}[y] \subseteq (U_{\mu}[x])^c. \quad (4)$$

Let $z \in U_{\mu}[y]$, then $\mu(z * y) > 0$ and $\mu(y * z) > 0$. If $z \in U_{\mu}[x]$, then $\mu(x * z) > 0$ and $\mu(z * x) > 0$. By Lemma 2.5 (iii), $\mu(x * y) > 0$ and $\mu(y * x) > 0$. It follows that $y \in U_{\mu}[x]$, which is a contradiction. Hence
\( z \in (U_\mu[x])^c \), and (4) holds. Therefore \((U_\mu[x])^c \) is open, that is \( U_\mu[x] \) is closed.

**Theorem 3.3** [1]. In a quasi right alternate BCI-algebra, fuzzy ideals and fuzzy subalgebra coincide.

**Corollary 3.4.** Let \( X \) be a quasi right alternate BCI-algebra, then
i) Every subalgebra of \( X \) is clopen set in \((X, u)\).
ii) If \( \mu \) is a fuzzy subalgebra of \( X \), then \( U_\mu[x] \) is a clopen set in \((X, u)\).

**Proof.** The proof follows from Theorems 3.12, 3.13 and Proposition 3.11.

**Proposition 3.1.** \( K \) is a discrete topology.

**Proof.** Let \( x \) be an arbitrary element of \( X \). Then
\[
\{x\} = \{y \in X : y \neq x\} = \{y \in X : x \neq y = 1, y \neq x = 0\} = \{y \in X : \chi_{\{x\}}(x \neq y) > 0, \chi_{\{y\}}(y \neq x) > 0\} = U_{\chi_{\{x\}}[x]}.
\]

Now, the proof follows from Theorem 3.2.

**Remark.** Clearly \((X \times X, \otimes, (0,0))\) is a BCK-algebra, where
\[
\otimes : (X \times X) \times (X \times X) \to X \times X
\]
\[
((x, y), (x', y')) \mapsto (x \otimes x', y \otimes y')
\]

Now, by \( u_{X \times X} \) and \( u_X \) we mean the uniform Topology on \( X \times X \) and \( X \) respectively.

**Theorem 3.6.** Let \( X \) be a medial BCI-algebra. Then the operation \( * : X \times X \to X \) is continuous.

**Proof.** Let \( f : X \times X \to X \) be defined by
\[
f(x, y) = x \ast y, \forall x, y \in X, G \in u_X \quad \text{and} \quad (x, y) \in f \^{-1}(G).
\]
Then there is \( U \in K_\chi \) such that \( U[x \ast y] \subseteq G \). Hence \( U_\mu \subseteq U \), for some \( \mu \in FI(X) \). Now we define fuzzy subset \( \eta \) of \( X \times X \) by
\[
\eta(x, y) = \mu(x \ast y).
\]
we show that \( \eta \in FI(X \times X) \).

\[
\eta(0,0) = \mu(0 \ast 0) = \mu(0) \geq \mu(x \ast y) = \eta(x, y), \forall x, y \in X. \quad \text{On the other hand}
\]
\[
\min \{\eta((x, y) \ast (z, u)), \eta((z, u))\} = \min \{\mu((x \ast z) \ast (y \ast u)), \mu(z \ast u)\}
\]
\[
= \min \{\mu((x \ast y) \ast (z, u)), \mu(z, u)\}
\]
\[
\leq \mu(x \ast y)
\]
\[
= \eta(x, y), \forall x, y, z, u \in X.
\]
Therefore \( \eta \in FI(X \times X) \). Now consider \( U_\eta \) in \( K^*_{X \times X} \).

We show that \( U_\eta[(x, y)] \subseteq f \^{-1}(G) \). Let
\[
(z, u) \in U_\eta[(x, y)],
\]
then
\[
\min \{\eta((x, y) \otimes (z, u)), \eta((z, u) \otimes (x, y))\} > 0.
\]
So,
\[
\min \{\eta((x \ast z, y \ast u)), \eta((z \ast x, u \ast y))\} > 0.
\]
In other words,
\[
\min \{\mu((x \ast z) \ast (y \ast u)), \mu((z \ast x) \ast (u \ast y))\} > 0.
\]
Hence
\[
\mu((x \ast y) \ast (z \ast u)) > 0
\]
and
\[
\mu((z, u) \ast (x, y)) > 0.
\]

It follows that,
\[
(x \ast y, z \ast u) \in U_\mu \subseteq U \quad \text{and so} \quad z \ast u \in U[x \ast y] \subseteq G.
\]
It means that \( (z \ast u) = f(z, u) \in G \) or \( (z, u) \in f \^{-1}(G) \). Consequently, \( f \ast (G) \in u_{X \times X} \).

REFERENCES