The Balanced Hamiltonian Cycle on the Toroidal Mesh Graphs

Wen-Fang Peng, Justie Su-Tzu Juan

Abstract—The balanced Hamiltonian cycle problem is a quiet new topic of graph theory. Given a graph $G = (V, E)$, whose edge set can be partitioned into $k$ dimensions, for positive integer $k$ and a Hamiltonian cycle $C$ on $G$. The set of all $i$-dimensional edge of $G$, which is a subset by $E(C)$, is denoted as $E_i(C)$. If $\|E_i(C)\| = \|E_j(C)\| \leq 1$ for $1 \leq i < j \leq k$, $C$ is called a balanced Hamiltonian cycle. In this paper, the proposed result shows that there exists a balanced Hamiltonian cycle for any Toroidal Mesh graph $T_{m,n}$, if and only if $m, n \geq 3$ and Toroidal Mesh graph $nm \neq 2$ (mod 4), and how to find a balanced Hamiltonian cycle on $T_{m,n}$ for $n, m \geq 3$ and $nm \neq 2$ (mod 4).

Keywords—Hamiltonian cycle; balanced; Cartesian product

I. INTRODUCTION

The research of optimal encode uses gray-code encode to signify the information of n-bit about the application of 3D scanning, which has been mentioned in the references [1], [2], [3], [5] and [8]. The utility of gray-code will decrease the consumption of resource and increase the precision. Nevertheless, there would be some problem when deal with those information of transforming between 0 and 1, such as it will spend much more cost in identification. How to decrease the cost in dealing with such problems is important. Hence, in this paper, it discusses a method to decrease the number of transformation between 0 and 1 in the some dimension.

Balanced Hamiltonian cycle (BHC) problems are widely discussed in recent years. Several issues about BHC have been proposed by other researchers [2]. Wang et al proposed the BHC on $C_m \times C_n$ for any positive integer $n \geq 3$. This paper proposes an extended research about the BHC on $C_m \times C_n$, also called $T_{m,n}$, for any positive integer $m \geq 3, n \geq 3$.

Next section introduces some background knowledge about the Hamiltonian cycle (HC) problem, Cartesian product, and some related definitions. Section 3 describes the main results, the research about the BHC problem on $T_{m,n}$, for $m, n \geq 3$, proposed by this paper. Finally, the last section makes a conclusion and lists the future work.

II. DEFINITION AND NOTATION

This paper denotes the symbols below by referring to [4], [6], [7] and [9]. Define a walk $W$, which is in a graph $G = (V, E)$, is a sequence $w = x_1e_1x_2e_2...x_te_t$ for $x_1, x_2,..., x_t, y \in V(G)$ and $e_1, e_2,..., e_t \in E(G)$. And let $x$ be the origin vertex of $W$, $y$ be the terminus vertex of $W$. If all of vertices in this walk are different, a walk $W$ is denoted a path. When the origin vertex and the terminus vertex are the same vertex, then this path is denoted a cycle.

A Hamiltonian path of graph $G = (V, E)$ is a path that contains all vertices. A Hamiltonian cycle of $G$ is a cycle that contains all vertices.

Given a Hamiltonian cycle $C$ on a graph $G = (V, E)$, whose edge set can be partitioned into $k$ dimensions, for positive integer $k$. And let $E_i(C)$ represents the set of all $i$-dimensional edge of $C$ which is a subset by $E(C)$. If $\|E_i(C)\| = \|E_j(C)\| \leq 1$ for $1 \leq i < j \leq k$, $C$ is called a balanced Hamiltonian cycle.

Let $C_m, n$ denote a cycle with $n$ vertices, given two graph $G_1, G_2$, the Cartesian product $G_1 \times G_2$ of $G_1$ and $G_2$ is a graph with vertex set $V(G_1 \times G_2) = \{ (x, y) \mid x \in V(G_1), y \in V(G_2) \}$ and the edge set $\{(u, v), (u', v') \mid u = u' \in V(G_1) \text{ and } (v, v') \in E(G_2) \text{ or } v = v' \in V(G_2) \text{ and } (u, u') \in E(G_1) \}$. The toroidal mesh graph $T_{m,n}$ is the graph $C_m \times C_n$.

The dimension of $T_{m,n}$ is 2. Given an Hamiltonian cycle $C$ of $T_{m,n}$, let $E_i(C) = \{ (x_i, y)(x_{i+1}, y) \mid 1 \leq i < n \}$ and $E_2(C) = \{ (x_i, y)(x_{(i+1)}), 1 \leq i \leq m, 1 \leq j \leq n \}$ mean the 1-dimension edge set and 2-dimension edge set of a Hamiltonian cycle $C$, respectively. Thus, the relation between vertex number and edge number of Hamiltonian cycle $C$ is $|V(C)| = |V(C_m \times C_n)| = |E_1(C)| + |E_2(C)| = nm$. If $C$ satisfied that $|E_1(C)| = |E_2(C)| \leq 1$, it presents that $C$ is balanced.

In this paper, when we draw a figure of $T_{m,n}$, $m$ is denoted the number of vertices on $x$-axis, and $n$ is the number of vertices on $y$-axis, respectively. Besides, for any vertices $(x, y)$ of $T_{m,n}$, $x$ is called the 1st-dimension and $y$ is called the 2nd-dimension. Furthermore, we define the lower-left vertex of $T_{m,n}$ to be the origin vertex and set it as (1, 1). Fig. 1 shows an example of $T_{3,4}$.

![Fig. 1 $T_{3,4}$](image)

The next section discusses the methods for getting the balanced Hamiltonian cycle on $T_{m,n}$ for $n, m \geq 3$.

III. MAIN RESULTS

This section gives theorems 1 for prove $\exists$, and gives some cases to prove Theorem 3 that there exists a BHC on $T_{m,n}$ for positive integers $n, m \geq 3$, except for the situation on $mn$ mod 4 $= 2$.

Theorem 1: For $mn$ mod 4 $= 2$, there is no any balanced Hamiltonian cycle $C$ exists on $T_{m,n}$.

Proof.

When $mn$ mod 4 $= 2$, one of the following cases will hold. (i) $n$
mod $4 = 2$ and $m$ is odd; (ii) $m \mod 4 = 2$ and $n$ is odd. Without loss of generality, we say $n \mod 4 = 2$ and $m$ is odd. Furthermore, let $n = 4k_1 + 2$ and $m = 2k_2 + 1$ for some positive $k_1$ and $k_2$. Assume that there exists a balanced Hamiltonian cycle $C'$ on $T_{m,n}$. Since $V(C') = mn = (4k_1 + 2)(2k_2 + 1) = 8k_1k_2 + 4k_2 + 4k_1 + 2 = 2(2k_2k_1 + k_1 + k_2) + 1$ is an odd integer.

We call a vertex $u$ in $V(C')$ is black if $u \in \{(x, y) \mid 1 \leq x \leq m, 1 \leq y \leq n \text{ and } y \text{ is even}\}$, and $u$ is white if $u \in \{(x, y) \mid 1 \leq x \leq m, 1 \leq y \leq n \text{ and } y \text{ is even}\}$. Hence the origin vertex is black. According to the definition of $E_2(C')$, $E_2(C')$ should trace black point to white point or white point to black point. After tracing all edges of $C'$, find the terminate vertex of $C'$ is white due to $|E_2(C')|$ is odd. Obviously, the origin vertex and the terminate vertex of $C'$ are different. That is a contradiction. So, there is no BHC on $T_{m,n}$ when $mn \mod 4 = 2$.

**Lemma 2:** For $n = 3, m \geq 3$ and $m$ is odd, there is a balanced Hamiltonian cycle on $T_{m,n}$.

**Proof.**

The proof is divided into two cases. Case 1 discusses the condition on $m \mod 4 = 1$ and $n = 3$; Case 2 discusses the state on $m \mod 4 = 3$ and $n = 3$.

**Case 1.** $m \mod 4 = 1$ and $n = 3$.

In this section, $T_{m,n}$ consists of the BHC on $T_{3,n}$ and the BHC on $T_{3,n}$, as shown in Fig. 2 and Fig. 3, respectively. Besides, Fig. 4 indicates how to connect all figures. First of all, let $x = (m - 5)/4$, and insert Fig. 2 for $x$ times on right side of Fig. 3 when $m > 5$ and $n = 3$. Then, delete edge set $E_1 = \{(6 + 4i, 3)(9 + 4i, 3) \mid 1 \leq i \leq (m - 8)/4\}$ and add edge set $E_2 = \{(5 + 4i, 3)(6 + 4i, 3) \mid 1 \leq i \leq (m - 3)/4\} \cup \{(1, 3)(3, 3), (1, 3)(m, 3)\}$. After these steps, a Hamiltonian cycle $C$ on $T_{m,n}$ is generated, whose $|E_1(C)| = 7 + 6x$ and $|E_2(C)| = 8 + 6x$. Consequently, $|E_1(C)| - |E_2(C)| = 1$. $C$ satisfies the definition of BHC.

**Case 2.** $m \mod 4 = 3$ and $n = 3$.

Compare Fig. 4 with Fig. 6, which indicates how to construct the BHC on $T_{m,n}$, there is only one difference at the beginning. As a result, refer to Case 3.1, replace Fig. 3 with Fig. 5, which is one of possible BHCs on $T_{3,3}$, and revise $x = (m - 3)/4$. Then correct the edge set $E_1 = \{(4 + 4i, 3)(7 + 4i, 3) \mid 1 \leq i \leq (m - 7)/4\} \cup \{(1, 3)(3, 3), (1, 3)(m, 3)\}$. Obviously, $C$ satisfies the definition of BHC as a result of $|E_1(C)| - |E_2(C)| = 1$.

**Theorem 3:** For $n, m \geq 3$, there is a balanced Hamiltonian cycle on $T_{m,n}$, except for the state on $mn \mod 4 = 2$.

**Proof.**

According to the condition of even or odd on $n, m$, the proof is divided into three cases. Case 1 proposes the condition on $n, m$ both are even; Case 2 proposes the condition one of $n, m$ is even and the other is odd; Case 3 discusses the condition on $n, m$ both are odd.

| TABLE I
| THE RESULT OF THIS THEOREM |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $m \mod 4 = 0$  | Case 1.1        | Case 1.2        | Case 1.3        | Case 2.1        | Case 2.2        | Case 3.1        |
| $m \mod 4 = 2$  |                |                |                |                |                |                |
| $n \mod 4 = 0$  |                |                |                |                |                |                |
| $n \mod 4 = 2$  |                |                |                |                |                |                |
| $n \mod 4 = 3, n \geq 7$ | Case 1.3 |                |                |                |                |                |
| $n \mod 4 = 3, n \geq 7$ | Case 2.1 | Case 2.2       |                |                |                |                |
| $n \mod 4 = 3, n \geq 7$ |                |                |                |                |                |                |
| $n \mod 4 = 3, n \geq 7$ |                |                |                |                |                |                |

**Case 1.** $n, m$ both are even

This case is separated into three subcases for discussion. Case 1.1, Case 1.2 and Case 1.3 consider the states on $m$ is even and $n \mod 4 = 0, m \mod 4 = 0$ and $n \mod 4 = 2, m \mod 4 = 2$ and $n \mod 4 = 2$, respectively.

**Case 1.1.** $m$ is even and $n \mod 4 = 0$

Fig. 7 and Fig. 8 show one of the possible HCs on $T_{2,4}$ and one of the possible BHCs on $T_{4,4}$, respectively. When $m > 4$ and...
n = 4, let \( x = (m - 4) / 2 \), and then duplicate Fig. 7 for \( x \) times. Next, inset them on the right side of Fig. 8 mentioned above. Then delete edge set \( E_3 = \{(i, 2)(i, 3) | 1 \leq i \leq m - 1 \} \), and insert edge set \( E_5 = \{(i, 2)(i + 1, 2) \cup (i, 3)(i + 1, 3) | 1 \leq i \leq m - 2 \) and \( i \) is even.\}.

Hamiltonian cycle \( C \) is produced, whose \( |E_3(C)| = 12 + 12x \) and \( |E_5(C)| = 12 + 12x \), as shown in Fig. 13. Because of \( |E_3(C)| - |E_5(C)| = 0 \), \( C \) satisfies the definition of BHC.

Case 1.2 \( m \text{ mod } 4 = 0 \) and \( n \text{ mod } 4 = 2 \)

Fig. 11 and Fig. 12 are isomorphic BHC on \( T \). The following steps illustrate how to find a BHC on \( T \) when \( m > 4 \). First, inset Fig. 12 on the right side of Fig. 11 for \( x \) times, where \( x = (m - 4) / 4 \). Second, delete edge set \( E_9 = \{(1 + 4i)(4 + 4i, 6) | 10 \leq i \leq x \} \). Third, add edge set \( E_{10} = \{(4 + 4i, 6)(5 + 4i, 6) | 10 \leq i \leq (n - 8) / 4 \} \). By implementing these steps above, a
Case 2. One of \( m, n \) is even, and the other is odd

For any positive integer \( n, m, T_{m, n} \) and \( T_{n, m} \) are isomorphic. Hence, if one of \( m \) is even, and the other is odd, without loss of generality, set that \( n \) is even and \( m \) is odd.

This case can be also divided into two subcases for discussion. Case 2.1 discusses the condition on \( m \) is odd and \( n \) mod 4 = 0; Case 2.2 discusses the state on \( m \) is odd and \( n \) mod 4 = 2.

Case 2.1. \( m \) is odd and \( n \) mod 4 = 0

Fig. 16 shows a possible BHC on \( T_{m, n} \). When \( m > 3 \), the BHC on \( T_{m, n} \) consists of Fig. 7 and Fig. 16. Fig. 17 illustrates the way of connecting. First, for \( \alpha = (m - 3) / 4 \), inset \( x \) duplicate BHC, which has been shown in Fig. 7, on the right side of Fig. 16. Second, eliminate edge set \( E_{15} = \{(3, 3)(3, 4)14 \leq i \leq m \} \cup (1, 4)(3, 4) \cup (1, 3) \). Third, put edge set \( E_{16} = \{(3 + 2i, 3)(4 + 2i, 3) \cup (3 + 2i, 4)(4 + 2i, 4)10 \leq i \leq (m - 4) / 2 \} \cup (1, 3)(m, 3) \cup (1, 4)(m, 4) \) on the graph produced by previous steps. After that, a Hamiltonian cycle \( C \) is established, whose \( |E(C)| = 6 + 2x \) and \( |E(C)| = 6 + 4x \). Due to \( |E(C)| - |E(C)| = 0 \), \( C \) satisfies the definition of BHC.

Case 2.2. \( n \) mod 4 = 2 and \( m \) is odd

According to theorem 1, there is no balanced Hamiltonian cycle on \( T_{m, n} \) for \( m \) mod 4 is odd and \( n \) mod 4 = 2.

Case 3. \( m, n \) both are odd

This case is also separated into eight subcases for discussion.

Case 3.1 discusses the state on \( n \) mod 4 = 1 and \( m \) mod 8 = 1; Case 3.2 proposes the state on \( n \) mod 4 = 3 and \( m \) mod 8 = 3; Case 3.3 considers the operations when \( n \) mod 4 = 1 and \( m \) mod 8 = 5; Case 3.4 concerns the details when \( n \) mod 4 = 1 and \( m \) mod 8 = 7.

The other four remaining cases propose the method under the condition of \( n > 3 \). Case 3.5 discusses the state on \( n \) mod 4 = 3 and \( m \) mod 8 = 1; Case 3.6 considers the state on \( n \) mod 4 = 3 and \( m \) mod 8 = 3; Case 3.7 considers the operations when \( n \) mod 4 = 3 and \( m \) mod 8 = 5; Case 3.8 consults the details when \( n \) mod 4 = 3 and \( m \) mod 8 = 7.

Case 3.1. \( m \) mod 8 = 1 and \( n \) mod 4 = 1

Fig. 19 and Fig. 20 show one of the possible HCIs on \( T_{m, 5} \) and one of possible BHCs on \( T_{m, 5} \), respectively. When \( m > 9 \) and \( n = 5 \), make \( \beta = (m - 9) / 8 \). First, use Fig. 20 as the beginning, and inset Fig. 19 for \( x \) times on its right side. Then eliminate edge set \( E_{19} = \{(10 + 8i, 4)(10 + 8i, 5) \cup (17 + 8i, 4)(17 + 8i, 5)10 \leq i \leq (m - 17) / 8 \} \cup (1, 4)(9, 4) \cup (1, 5)(9, 5) \), and add edge set \( E_{20} = \{(9 + 8i, 4)(10 + 8i, 4) \cup (9 + 8i, 5)(10 + 8i, 5)10 \leq i \leq (m - 17) / 8 \} \cup (1, 4)(m, 4) \cup (1, 5)(1, m) \). Finally, a Hamiltonian cycle \( C \) on \( T_{m, 5} \) is produced, which is shown in Fig. 21. For \( |E(C)| = 22 + (18 + 2)x = 22 + 20x \) and \( |E(C)| = 23 + 20x \), \( C \) satisfies that \( |E(C)| - |E(C)| = 1 \). Undoubtedly, \( C \) is a BHC of \( T_{m, 5} \).

When \( n > 5 \), let \( y = (n - 5) / 4 \). Use Fig. 21 as base, then stack \( y \) BHCs, which is shown in Fig. 12. Next, remove edge set \( E_{21} = \{(1, 6 + 4i)(1, 9 + 4i)10 \leq i \leq (n - 9) / 4 \} \cup (1, 1)(1, 5) \), and insert edge set \( E_{22} = \{(1, 1)(1, n)10 \leq i \leq (n - 9) / 4 \} \cup (1, 5 + 4i)(1, 6 + 4i) \). After complete all of the steps, a BHC on \( T_{m, n} \) for \( m \) mod 8 = 1 and \( n \) mod 4 = 1 is established.
Case 3.2. \( m \mod 8 = 3 \) and \( n \mod 4 = 1 \)

Fig. 23 represents the way of constructing a BHC on \( T_{m,5} \).

When \( m > 3 \), let \( x = (m - 3) / 8 \), and then duplicate Fig. 19 for \( x \) times. Next, inset them on the right side of the BHC on \( T_{3,5} \), which is shown in Fig. 22. In order to connect every figure, delete edge set \( E_{23} \) and add edge set \( E_{24} \) that make it be a BHC of \( T_{3,5} \).

When \( n > 5 \), the way of constructing the BHC on \( T_{m,n} \) is similar to Case 3.1. Only one difference is to replace Fig. 21 with Fig. 23.

Case 3.3. \( m \mod 8 = 5 \) and \( n \mod 4 = 1 \)

Fig. 24 represents a BHC on \( T_{5,5} \), which is used to construct the BHC on \( T_{m,5} \). When \( m > 5 \), let \( x = (m - 5) / 8 \). To begin with, inset Fig. 19 for \( x \) times on the right side of Fig. 24. Next, delete edge set \( E_{25} \) and add edge set \( E_{26} \) that make it be a BHC of \( T_{m,5} \) for \( m \mod 8 = 5 \) and \( n \mod 4 = 1 \).
Fig. 27 The BHC on $T_{m,5}$

Case 3.5. $m \mod 8 = 1$ and $n \mod 4 = 3$

Fig. 27 and Fig. 28 show one of the possible HCs on $T_{8,7}$ and one of possible BHCs on $T_{9,7}$, respectively. Furthermore, Fig. 29 illustrates the way of constructing a BHC on $T_{m,7}$, which is described in the content below. When $m > 9$, let $x = (m - 9) / 8$. First, inset $x$ HCs, which has been mentioned above, on the right side of Fig. 28. Second, remove edge set $E_{29} = \{(10 + 8i, 6)(10 + 8i, 7) \cup (17 + 8i, 6)(17 + 8i, 7) | 10 \leq i \leq (m - 17) / 8 \} \cup (1, 6)(9, 6) \cup (1, 7)(9, 7)$. Third, add edge set $E_{30} = \{(9 + 8i, 6)(10 + 8i, 6) \cup (9 + 8i, 7)(10 + 8i, 7) | 10 \leq i \leq (m - 17) / 8 \} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$. Finally, a Hamiltonian cycle $C$ is yielded, whose $|E(C)| = 32 + (26 + 2)x = 32 + 28x$ and $|I(E(C))| = 31 + 28x$. As a result of $|E(C)| - |I(E(C))| = 1$, it verify that $C$ is a BHC.

Fig. 28 The BHC on $T_{9,7}$

When $n > 7$, let $y = (n - 7) / 4$. Stack $y$ BHCs, which is shown in Fig. 12, above Fig. 28. Then delete edge set $E_{31} = \{(1, 8 + 4i)(1, 11 + 4i) | 10 \leq i \leq (n - 11) / 4 \} \cup (1, 1)(1, 7)$, and add edge set $E_{32} = \{(1, 7 + 4i)(1, 8 + 4i) | 10 \leq i \leq (n - 11) / 4 \} \cup (1, 1)(1, n)$. After that, a BHC on $T_{m,n}$ is generated.

Case 3.6. $m \mod 8 = 3$ and $n \mod 4 = 3$

Fig. 29 The BHC on $T_{m,7}$

Fig. 30 represents a BHC on $T_{3,7}$, which is used to construct a BHC on $T_{m,7}$. Besides, Fig. 31 illustrates how to connect Fig. 27 and Fig. 30. Let $x = (m - 3) / 8$. Then, copy Fig. 27 for $x$ times, and inset them on the right side of Fig. 30. Next, eliminate edge set $E_{33} = \{(4 + 8i)(4 + 8i, 7) \cup (11 + 8i, 6)(11 + 8i, 7) | 10 \leq i \leq (m - 11) / 8 \} \cup (1, 6)(3, 6) \cup (1, 7)(3, 7)$, and put edge set $E_{34} = \{(3 + 8i, 6)(4 + 8i, 6) \cup (3 + 8i, 7)(4 + 8i, 7) | 0 \leq i \leq (m - 11) / 8 \} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$. Because of $|E(C)| - |I(E(C))| = 1$, $C$ satisfies the definition of BHC.

Fig. 30 The BHC on $T_{3,7}$

Fig. 31 The BHC on $T_{m,7}$

Refer to Case 3.5 when $n > 7$. Fig. 31 substitutes for Fig. 29, and the else parts are same as Case 3.5. Eventually, a BHC on $T_{m,n}$ is built.

Case 3.7. $m \mod 8 = 5$ and $n \mod 4 = 3$

In this section, $T_{m,7}$ consists of the HC on $T_{8,7}$, as shown in Fig. 27, and the BHC on $T_{9,7}$, as shown in Fig. 32. When $m > 5$, make $x = (m - 5) / 8$. First of all, inset Fig. 27 for $x$ times on the right side of Fig. 32. So as to connect all figures, delete edge set $E_{35} = \{(6 + 8i)(6 + 8i, 7) \cup (13 + 8i, 6)(13 + 8i, 7) | 10 \leq i \leq (m - 13) / 8 \} \cup (1, 6)(5, 6) \cup (1, 7)(5, 7)$, and add edge set $E_{36} = \{(5 + 8i, 6)(6 + 8i, 6) \cup (5 + 8i, 7)(6 + 8i, 7) | 10 \leq i \leq (m - 13) / 8 \} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$. As a result, a Hamiltonian cycle $C$ on $T_{m,7}$ is yielded, whose $|E(C)| = 18 + (26 + 2)x = 18 + 28x$ and $|I(E(C))| = 17 + 28x$, as shown in Fig. 33. Without a doubt, $C$ is a BHC due to $|E(C)| - |I(E(C))| = 1$.

Fig. 32 The BHC on $T_{3,7}$
When \( n > 7 \), the way of constructing the BHC on \( T_{m,n} \) is similar to Case 3.5. Only one difference is to replace Fig. 29 with Fig. 33.

Case 3.10. \( m \mod 8 = 7 \) and \( n \mod 4 = 3 \)

There is a BHC on \( T_{7,7} \) as shown in Fig. 34. When \( m > 7 \), use the BHC on \( T_{7,7} \) mentioned above as the beginning. Make \( x = (m - 7) / 8 \), then inset \( x \) HCs, which is shown in Fig. 27, on the right side of Fig. 34. Then, remove edge set \( E_{37} = \{(15 + 8i, 6)(15 + 8i, 7) \mid 0 \leq i \leq (m - 15) / 8\} \cup \{(1, 6)(7, 6) \cup (1, 7)(7, 7) \cup (8 + 8i, 6)(8 + 8i, 7), \) and add edge set \( E_{38} = \{(7 + 8i, 6)(8 + 8i, 6) \cup (7 + 8i, 7)(8 + 8i, 7) \mid 0 \leq i \leq (m - 15) / 8\} \cup \{(1, 6)(m, 6) \cup (1, 7)(m, 7). \) Thus, a Hamiltonian cycle \( C \) on \( T_{m,7} \) is built, which is shown in Fig. 35.

For \( |E_1(C)| = 24 + (26 + 2)x = 24 + 28x \) and \( |E_2(C)| = 25 + 28x \), \( C \) satisfies that \( |E_1(C)| - |E_2(C)| = 1 \). Undoubtedly, \( C \) is a BHC of \( T_{m,7} \).

Compare Case 3.5 with Case 3.8 for \( n > 7 \), the only difference is that Fig. 29 is replaced with Fig. 35, and the else parts of operating are all the same. Then a BHC on \( T_{m,n} \) is constructed.

IV. CONCLUSION

By giving Theorem 1 and 2 in this paper, the main result below can be verified. In general cases, there exists a BHC on \( T_{m,n} \) for positive integers \( n, m \), except for the situation of \( mn = 2 \) (mod 4). How to find a balanced Hamiltonian cycle in the graph of \( k \)-dimension Cartesian product for any positive integer \( k \), will be the future work.

REFERENCES