The Balanced Hamiltonian Cycle on the Toroidal Mesh Graphs

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Abstract—The balanced Hamiltonian cycle problem is a quiet new topic of graph theory. Given a graph \( G = (V, E) \), whose edge set can be partitioned into \( k \) dimensions, for positive integer \( k \) and a Hamiltonian cycle \( C \) on \( G \). The set of all \( i \)-dimensional edge of \( C \), which is a subset by \( E(C) \), is denoted as \( E_i(C) \). If \( \| E_i(C) \| - | E_i(C) \| \leq 1 \) for \( 1 \leq i \leq k \), \( C \) is called a balanced Hamiltonian cycle. In this paper, the proposed result shows that there exists a balanced Hamiltonian cycle for any Toroidal Mesh graph \( T_{m,n} \), if and only if \( n, m \geq 3 \) and Toroidal Mesh graph \( nm \neq 2 \) (mod 4), and how to find a balanced Hamiltonian cycle on \( T_{m,n} \) for \( n, m \geq 3 \) and \( nm \neq 2 \) (mod 4).

Keywords—Hamiltonian cycle; balanced; Cartesian product

I. INTRODUCTION

The research of optimal encode uses gray-code encode to signify the information of n-bit about the application of 3D scanning, which has been mentioned in the references [1], [2], [3], [5] and [8]. The utility of gray-code will decrease the consumption of resource and increase the precision. Nevertheless, there would be some problem when deal with those information of transforming between 0 and 1, such as it will spend much more cost in identification. How to decrease the cost in dealing with such problems is important. Hence, in this paper, it discusses a method to decrease the number of transformation between 0 and 1 in the same dimension.

Balanced Hamiltonian cycle (BHC) problems are widely discussed in recent years. Several issues about BHC have been proposed by other researchers [2]. Wang et al proposed the BHC on \( C_m \times C_n \) for any positive integer \( n \geq 3 \). This paper propose an extended research about the BHC on \( C_m \times C_n \), also called \( T_{m,n} \), for any positive integer \( m \geq 3, n \geq 3 \).

Next section introduces some background knowledge about the Hamiltonian cycle (HC) problem, Cartesian product, and some related definitions. Section 3 describes the main results, the research about the BHC problem on \( T_{m,n} \), for \( m, n \geq 3 \), proposed by this paper. Finally, the last section makes a conclusion and lists the future work.

II. DEFINITION AND NOTATION

This paper denotes the symbols below by referring to [4], [6], [7] and [9]. Define a walk \( W \), which is in a graph \( G = (V, E) \), is a sequence \( w = x_1e_1x_2e_2...x_ne_n \) for \( x_i, x_2, ..., x_n, y \in V(G) \) and \( e_1, e_2, ..., e_n \in E(G) \). And let \( x \) be the origin vertex of \( W \), \( y \) be the terminus vertex of \( W \). If all of vertices in this walk are different, a walk \( W \) is denoted a path. When the origin vertex and the terminus vertex are the same vertex, then this path is denoted a cycle.

A Hamiltonian path of graph \( G = (V, E) \) is a path that contains all vertices. A Hamiltonian cycle of \( G \) is a cycle that contains all vertices.

Given a Hamiltonian cycle \( C \) on a graph \( G = (V, E) \), whose edge set can be partitioned into \( k \) dimensions, for positive integer \( k \). And let \( E_i(C) \) represents the set of all \( i \)-dimensional edge of \( C \) which is a subset by \( E(C) \). If \( \| E_i(C) \| - | E_i(C) \| \leq 1 \) for \( 1 \leq i \leq k \), \( C \) is called a balanced Hamiltonian cycle.

Let \( C_n \) denote a cycle with \( n \) vertices, given two graph \( G_1, G_2 \), the Cartesian product \( G_1 \times G_2 \) of \( G_1 \) and \( G_2 \) is a graph with vertex set \( V(G_1 \times G_2) = \{ (x, y) | (x \in V(G_1), y \in V(G_2)) \} \) and the edge set \( \{(u, v), (u', v') | u = u' \in V(G_1) \text{ and } (v, v') \in E(G_2) \text{ or } v = v' \in V(G_2) \text{ and } (u, u') \in E(G_1)\} \). The toroidal mesh graph \( T_{m,n} \) is the graph \( C_m \times C_n \).

The dimension of \( T_{m,n} \) is 2. Given an Hamiltonian cycle \( C \) of \( T_{m,n} \), let \( E_1(C) = \{ (x_i, y) | 1 \leq i \leq m \text{ and } 1 \leq j \leq n \} \) and \( E_2(C) = \{ (x_i, y) | 1 \leq i \leq m \text{ and } 1 \leq j \leq n \} \) mean the 1-dimension edge set and 2-dimension edge set of a Hamiltonian cycle \( C \), respectively. Thus, the relation between vertex number and edge number of Hamiltonian cycle \( C \) is \( | V(C) | = | V(C_m \times C_n) | = \| E_1(C) \| + \| E_2(C) \| = mn \). If \( C \) satisfied that \( \| E_1(C) \| - \| E_2(C) \| \leq 1 \), it presents that \( C \) is balanced.

In this paper, when we draw a figure of \( T_{m,n} \), \( m \) is denoted the number of vertices on x-axis, and \( n \) is the number of vertices on y-axis, respectively. Besides, for any vertices \((x, y) \) of \( T_{m,n} \), \( x \) is called the 1st-dimension and \( y \) is called the 2nd-dimension.

Furthermore, we define the lower-left vertex of \( T_{m,n} \) to be the origin vertex and set it as \((1, 1)\). Fig. 1 shows an example of \( T_{3,4} \).

![Fig. 1 T_{3,4}](image)

The next section discusses the methods for getting the balanced Hamiltonian cycle on \( T_{m,n} \) for \( n, m \geq 3 \).

III. MAIN RESULTS

This section gives theorems for prove \( \# \), and gives some cases to prove Theorem 3 that there exists a BHC on \( T_{m,n} \) for positive integers \( n, m \geq 3 \), except for the situation on \( mn \mod 4 = 2 \).

**Theorem 1:** For \( mn \mod 4 = 2 \), there is no any balanced Hamiltonian cycle \( C \) exists on \( T_{m,n} \).

**Proof.**

When \( mn \mod 4 = 2 \), one of the following case will hold. (i) \( n \)
mod 4 = 2 and m is odd; (ii) m mod 4 = 2 and n is odd. Without loss of generality, we say n mod 4 = 2 and m is odd. Furthermore, let \( n = 4k_1 + 2 \) and \( m = 2k_2 + 1 \) for some positive \( k_1 \) and \( k_2 \). Assume that there exists a balanced Hamiltonian cycle \( C \) on \( T_{m,n} \). Since \( V(C) = mn = (4k_1 + 2)(2k_2 + 1) = 8k_1k_2 + 4k_2 + 4k_1 + 2 = 2(2k_1k_2 + k_1 + k_2) + 1 \) is an odd integer.

We call a vertex \( u \) in \( V(C) \) is black if \( u \in \{ (x, y) \mid 1 \leq x \leq m, 1 \leq y \leq n \ \text{and} \ y \ \text{is odd} \} \); white if \( u \in \{ (x, y) \mid 1 \leq x \leq m, 1 \leq y \leq n \ \text{and} \ y \ \text{is even} \} \). Hence the origin vertex is black. After tracing all edges of \( C' \), find the terminate vertex of \( C' \) is white due to \( |E_2(C')| \) is odd. Obviously, the origin vertex and the terminate vertex of \( C' \) are different. That is a contradiction. So, there is no BHC on \( T_{m,n} \) when \( mn \mod 4 = 2 \).

**Lemma 2**: For \( n = 3, m \geq 3 \) and \( m \) is odd, there is a balanced Hamiltonian cycle on \( T_{m,n} \).

**Proof.**

The proof is divided into two cases. Case 1 discusses the condition on \( m \mod 4 = 1 \) and \( n = 3 \); Case 2 discusses the state on \( m \mod 4 = 3 \) and \( n = 3 \).

**Case 1. \( m \mod 4 = 1 \) and \( n = 3 \)**

In this section, \( T_{m,3} \) consists of the BHC on \( T_{3,3} \) and the BHC on \( T_{3,n} \), as shown in Fig. 2 and Fig. 3, respectively. Besides, Fig. 4 indicates how to connect all figures. First of all, let \( x = (m - 5) / 4 \), and inset Fig. 2 for \( x \) times on right side of Fig. 3 when \( m > 5 \) and \( n = 3 \). Then, delete edge set \( E_1 = \{ (6 + 4i, 3)(9 + 4i, 3) 10 \leq i \leq (m - 9) / 4 \} \cup (1, 3)(5, 3) \), and add edge set \( E_2 = \{ (5 + 4i, 3)(6 + 4i, 3) 10 \leq i \leq (m - 9) / 4 \} \cup (1, 3)(m, 3) \). After these steps, a Hamiltonian cycle \( C \) on \( T_{m,3} \) is generated, whose \( |E_1(C)| = 7 + 6x \) and \( |E_2(C)| = 8 + 6x \). Consequently, \( |E_1(C)| - |E_2(C)| = 1 \). C satisfies the definition of BHC.

**Case 2. \( m \mod 4 = 3 \) and \( n = 3 \)**

Compare Fig. 4 with Fig. 6, which indicates how to construct the BHC on \( T_{m,3} \), there is only one difference at the beginning. As a result, refer to Case 3.1, replace Fig. 3 with Fig. 5, which is one of possible BHCs on \( T_{m,3} \), and revise \( x = (m - 3) / 4 \). Then correct the edge set \( E_1 = \{ (5 + 4i, 3)(4 + 4i, 3) 10 \leq i \leq (m - 7) / 4 \} \cup (1, 3)(3, 3) \) and \( E_2 = \{ (3 + 4i, 3)(4 + 4i, 3) 10 \leq i \leq (m - 7) / 4 \} \cup (1, 3)(m, 3) \), respectively. In the end, a Hamiltonian cycle \( C \) on \( T_{m,3} \) is built, which \( |E_1(C)| = 5 + 6x \) and \( |E_2(C)| = 4 + 6x \). Obviously, \( C \) satisfies the definition of BHC as a result of \( |E_1(C)| - |E_2(C)| = 1 \).

**Theorem 3**: For \( n, m \geq 3 \), there is a balanced Hamiltonian cycle on \( T_{m,n} \) except for the state on \( mn \mod 4 = 2 \).

**Proof.**

According to the condition of even or odd on \( n, m \), the proof is divided into three cases. Case 1 proposes the condition on \( m, n \) both are even; Case 2 proposes the condition one of \( m, n \) is even and the other is odd; Case 3 discusses the condition on \( n, m \) both are odd.

**Case 1. \( n, m \) both are even**

This case is separated into three subcases for discussion. Case 1.1, Case 1.2 and Case 1.3 consider the states on \( n, m \) is even and \( n \mod 4 = 0, m \mod 4 = 0 \) and \( n \mod 4 = 2, m \mod 4 = 2 \) and \( n \mod 4 = 2, m \mod 4 = 2 \), respectively.

**Case 1.1. \( m \) is even and \( n \mod 4 = 0 \)**

Fig. 7 and Fig. 8 show one of the possible HCs on \( T_{2,4} \) and one of the possible BHCs on \( T_{4,4} \), respectively. When \( m > 4 \) and
n = 4, let x = (m – 4) / 2, and then duplicate Fig. 7 for x times. Next, inset them on the right side of Fig. 8 mentioned above. Then delete edge set \( E_3 = \{(i, 2)(i, 3) | 4 \leq i \leq m - 1\} \), and insert edge set \( E_6 = \{(i, 2)(i + 1, 2) \cup (i, 3)(i + 1, 3) | 4 \leq i \leq m - 2 \) and \( i \) is even.\].

A Hamiltonian cycle \( C \) is produced, whose \( |E_3(C)| = 12 + 12x \) and \( |E_2(C)| = 12 + 12n \), as shown in Fig. 13. Because of \( |E_3(C)| - |E_2(C)| = 0 \), \( C \) satisfies the definition of BHC.

**Case 1.2** \( m \) mod 4 = 2 and \( n \) mod 4 = 2

Fig. 11 and Fig. 12 are isomorphic BHC on \( T_{4,6} \). The following steps indirect how to find a BHC on \( T_{m,6} \) when \( m > 4 \). First, inset Fig. 12 on the right side of Fig. 11 for \( x \) times, where \( x = (m - 4) / 4 \). Second, delete edge set \( E_9 = \{(1 + 4i)(4 + 4i, 6) | 10 \leq i \leq x\} \). Third, add edge set \( E_{10} = \{(4 + 4i)(5 + 4i, 6) | 10 \leq i \leq (n - 8) / 4\} \). By implementing these steps above, a Hamiltonian cycle \( C \) is produced, whose \( |E_9(C)| = 12 + 12x \) and \( |E_2(C)| = 12 + 12n \), as shown in Fig. 13. Because of \( |E_9(C)| - |E_2(C)| = 0 \), \( C \) satisfies the definition of BHC.

**Case 1.3** \( m \) mod 4 = 2 and \( n \) mod 4 = 2

Fig. 14 shows a BHC on \( T_{6,6} \) which can be used to build a BHC on \( T_{m,6} \). Consider \( m > 6 \), make \( x = (m - 6) / 4 \). Use Fig. 14 as the beginning, and inset Fig. 12 on the right side for \( x \) times. After that, remove edge set \( E_{13} = \{(7 + 4i, 6)| 10 \leq i \leq (m - 10) / 4\} \cup (1, 6)(6, 6) \), and add edge set \( E_{14} = \{(1, 6)(6 + 4i, 6)(7 + 4i, 6) | 10 \leq i \leq (m - 10) / 4\} \cup (1, 6)(m, 6) \). Finally, a Hamiltonian cycle \( C \) on \( T_{m,6} \) is built as shown in Fig. 15, whose \( |E_9(C)| = 18 + 12x \) and \( |E_2(C)| = 18 + 12n \). Obviously, \( C \) satisfies the definition of BHC owing to \( |E_9(C)| - |E_2(C)| = 0 \).

When \( n > 6 \), the way of constructing the BHC on \( T_{m,n} \) is similar to Case 1.2. Only difference is to replace Fig. 13 with Fig. 15, else parts are the same.
Case 2. One of $m$, $n$ is even, and the other is odd

For any positive integer $n$, $m$, $T_{m,n}$ and $T_{n,m}$ are isomorphic.

Hence, if one of $m$, $n$ is even, and the other is odd, without loss of generality, set that $n$ is even and $m$ is odd.

This case can be also divided into two subcases for discussion. Case 2.1 discusses the condition on $m$ is odd and $n \mod 4 = 0$; Case 2.2 discusses the state on $m$ is odd and $n \mod 4 = 2$.

Case 2.1. $m$ is odd and $n \mod 4 = 0$

Fig. 16 shows a possible BHC on $T_{3,n}$. When $m > 3$, the BHC on $T_{m,n}$ consists of Fig. 7 and Fig. 16. Fig. 17 illustrates the way of connecting. First, for $x = (m-3)/4$, inset $x$ duplicate BHC, which has been shown in Fig. 7, on the right side of Fig. 16. Second, eliminate edge set $E_{15} = \{(3, 3)(i, 3)(i, 4) | 14 \leq i \leq m \} \cup (1, 4)(3, 4) \cup (1, 3)$. Third, put edge set $E_{16} = \{(3 + 2i, 3)(4 + 2i, 3) \cup (3 + 2i, 4)(4 + 2i, 4) | 10 \leq i \leq (m-4)/2 \} \cup (1, 3)(m, 3) \cup (1, 4)(m, 4)$ on the graph produced by previous steps. After that, a Hamiltonian cycle is established, whose $|E(C)| = 6 + 4x$ and $|E_2(C)| = 6 + 4x$. Due to $|E_1(C)| - |E_2(C)| = 0$, $C$ satisfies the definition of BHC.

Case 2.2: $n \mod 4 = 2$ and $m$ is odd

According to theorem 1, there is no balanced Hamiltonian cycle on $T_{m,n}$ for $m$ mod 4 is odd and $n$ mod 4 is 2.

Case 3. $m, n$ both are odd

This case is also separated into eight subcases for discussion. Case 3.1 discusses the state on $n \mod 4 = 1$ and $m$ mod 8 = 1; Case 3.2 proposes the state on $n \mod 4 = 3$ and $m$ mod 8 = 3; Case 3.3 considers the operations when $n \mod 4 = 1$ and $m$ mod 8 = 5; Case 3.4 concerns the details when $n \mod 4 = 1$ and $m$ mod 8 = 7.

The other four remaining cases propose the method under the condition of $n > 3$. Case 3.5 discusses the state on $n \mod 4 = 3$ and $m$ mod 8 = 1; Case 3.6 considers the state on $n \mod 4 = 3$ and $m$ mod 8 = 3; Case 3.7 concerns the operations when $n \mod 4 = 3$ and $m$ mod 8 = 5; Case 3.8 details the cases when $n \mod 4 = 3$ and $m$ mod 8 = 7.

Case 3.1. $m \mod 8 = 1$ and $n \mod 4 = 1$

Fig. 19 and Fig. 20 show one of the possible HCs on $T_{8,5}$ and one of possible BHCs on $T_{8,5}$, respectively. When $m > 9$ and $n = 5$, make $x = (m-9)/8$. First, use Fig. 20 as the beginning, and insert Fig. 19 for $x$ times on its right side. Then eliminate edge set $E_{19} = \{(10 + 8i, 4)(10 + 8i, 5) \cup (17 + 8i, 4)(17 + 8i, 5) | 10 \leq i \leq (m-17)/8 \} \cup (1, 4)(9, 4) \cup (1, 5)(9, 5)$, and add edge set $E_{20} = \{(9 + 8i, 4)(10 + 8i, 4) \cup (9 + 8i, 5)(10 + 8i, 5) | 10 \leq i \leq (m-17)/8 \} \cup (1, 4)(m, 4) \cup (1, 5)(1, m)$. Finally, a Hamiltonian cycle $C$ on $T_{m,n}$ is produced, which is shown in Fig. 21. For $|E_1(C)| = 22 + 18x = 22 + 20x$ and $|E_2(C)| = 23 + 20x$, $C$ satisfies that $|E_1(C)| - |E_2(C)| = 1$. Undoubtedly, $C$ is a BHC of $T_{m,n}$.

When $n > 5$, let $y = (n-5)/4$. Use Fig. 21 as base, then stack $y$ BHCs, which is shown in Fig. 12. Next, remove edge set $E_{21} = \{(1, 6 + 4i)(1, 9 + 4i) | 10 \leq i \leq (n-9)/4 \} \cup (1, 1)(1, 5)$, and insert edge set $E_{22} = \{(1, 1)(1, n) | 10 \leq i \leq (n-9)/4 \} \cup (1, 5 + 4i)(1, 6 + 4i)$. After complete all of the steps, a BHC on $T_{m,n}$ for $m$ mod 8 = 1 and $n$ mod 4 = 1 is established.
Case 3.2. \( m \mod 8 = 3 \) and \( n \mod 4 = 1 \)

Fig. 23 represents the way of constructing a BHC on \( T_{m,5} \). When \( m > 5 \), let \( x = (m - 3) / 8 \), and then duplicate Fig. 19 for \( x \) times. Next, insert them on the right side of the BHC on \( T_{3,5} \), which is shown in Fig. 22. In order to connect every figure, delete edge set \( E_{23} = \{(4 + 8i, 4)(4 + 8i, 5) \cup (11 + 8i, 4)(11 + 8i, 5) : 10 \leq i \leq (m - 11) / 8 \} \cup \{(1, 4)(3, 4) \cup (1, 5)(3, 5), \} \), and add edge set \( E_{24} = \{(3 + 8i, 4)(4 + 8i, 4) \cup (3 + 8i, 5)(4 + 8i, 5) : 10 \leq i \leq (m - 11) / 8 \} \cup \{(1, 4)(m, 4) \cup (1, 5)(m, 5)\). By implementing the steps above, a Hamiltonian cycle \( C \) is generated, whose \( ||E_1(C)|| = 8 + (18 + 2)x = 8 + 20x \) and \( ||E_2(C)|| = 7 + 20x \). Due to \( ||E_1(C)|| - ||E_2(C)|| = 1 \), \( C \) satisfies the definition of BHC.

Case 3.3. \( m \mod 8 = 5 \) and \( n \mod 4 = 1 \)

Fig. 24 represents a BHC on \( T_{5,5} \), which is used to construct the BHC on \( T_{m,5} \). When \( m > 5 \), let \( x = (m - 5) / 8 \). To begin with, insert Fig. 19 for \( x \) times on the right side of Fig. 24. Next, delete edge set \( E_{25} = \{(6 + 8i, 4)(6 + 8i, 5) \cup (13 + 8i, 4)(13 + 8i, 5) : 10 \leq i \leq (m - 13) / 8 \} \cup \{(1, 4)(5, 4) \cup (1, 5)(5, 5)\), and put edge set \( E_{26} = \{(5 + 8i, 4)(6 + 8i, 4) \cup (5 + 8i, 5)(6 + 8i, 5) : 10 \leq i \leq (m - 13) / 8 \} \cup \{(1, 4)(m, 4) \cup (1, 5)(m, 5)\). Thus, a Hamiltonian cycle \( C \) is established, which is shown in Fig. 24. For \( ||E_1(C)|| = 12 + (18 + 2)x = 12 + 20x \) and \( ||E_2(C)|| = 13 + 20x \), \( C \) obviously satisfies \( ||E_1(C)|| - ||E_2(C)|| = 1 \) that make it be a BHC of \( T_{m,5} \) for \( m \mod 8 = 5 \) and \( n \mod 4 = 1 \).

Case 3.4. \( m \mod 8 = 7 \) and \( n \mod 4 = 1 \)

The following steps indirect how to construct a BHC on \( T_{m,5} \). A BHC on \( T_{5,5} \) is shown in Fig. 25. When \( m > 7 \), make \( x = (m - 7) / 8 \). Then use Fig. 25 as the beginning, and inset Fig. 19 for \( x \) times on the right side. In order to connect all figures, eliminate edge set \( E_{27} = \{(8 + 8i, 4)(8 + 8i, 5) \cup (15 + 8i, 4)(15 + 8i, 5) : 10 \leq i \leq (m - 15) / 8 \} \cup \{(1, 4)(7, 4) \cup (1, 5)(7, 5), \} \), and insert edge set \( E_{28} = \{(7 + 8i, 4)(8 + 8i, 4) \cup (7 + 8i, 5)(8 + 8i, 5) : 10 \leq i \leq (m - 15) / 8 \} \cup \{(1, 4)(m, 4) \cup (1, 5)(m, 5)\). Therefore, a Hamiltonian cycle \( C \) is built, which as shown in Fig. 26. For \( ||E_1(C)|| = 18 + (18 + 2)x = 18 + 20x \) and \( ||E_2(C)|| = 17 + 20x \), \( C \) satisfies \( ||E_1(C)|| - ||E_2(C)|| = 1 \). Without a doubt, \( C \) is a BHC of \( T_{m,5} \) for \( m \mod 8 = 7 \) and \( n \mod 4 = 1 \).

When \( n > 5 \), the way of constructing the BHC on \( T_{m,5} \) is similar to Case 3.1. Only one difference is to replace Fig. 21 with Fig. 23.

Refer to Case 3.1 when \( n > 5 \). Replace Fig. 21 with Fig. 26, else parts are similar to Case 3.1. Finally, a BHC on \( T_{m,5} \) is produced.
Case 3.5. \( m \equiv 8 = 1 \) and \( n \equiv 4 = 3 \)

Fig. 27 and Fig. 28 show one of the possible HC s on \( T_{8,7} \) and one of possible BHC s on \( T_{m,7} \), respectively. Furthermore, Fig. 29 illustrates the way of constructing a BHC on \( T_{m,7} \), which is described in the content below. When \( m > 9 \), let \( x = \frac{(m - 9)}{8} \). First, inset \( x \) HC s, which has been mentioned above, on the right side of Fig. 28. Second, remove edge set \( E_{29} = \{(10 + 8i, 6)(10 + 8i, 7) \cup (17 + 8i, 6)(17 + 8i, 7) | 0 \leq i \leq \frac{(m - 17)}{8}\} \cup \{(1, 6)(9, 6) \cup (1, 7)(9, 7)\}. \) Third, add edge set \( E_{30} = \{(9 + 8i, 6)(9 + 8i, 7)(10 + 8i, 7) | 0 \leq i \leq \frac{(m - 17)}{8}\} \cup \{(1, 6)(m, 6) \cup (1, 7)(m, 7)\}. \) Finally, a Hamiltonian cycle \( C \) is yielded, whose \( |E_1(C)| = 32 + (26 + 2) = 32 + 28 \) and \( |E_2(C)| = 31 + 28x \). As a result of \( |E_1(C)| - |E_2(C)|\| = 1 \), it verify that \( C \) is a BHC.

When \( n > 7 \), let \( y = \frac{(n - 7)}{4} \). Stack \( y \) BHCs, which is shown in Fig. 12, above Fig. 28. Then delete edge set \( E_{31} = \{(1, 8 + 4i)(1, 11 + 4i) | 10 \leq i \leq \frac{(n - 11)}{4}\} \cup \{(1, 1)(1, 7)\}, \) and add edge set \( E_{32} = \{(1, 7 + 4i)(1, 8 + 4i) | 10 \leq i \leq \frac{(n - 11)}{4}\} \cup \{(1, 1)(1, n)\}. \) After that, a BHC on \( T_{m,n} \) is generated.

Case 3.6. \( m \equiv 8 = 3 \) and \( n \equiv 4 = 3 \)

Fig. 30 represents a BHC on \( T_{3,7} \), which is used to construct a BHC on \( T_{m,7} \). Besides, Fig. 31 illustrates how to connect Fig. 27 and Fig. 30. Let \( x = \frac{(m - 3)}{8} \) for \( m > 3 \). Then, copy Fig. 27 for \( x \) times, and inset them on the right side of Fig. 30. Next, eliminate edge set \( E_{33} = \{(4 + 8i, 6)(4 + 8i, 7) \cup \{(11 + 8i, 6)(11 + 8i, 7) | 10 \leq i \leq \frac{(m - 11)}{8}\} \cup \{(1, 6)(3, 6) \cup (1, 7)(3, 7)\}, \) and put edge set \( E_{34} = \{(3 + 8i, 6)(4 + 8i, 6) \cup (3 + 8i, 7)(4 + 8i, 7) | 0 \leq i \leq \frac{(m - 11)}{8}\} \cup \{(1, 6)(m, 6) \cup (1, 7)(m, 7)\}. \) Therefore, a Hamiltonian cycle \( C \) is established, whose \( |E_1(C)| = 10 + (26 + 2)x = 10 + 28x \) and \( |E_2(C)| = 11 + 28x \). Because of \( |E_1(C)| - |E_2(C)|\| = 1 \), \( C \) satisfies the definition of BHC.

Case 3.7. \( m \equiv 8 = 5 \) and \( n \equiv 4 = 3 \)

In this section, \( T_{m,7} \) consists of the HC on \( T_{8,7} \), as shown in Fig. 27, and the BHC on \( T_{3,7} \), as shown in Fig. 32. When \( m > 5 \), make \( x = \frac{(m - 5)}{8} \). First of all, inset Fig. 27 for \( x \) times on the right side of Fig. 32. So as to connect all figures, delete edge set \( E_{35} = \{(6 + 8i, 6)(6 + 8i, 7) \cup \{(13 + 8i, 6)(13 + 8i, 7) | 10 \leq i \leq \frac{(m - 13)}{8}\} \cup \{(1, 6)(5, 6) \cup (1, 7)(5, 7)\}, \) and add edge set \( E_{36} = \{(5 + 8i, 6)(6 + 8i, 6) \cup (5 + 8i, 7)(6 + 8i, 7) | 10 \leq i \leq \frac{(m - 13)}{8}\} \cup \{(1, 6)(m, 6) \cup (1, 7)(m, 7)\}. \) As a result, a Hamiltonian cycle \( C \) on \( T_{m,7} \) is yielded, whose \( |E_1(C)| = 18 + (26 + 2)x = 18 + 28x \) and \( |E_2(C)| = 17 + 28x \), as shown in Fig. 33. Without a doubt, \( C \) is a BHC due to \( |E_1(C)| - |E_2(C)|\| = 1 \).
When \( n > 7 \), the way of constructing the BHC on \( T_m, n \) is similar to Case 3.5. Only one difference is to replace Fig. 29 with Fig. 33.

**Case 3.10.** \( m \) mod 8 = 7 and \( n \) mod 4 = 3

There is a BHC on \( T_7, 7 \) as shown in Fig. 34. When \( m > 7 \), use the BHC on \( T_7, 7 \) mentioned above as the beginning. Make \( x = (m - 7) / 8 \), then inset \( x \) HCs, which is shown in Fig. 27, on the right side of Fig. 34. Then, remove edge set \( E_{37} = \{(15 + 8i, 6)(15 + 8i, 7) | 0 \leq i \leq (m - 15) / 8\} \cup (1, 6)(7, 6) \cup (1, 7)(7, 7) \cup (8 + 8i, 6)(8 + 8i, 7) \) and add edge set \( E_{38} = \{(7 + 8i, 6)(8 + 8i, 6) \cup (7 + 8i, 7)(8 + 8i, 7) | 0 \leq i \leq (m - 15) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7) \). Thus, a Hamiltonian cycle \( C \) on \( T_m, 7 \) is built, which is shown in Fig. 35.

**REFERENCES**


