Assessing the Relation between Theory of Multiple Algebras and Universal Algebras

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Abstract—In this study, we examine multiple algebras and algebraic structures derived from them and by stating a theory on multiple algebras; we will show that the theory of multiple algebras is a natural extension of the theory of universal algebras. Also, we will treat equivalence relations on multiple algebras, for which the quotient constructed modulo them is a universal algebra and will study the basic relation and the fundamental algebra in question.

In this study, by stating the characteristic theorem of multiple algebras, we show that the theory of multiple algebras is a natural extension of the theory of universal algebras.

Keywords—multiple algebras, universal algebras

I. INTRODUCTION

The multiple algebras theory was discussed for the first time in 1931, by Marty, the French Mathematician, in an article presented to the 8th Scandinavian mathematical congress. In that article, the author proposed an extension of groups, called super groups, and stated some characteristics. Considering many other papers by Marty, one can predict that multiple algebras can be used as tools in other fields of mathematics theory.

Geratz and Pickett have provided important articles on the theory of multiple algebras. In those articles, multiple algebras are seen as relational systems that are an extension of the theory of the study provided by Gratzer, Pickett and Hansoul [1], [2], Hoft and Howard [3], Schweigert [4], Walicki and Bialaslak [5], and, recently, Breaz and Pelea [6] and Pelea [7]-[10] have published articles.

II. THE QUOTIENT OF A GLOBAL ALGEBRA, A THEOREM SPECIFIC FOR MULTIPLE ALGEBRAS

As it was mentioned in the introduction, Geratz showed in [11] that each multiple algebra can be obtained from the quotient of a global algebra to a definite equivalence relation. In this research we mention this matter by expressing the theorem specific for the multiple algebras.

Suppose that $B = \{B_r, (f_r)_{r \in \tau}\}$ is a global algebra and $\rho$ is an equivalence relation on $B$. Let $B/\rho = \{\rho < b > | b \in B\}$ set, and Geratz defines the multioperation of $f_r$ for every $r < o(r)$ as follows:

$$f_r(\rho < b_0 >, \ldots, \rho < b_{n-1} >) = \{\rho < c > | c = f_r(c_0, \ldots, c_{n-1}), c_i \in \rho < b_i >, i = 0, \ldots, n-1\}.$$  

The $B/\rho$ set together with the above-defined multiple operations form the multiple algebra of $B/\rho$ which is called the multiple concrete algebra.

Theorem 1

Each multiple algebra is a concrete one.

The above theorem can be expressed as follows:

For every multiple algebra $A$ of type $\tau$ there is a universal algebra $B$ of type $\tau$ and the equivalence relation $\rho$ on $B$ such that $A \cong B/\rho$ which is proven in [11]

Remark 1

Let $B = \{B_r, (f_r)_{r \in \tau}\}$ be a universal algebra and $\mathfrak{P}$ be an equivalence relation on $B$. $B/\rho$ is the quotient algebra corresponding to $\rho$. If $p \in P^\sim(A)$ then,

$$p(\rho < b_0 >, \ldots, \rho < b_{n-1} >) \supseteq \{\rho < c > | c = p(c_0, \ldots, c_{n-1}), c_i \rho b_i, i \in \{0, \ldots, n-1\}\}.$$  

Proof:

Since $\pi_\rho$ is a homomorphism between $B$ and $B/\rho$, we have $p(\pi_\rho < b_0 >, \ldots, \pi_\rho < b_{n-1} >) \supseteq \pi_\rho(p(b_0, \ldots, b_{n-1}))$

Therefore, $p(\pi_\rho < b_0 >, \ldots, \pi_\rho < b_{n-1} >) \supseteq \{\rho < b > | b \in p(b_0, \ldots, b_{n-1})\}$.

On the other hand, since $B$ is a universal algebra, therefore, the defined $p$ on the right side of (*) is monovalued, and, consequently,
Let $A = \left( A, f_r \right)_{r \in \mathbb{N}}$ be a multiple algebra of type $\tau$.

Among the equivalence relations defined on $A$, there are relations that the made quotients of the multiple algebra $A$. We proceed by examining of conditions equivalent to the universality of the quotient $A / \rho$. Let $\rho$ be an equivalence relation on $A$. We define the relation $\rho$ on $P^*(A)$ as follows:

\[
\begin{align*}
X \rho Y & \iff \forall x \in X, \forall y \in Y \ x \rho y \quad \text{Or} \\
X \rho Y & \iff X \times Y \subseteq \rho
\end{align*}
\]

the relation $\rho$ is translational and symmetric, but not necessarily reflexive.

Lemma 1

Let $A = \left( A, (f_r)_{r \in \mathbb{N}} \right)$ be a multiple algebra. Let $\rho$ be an equivalence relation on $A$ and $A / \rho$ be a universal algebra. For every $n \in \mathbb{N}$, $p \in P^n(\phi^* \left( A \right))$ and $a_1, \ldots, a_{n-1} \in A$. if $x, y \in p(a_1, \ldots, a_{n-1})$ then $x \rho y$.

Proof:

We prove the lemma by induction on the process of constructing $n$-nominal functions.

If $p = c^a$ and $x, y \in c^a(a_1, \ldots, a_{n-1})$, we conclude that $x = y = a$. Therefore, $x \rho y$.

Theorem 2

Let $A = \left( A, (f_r)_{r \in \mathbb{N}} \right)$ be a multiple algebra of type $\tau$ and $\rho$ be an equivalence relation on $A$. All the conditions states below are equivalent.

1. $A / \rho$ is a universal algebra.

2. If $r < o(\tau), a, b, x_i \in A \ (i \in \{0, \ldots, n-1\})$ and $a \rho b$ , then

\[
\left( f_r(x_1, \ldots, x_{n-1}, a, x_{n+1}, \ldots, x_{n-1}) \right) \rho f_r(x_1, \ldots, x_{n-1}, b, x_{n+1}, \ldots, x_{n-1})
\]

For every $i \in \{0, \ldots, n-1\}$

3. if $r < o(\tau)$ and $x_i, y_i \in A$ such that $x_i \rho y_i (i \in \{0, \ldots, n-1\})$, then

\[
f_r(x_1, \ldots, x_{n-1}) \rho f_r(y_1, \ldots, y_{n-1})
\]
4. if $n \in \mathbb{N}$, $\rho \in P_{\alpha}^n (\mathcal{P}^* (A))$, $x_i, y_i \in A$ such that, for every $i \in \{0, \ldots, n-1\}$, $x_i, y_i \in A$, then

$$p(x_0, \ldots, x_{n-1}) \rho p(y_0, \ldots, y_{n-1}).$$

Proof:

(2) $\iff$ (1) we have, from $a \rho b$, that $\rho < a => \rho < b$. As a result, since $f_r$ is well-defined, for every $r < o(\tau)$,

$$f_r(\rho < x_0, \ldots, \rho < a_0, \ldots, \rho < x_{n-1}) =$

$$f_r(\rho < x_0, \ldots, \rho < b_0, \ldots, \rho < x_{n-1}).$$

On the other hand, by the definition of multiple operations on the quotients of multiple algebras, for every $x \in f_r(x_0, x_1, \ldots, x_{n-1})$

And for every $y \in f_r(x_0, x_1, \ldots, y_{n-1})$

$$\rho < x < y \Rightarrow f_r(\rho < x_0, \ldots, \rho < a_0, \ldots, \rho < x_{n-1} < y).$$

According to assumption, $A / \rho$ is a universal algebra. Therefore, for every $r < o(\tau)$, the defined $f_r$ on $A / \rho$ is mono-valued. Thus, $\rho < x >= \rho < y$ and, consequently,

$$f_r(x_0, x_1, \ldots, x_{n-1}) \rho f_r(x_0, x_1, \ldots, x_{n-1})$$

(2) $\Rightarrow$ (1) Let $\rho < x_0, \rho < y_0 \Rightarrow$

$$f_r(\rho < x_0, \ldots, \rho < x_0, \ldots, \rho < x_{n-1}).$$

Where $x_0, \ldots, x_{n-1} \in A$ are arbitrary. By the definition of $f_r$ on $A / \rho$, there are elements $b_0, \ldots, b_{n-1}, a_0, \ldots, a_{n-1} \in A$ such that, for every $i \in \{0, \ldots, n-1\}$, $y \in f_r(b_0, \ldots, b_{n-1})$ and $x \in f_r(a_0, \ldots, a_{n-1})$. As a result we have, by the hypothesis,

$$f_r(b_0, \ldots, b_{n-1}) \rho f_r(a_0, b_0, \ldots, b_{n-1}),$$

$$f_r(a_0, b_0, b_1, \ldots, b_{n-1}) \rho f_r(a_0, a_0, b_0, \ldots, b_{n-1}),$$

$$f_r(a_0, \ldots, a_{n-2}, b_{n-1}) \rho f_r(a_0, \ldots, a_{n-2}, a_{n-1}).$$

On the other hand, $\rho$ is translational, Therefore,

$$f_r(b_0, \ldots, b_{n-1}) \rho f_r(a_0, \ldots, a_{n-1}),$$

And thus $\rho < x \Rightarrow \rho < y$. Therefore, for every $r < o(\tau)$, $f_r$ is mono-valued. As a result, $A / \rho$ is a universal algebra.

(2) $\iff$ (3) since we assumed that (2) is satisfied, therefore, $A / \rho$ is universal algebra. Thus, like the proof of (1) $\Rightarrow$ (2), we have

$$f_r(x_0, x_1, \ldots, x_{n-1}) \rho f_r(y_0, x_1, \ldots, y_{n-1}).$$

(1) $\Rightarrow$ (3) the stages of proof are ((1) $\Rightarrow$ (2) $\Rightarrow$ (3))

(1) $\iff$ (3) the proof is as (1) $\iff$ (2)

(4) $\iff$ (3) if we consider each element as a single element set, in other words, if we consider $\{x_i\}$ instead of $x_i$, for every $r < o(\tau)$, we can consider $f_r$ as a map of $(P^* (A))$ to $P^* (A)$ as follows:

$$f_r = f_r(e_0^*, \ldots, e_{n-1}^*) \in P_{\alpha}^n (P^* (A))$$

(4) $f_r$ is an operation on the universal algebra $\mathcal{P}^* (A)$.

Therefore for every $r < o(\tau)$, the result is satisfied for $f_r$.

(4) $\Rightarrow$ (3) if $p = e_i$ such that $i \in \{0, \ldots, n-1\}$, then

$$p(x_0, \ldots, x_{n-1}) = e_i^* (x_0, \ldots, x_{n-1}) = x_i,$$

$$p(y_0, \ldots, y_{n-1}) = e_i^* (y_0, \ldots, y_{n-1}) = y_i.$$ 

On the other hand, since for every $i \in \{0, \ldots, n-1\}$, $y_i, \rho x_i$ therefore,

$$p(x_0, \ldots, x_{n-1}) \rho p(y_0, \ldots, y_{n-1}).$$

Assume that the result is satisfied for polynomial functions $p_0, \ldots, p_{n-1} \in P_{\alpha}^n (\mathcal{P}^* (A))$, we prove it for
\[ p = f_r(p, \ldots, p_{n-1}) \quad \text{for every} \quad x \in p(x_1, \ldots, x_{n-1}) \quad \text{and} \quad y \in p(y_1, \ldots, y_{n-1}) \], there are elements \( a_i \in p_i(x_1, \ldots, x_{n-1}) \) and \( b_j \in p_j(y_1, \ldots, y_{n-1}) \) such that

\[ x \in f_r(a_1, \ldots, a_{n-1}), \quad y \in f_r(b_1, \ldots, b_{n-1}). \]

Since we assume that the result is satisfied for \( p_i \), for every \( i \in \{1, \ldots, n_r - 1\} \), \( a_i \) is a relation. Thus, by (1) \( \Rightarrow \) (3), we have

\[ f_r(a_1, \ldots, a_{n-1}) = f_r(b_1, \ldots, b_{n-1}). \]

Therefore, \( x \rho y \).

**Corollary 1**

Let \( A \) be a multiple algebra of type \( \tau \) and \( \rho \) is an equivalence relation on \( A \). \( A / \rho \) is a universal algebra, if and only if for every \( a, b \in A \), where \( a \rho b \) and for every \( r \in o(\tau) \) and \( x, \ldots, x_{n-1} \in A \), we have

\[ f_r(a_1, \ldots, a_{n-1}) = f_r(b_1, \ldots, b_{n-1}). \]

For every \( i \in \{1, \ldots, n_r - 1\} \)

**Corollary 2**

If \( \rho \) is an equivalence relation on \( A \) and \( A / \rho \) is a universal algebra, then

\[ f_r(\rho < a_1 >, \ldots, \rho < a_{n-1}>) = \]

\[ \{ \rho < b > | b \in f_r(a_1, \ldots, a_{n-1}) \}. \]

Also we can write

\[ f_r(\rho < a_1 >, \ldots, \rho < a_{n-1}>) = \]

\[ \rho < b >, b \in f_r(a_1, \ldots, a_{n-1}). \]

**Proof:**
Considering the above definitions, if $A = \left( A, (f_r)_{r \in \tau} \right)$ is a universal algebra, the congruence relation and the strongly regular relations on $A$ are equivalent.

**Remark 2**

If $\rho$ is an equivalence relation on the multiple algebra $A$ such that $A / \rho$ is universal, then $\rho$ is a strongly regular relation on the multiple algebra $A$ and vice versa.

**Remark 3**

Let $A = \left( A, (f_r)_{r \in \tau} \right)$ be a multiple algebra of type $\tau$. If $\rho$ is an equivalence relation on $A$ and $A / \rho$ is a universal algebra, then $\rho$ is an ideal equivalence relation. Therefore, the map $\pi_\rho$ is an ideal homomorphism, and therefore, the inclusion stated in remark 1 turns into an equality.

**REFERENCES**