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Parametric Modeling Approach for Call Holding Times for IP based Public Safety Networks via EM Algorithm

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***Abstract*—**This paper presents parametric probability density models for call holding times (CHTs) into emergency call center based on the actual data collected for over a week in the public Emergency Information Network (EIN) in Mongolia. When the set of chosen candidates of Gamma distribution family is fitted to the call holding time data, it is observed that the whole area in the CHT empirical histogram is underestimated due to spikes of higher probability and long tails of lower probability in the histogram. Therefore, we provide the Gaussian parametric model of a mixture of lognormal distributions with explicit analytical expressions for the modeling of CHTs of PSNs. Finally, we show that the CHTs for PSNs are fitted reasonably by a mixture of lognormal distributions via the simulation of expectation maximization algorithm. This result is significant as it expresses a useful mathematical tool in an explicit manner of a mixture of lognormal distributions.

***Keywords*—**A mixture of lognormal distributions, modeling call holding times, public safety network.

I. Introduction

EMERGENCY communication networks that serve various safety personnel, including medical responders, police, hazard, and fire fighters, play a critical role in responding to Emergency calls and managing voice traffic.

Studying call holding time provides the ability to estimate the probability of an ongoing call to hang up the phone during the next t seconds [1]. Therefore, the call holding time distribution plays a prominent role that is used to analyze the traffic and accurate design of the system resources, simulation, performance, and resource allocation strategy.

The call duration over fixed, cellular, and trunked radio networks is traditionally assumed to be negative exponentially distributed. However, this distribution approximation is not valid for communication networks because many empirical approaches have proved that lognormal distribution and two and three mixed lognormal distributions fit empirical data much better [2]- [9]. For instance, the exponential distribution in cellular networks is quite inaccurate in capturing the empirical data compared to the mixture of lognormal distributions.

It is observed that the probability of very short occurrences is overestimated, while the area with the highest probability in the empirical histogram is underestimated [3], [10].

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Whereas there exists a literature such as call holding time distribution modeling in fixed PSTN network [4], in PCS network [6], in cellular networks [7], in private mobile radio PAMR [8], in public safety network [11], and in various networks [2], in this paper, we present the study of more flexible distribution of CHTs with its mathematical tools in PSNs, which has not has been presented before.

It is proved that the parametric mixture model has been implemented to show that average CHTs of PSNs may be approximated accurately by a mixture of lognormal distributions compared to the fitting of the statistical probability distributions of Lognormal, shifted Lognormal, and Weibull distributions, which provides the example of visual view of non superimposed distribution graphs (Fig. 2).

This article is structured as follows: Section II presents the IP based EIN as a PSN, its system structure and components. Section III presents a measurement data exploration and data statistics. Section IV describes the statistical methods of the deriving CHT distributions and fitting with statistical candidate distributions. Section V presents the proposed modeling of a mixture of lognormal distributions. Section VI presents the EM algorithm performance. In section VII, we demonstrate some results of the proposed method for PSN. Finally, we conclude the research.

II. IP Based Emergency Information Network

The ability to access emergency services by dialing fixed numbers is a vital component of public safety and emergency preparedness. The Federal Communications Commission (FCC) defines voice over Internet Protocol (VoIP) as a technology that supports some IP based services to allow a user to call anyone who has a telephone number, including mobile and fixed network numbers. Gradually, the Emergency telephonic services are migrating to “interconnected” Voice over Internet Protocol (VoIP).

The IP based EIN is the provider network with state-of-the-art ICT technology solution compared with previous Emergency services which were handled by the national medical, police, and emergency management agencies in Mongolia separately. There are four Emergency telephone numbers in the country. They are:103 (ambulance), 102 (police), 101 (fire), and 105 (hazards, disaster). When dialing any one of these numbers from telecommunications networks (PSTN, GSM, CDMA) through PSTN, the emergency call is pushed/forwarded to an emergency call center agency desk of the EIN.

In addition to the public emergency call services, the EIN supports the Government Emergency Telecommunication Services (GETS) which are transferred through the connected mutually citizen and government agencies such as the National Defense Agency, National Emergency Management Agency, National Security Agency, National Medical, National Transportation Agency, and all the branches of the respective agencies.

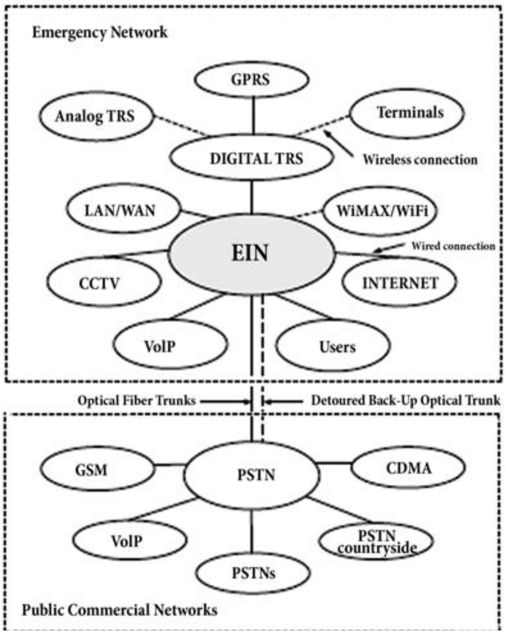

Fig. 1 Communication Architecture of EIN

Fig. 1 presents the EIN architecture that provides Emergency call services through wired (PSTN) and wireless (GSM, CDMA) system services for the main capital Ulaanbaatar city as well as provinces within 1100 km. The EIN system is composed of four main networks including Fiber optical (F/O) backbone network, IP based telecommunications and computer networks (IP PABX/VoIP/LAN/WAN/WiMax/WiFi), Trunk Radio System (TRS), and CCTV(Digital Camera TV) networks for data, video, and image transfers in urban area coverage.

However, the life-threatening emergency voice calls which are transferred through the bulk of the existing E1 optical trunks across long haul telecommunication networks are the main part of the system used for the peak period traffic analysis because of the dedicated limited number of inbound and outbound E1 trunks under the government regulation.

If the number of calls addressed to the system is too large during the peak period, the incoming trunks will be overloaded and operators cannot handle the volume of emergency calls. Hence, one aspect of particular interest is the peak period network analysis to model the call holding time for the traffic analysis and to estimate the call holding time parameters.

III. Data Exploration and Statistic Results

We explore the real data from a number of incoming bundled digital trunks that connect into the main edge port of IP PBX of EIN, a national emergency service provider in Mongolia.

The statistics indicate that the mean of CHT of Emergency ambulance, police, fire, and hazards incoming calls are 61.16s ($cv = 0.14$), 42.56s (cv = 0.22), 27.17s (cv = 0.42), and 27.76s (cv = 0.58), respectively. The mean CHT data for all emergency incoming call samples measured is 39.66s. This is a shorter call duration and less value of coefficient of variance cv compared to 201s with cv = 1.23 in non-VoIP call center [13], 110s with cv = 2.7 in Taiwan-mobile [5], 113s with cv = 1.4 in public telephone system [4], 63.3s with cv = 2.91 in PAMR-PCS [10], 40.6s with cv = 1.7 in non VoIP cellular network [3].

From this comparison, emergency VoIP calls are considered more efficient time-wise than those of commercial networks. The ambulance customers keep longer holding call behavior than the other emergency customers, such as police, fire and hazards; 35s ambulance's mean CHT is longer than the fire's mean CHT.

IV. General Methods on Deriving Call Holding Distributions

In this section, the set of probability density functions (pdfs) as a main statistical candidate to fit the CHT empirical distributions for ambulance, police, fire, and hazards call classes is considered.

For the fitting of these candidate distributions, we use the tests: K-S tends to be more sensitive near the center of the distribution than at the tails. Due to this limitation above, we prefer to use the Anderson-Darling goodness-of-fit test which gives more weight to the tails than does the K-S test. The pdf of the these candidate distributions:

1. The pdf of Exponential Distribution:

$$f(x;\lambda) = \lambda \exp(-\lambda x) \tag{1}$$

where $\lambda > 0$ is the rate parameter of distribution.

2. The pdf of Weibull Distribution:

Weibull distribution is a continuous probability distribution with wide applicability, primarily due to its relation to the Gamma distribution family.

The pdf of Weibull distribution has the form:

$$F(x;\beta,\sigma) = \exp\left(x/\sigma\right)^{\beta} \tag{2}$$

where β and σ are the scale and shape parameters of distribution, respectively.

3. The pdf of Shifted Lognormal Distribution:

The pdf of a shifted lognormal distribution has the form:

$$\phi_x(x) = \frac{1}{(x-\sigma)\gamma\sqrt{2\pi}} \exp\left(-\frac{(Ln(x-\sigma)-\mu)^2}{2\gamma^2} \right) \quad (3)$$

where $\gamma > 0$, $x > 0$, $and\ \sigma > 0$. We use the likelihood function :

$$L(\phi_x(x)) = \prod_{i=1}^{n} \phi_x(x_i) = \sum_{i=1}^{n} Ln\phi_x(x_i) \quad (4)$$

where $\gamma > 0$, $x > 0$, $and\ x > \sigma$.

4. The pdf of Lognormal Distribution:

Lognormal distribution is a probability distribution of any random variable whose logarithm is normally distributed. The pdf of a lognormal distribution has the form:

$$\phi_x(x) = \frac{1}{x\gamma\sqrt{2\pi}} \exp\left(-\frac{(Lnx-\mu)^2}{2\gamma^2} \right) \quad (5)$$

The log-likelihood function is given by

$$\begin{aligned} Ln(\phi_x(x)) &= \sum_{i=1}^{n} Ln\left[\frac{1}{x_i\gamma\sqrt{2\pi}} \exp\left(-\frac{(Lnx_i-\mu)^2}{2\gamma^2} \right) \right] \\ &= -nLn\gamma - \frac{n}{2}Ln2\pi - \sum_{i=1}^{n} Lnx_i - \frac{1}{2\gamma^2}\sum_{i=1}^{n}(Lnx_i-\mu)^2 \end{aligned} \quad (6)$$

The parameter values are computed through Maximum Likelihood Estimate (MLE) for a lognormal distribution [18], [19]. The statistical fitting results of police, fire, hazard's CHTs are similar to the ambulance's CHT pdf, which is depicted in Fig. 2.

The relative tail frequencies of the real data are much larger than the values of the Exponential, Weibull, Lognormal, and Shifted Lognormal distributions. The lognormal and shifted lognormal distributions are assumed to be the better fitted distributions with similar parameters for D.max values and p values at the 0.02 significance level under K-S, Chi-Sq, and A-D tests (see Table I).

The significance resulting from the Kolmogorov- Smirnov (K-S) goodness-of-fit test which is described by the modified K-S distance $Dmax = \epsilon(\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}})$, where ϵ for the maximum difference between the fitting distribution and the empirical cdf and the level of significance $\alpha = 2\sum_{i-1}^{\infty}(-1)^{i-1}e^{-2i^2n\epsilon^2}$[12], [17].

However MLE does not simplify a solution of a mixture of lognormal distributions for call holding time. Therefore, the EM approach [14], a specialized approach designed for MLE problems, is described for a mixture of lognormal distributions in this paper.

Based on the assumption of smooth histograms of an empirical data, the standard distributions for the empirical data fit quite accurate. However, in practice, a jagged probability histogram with random spikes and long tails frequently occurred. Therefore, the spikes and tails may cause the call holding time to have a jagged histogram. The most comprehensive method of a mixture of lognormal distributions with two or more parameters can be used to approximate the whole call duration, including the spikes and tails, detailed in the section V.

TABLE I
"103" CHT: PARAMETERS OF FITTED DISTRIBUTIONS AT SIGNIFICANCE LEVEL

m=61.16 cv=0.14	"D. max" K-S	"D. max" Chi-sq	p value K-S	p value Chi-sq
Shifted Lognormal	0.037	2.232	0.9631	0.9458
Log normal	0.038	2.23	0.96	0.9459
Weibull	0.067	14.7	0.4	0.0388
	p val (A-D)	"scale"	"shape"	"loc"
Shifted Lognormal	0.1936	0.14	μ=4.1	-1.63
Log normal	0.1959	σ=0.15	γ=4.1	0
Weibull	2.156	β=64.45	α=8.64	0

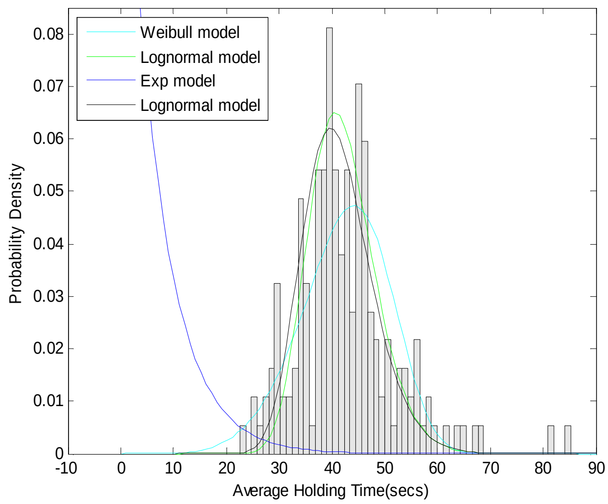

Fig. 2 PDF of CHT of Emergency Ambulance Calls

V. PROPOSED MODELING OF MIXED LOGNORMAL DISTRIBUTIONS BASED ON EXPECTATION MAXIMIZATION ALGORITHM

In the EM approach, when CHTs $x = (x_i, ..., x_n)$ are observed, we may consider the component indicators $y = (y_1, ..., y_n)$ as missing like as in the usual mixture situation, so that $z = (x, y)$ becomes the complete data [20].

In literature it might happen that the CHTs in networks are not explicitly described by a mixture of parametric multivariate distributions [14].

When each mixture component is mapped to a lognormal call holding time distribution using priority probabilities π_i, we notice that the n independent and identically distributed $(i.i.d)$ call holding time observations $x = (x_1, ..., x_n)$ come from a finite mixture of k lognormal CHT components as follows:

$$\phi_\theta(x) = \sum_{j=1}^{k} \pi_j \phi_{\theta_j}(x) = \sum_{j=1}^{k} \pi_j N(x \mid \mu_k, \gamma_k) \tag{7}$$

where $\phi_{\theta_j}(x) = (\phi_{\theta_1}(x), \dots, \phi_{\theta_k}(x))$ are the component lognormal densities, $\theta_j = (\mu_{i=1}, \dots, \mu_k, \gamma_{i=1}, \dots, \gamma_k)$ are the parameters, and $\pi_j = (\pi_1, \dots, \pi_k)$ are the component weights satisfying $\sum_{j=1}^{k} \pi_j = 1$. Therefore, the probability density function of the average CHTs may be modeled by a mixture of k random variables of Gaussian densities on a logarithmic time scale:

$$\begin{aligned}\phi_\theta(x) = \frac{1}{x\sqrt{2\pi}}[\pi_1 \frac{1}{\gamma_1}\exp\left(-\frac{(Lnx-\mu_1)^2}{2\gamma_1^2}\right) + \dots \\ \pi_k \frac{1}{\gamma_k}\exp\left(-\frac{(Lnx-\mu_k)^2}{2\gamma_k^2}\right)]\end{aligned} \tag{8}$$

It is often assumed that the parametric MLE is to estimate the set of parameters θ for the density of the samples $\phi_\theta(x)$ that maximizes the likelihood function $L(\phi_\theta(x)) = \prod_{i=1}^{n} \phi_\theta(x_i)$ [15]. Hence the likelihood of the lognormal CHT distribution becomes:

$$\begin{aligned}L = \prod_{i=1}^{n} \frac{1}{x_i\sqrt{2\pi}}[\pi_1 \frac{1}{\gamma_1}\exp\left(-\frac{(Lnx-\mu_1)^2}{2\gamma_1^2}\right) + \dots \\ \pi_k \frac{1}{\gamma_k}\exp\left(-\frac{(Lnx-\mu_k)^2}{2\gamma_k^2}\right)\end{aligned} \tag{9}$$

The complete CHT data set (observed CHTs x plus un-observed CHTs y datasets) exists $z = (x, y)$ with density $\phi_\theta(z) = \prod_{i=1}^{n} \phi_\theta(z_i)$ where $z = (z_i, \dots, z_n)$. When a many-to-one mapping from z to $x, y_i = (y_{ij}, i = 1, \dots, n; j = 1, \dots, k)$ is the unobserved component of the origin of the n call holding times. Hence, it is clear that the indicator y_{ij} implies $\sum_{j=1}^{k} y_{ij} = 1, a$ Bernoulli random variable indicating that the CHT observation x_i comes from the exact lognormal distribution with parameter θ_j.

The parametric MLE problem is to estimate the set of Gaussian parameters for the lognormal distribution that maximizes the likelihood function [15].

In this multivariate model case, the likelihood function of this complete density for one data observation becomes

$$\begin{aligned}L\phi_\theta(z) = L\prod_{i=1}^{n} \phi_\theta(x_i, y_i) = \prod_{i=1}^{n}\sum_{j=1}^{k} \pi_j I_{y_{ij}} \phi_{\theta_j}(x_i) = \\ \prod_{i=1}^{n}\sum_{j=1}^{k}\left[\frac{\pi_j}{x_i\gamma_j\sqrt{2\pi}}\exp\left(-\frac{(Lnx_i-\mu_j)^2}{2\gamma_j^2}\right)\right]^{y_{ij}}.\end{aligned} \tag{10}$$

where the indicator function $I_{y_{ij}} = 1$ for individual i that comes from the lognormal component with θ_j and is zero elsewhere.

Since the logarithm is a convex increasing function, maximizing the likelihood is equivalent to maximizing the log-likelihood, thus the likelihood (10) is updated in the logarithm form as log- likelihood function for the complete CHT data:

$$\begin{aligned}Ln\left[L\phi_\theta(z)\right] = \sum_{j=1}^{k} I_{y_{ij}}[nLn\pi_j - nLn\gamma_j - \frac{n}{2}Ln2\pi \\ -\sum_{i=1}^{n} Lnx_i - \frac{1}{2\gamma_j^2}\sum_{i=1}^{n}(Lnx_i - \mu_j)^2] \\ = \sum_{i=1}^{n}\sum_{j=1}^{k} I_{y_{ij}}[Ln\pi_j - (Ln\gamma_j + Lnx_i) - \frac{1}{2}Ln2\pi \\ -\frac{1}{2\gamma_j^2}(Lnx_i - \mu_j)^2]\end{aligned} \tag{11}$$

where n $(i.i.d)$ and $x = (x_1, \dots, x_n)$. The second component of the expected value of the complete CHT data log-likelihood is the marginal distribution of the unobserved CHT data $y = (y_1, \dots, y_n)$ on both the observed CHT data x and on the current estimates $\theta^t = (\pi_{y_i=1}^t, \dots, \pi_k^t, \mu_{y_i=1}^t, \dots, \mu_k^t, \gamma_{y_i=1}^t, \dots, \gamma_k^t)$. The weight π_j is often assumed that $p(y_{ij} = 1) = \pi_j$ .Therefore, we use Baysian theorem to compute the expression for the distribution of the unobserved CHT:

$$\begin{aligned}r_{ij}^t = \phi_j^t(x_i) = p_{\theta^t}(y_{ij} = 1 \mid x_i; \theta, \pi) \\ = \frac{\pi_j^t \phi_{\theta_j^t}(x_i)}{\phi_{\theta^t}(x_i)} = \frac{\pi_j^t \phi_{\theta_j^t}(x_i)}{\sum_{\tilde{j}}^{k} \pi_{\tilde{j}} \phi_{\theta_{\tilde{j}}^t}(x_i)}\end{aligned} \tag{12}$$

where $\phi_\theta(x_i)$ is simply the CHT lognormal distribution evaluated at x_i and likelihood function given θ^t which is similar to the expression in the right side of (9).

Given θ^t, a mixture of lognormal distribution is computed for each i and j .

The expected value of the "complete data $z = (x, y)$" log-likelihood $\phi_\theta(z)$ would be the iterative process. It is formed as a function of the estimate θ from (11) and (12) where θ^t is the current value at iteration t .

$$q(\theta,\theta^t)=\sum_{j=1}^{k}\sum_{i=1}^{n}Ln\pi_j\phi_{\theta_j}(x_i)\phi_j^t(x_i)$$
$$=\sum_{j=1}^{k}\sum_{i=1}^{n}Ln(\pi_j)\phi_j^t(x_i)+\sum_{j=1}^{k}\sum_{i=1}^{n}[Ln\pi_j-Lnx_i-Ln\gamma_j \quad (13)$$
$$-\frac{1}{2}Ln2\pi-\frac{1}{2\gamma_j^2}(Lnx_i-\mu_j)^2]\phi_j^t(x_i)$$

Therefore, taking derivatives with respect to these $\pi_j,\ \mu_j,\gamma_j$ and equating them to zero, we define the new estimates $\pi_j^t\ ,\mu_j^t\ ,\gamma_j^t$ respectively as follows:

1. The Weights of CHT Lognormal Components:

To derive the expression for π_j^t, we need the Lagrange multiplier λ with the constraint that $\sum_j \pi_j = 1$, then we can iteratively get the result:

$$\pi_j^t=\frac{\sum_{i=1}^{n}\phi_j^t(x_i)}{n} \quad (14)$$

2. The Parameter μ_j of CHT Lognormal Components:

$$\sum_{j=1}^{k}\sum_{i=1}^{n}[\frac{d}{d\mu_j}(Ln\pi_j-Lnx_i-Ln\gamma_j$$
$$-\frac{1}{2}Ln2\pi-\frac{1}{2\gamma_j^2}(Lnx_i-\mu_j)^2)]\phi_j^t(x_i)=0 \quad (15)$$

, it is clear that the parameter can be written in the form:

$$\mu_j^t=\frac{\sum_{i=1}^{n}\phi_j^t(x_i)Lnx_i}{\sum_{i=1}^{n}\phi_j^t(x_i)} \quad (16)$$

3. The Parameter γ_j of CHT Lognormal Components:

$$\sum_{j=1}^{k}\sum_{i=1}^{n}[\frac{d}{d\gamma_j}(Ln\pi_j-Lnx_i-Ln\gamma_j$$
$$-\frac{1}{2}Ln2\pi-\frac{1}{2\gamma_j^2}(Lnx_i-\mu_j)^2)]\phi_j^t(x_i)=0 \quad (17)$$

, we get the result:

$$\gamma_j^t=\frac{\sum_{i=1}^{n}\phi_j^t(x_i)(Lnx_i-\mu_j^t)(Lnx_i-\mu_j^t)^T}{\sum_{i=1}^{n}\phi_j^t(x_i)} \quad (18)$$

VI. EM Algorithm Performance of the Proposed Mixed Model

The EM algorithm provides computational techniques for distributions that are almost completely unspecified [14], [16].

The algorithm performs the process with respect to the parametric part of this expected value of the completed CHT data log-likelihood by assuming the existence of the hidden variables and making a guess at the initial parameters of the distribution [15]. The EM algorithm iteratively maximizes $Q(\theta,\theta^t)$ by the following two steps.

1. E-step: Compute $Q(\theta,\theta^t)$

E- step calculates the expected value of the "complete data" log likelihood from Equation (13) with respect to the unobserved call holding times y for all $i=1,\dots,n$ and $j=1,\dots,k$. It means we calculate the expected value of the unobserved CHT data using the observed incomplete CHT data. Computing this expectation requires the posterior probability $\phi_j^t(x_i)$ as in equation (12) and π_j^t for the parameter value (θ^t) (14).

2. M-step: set $\theta^{(t+1)}=argmax_\theta Q(\theta|\theta^t)$

This step maximizes the expectation we performed in E-step. More specifically, M-step performs the first part containing π_j^t and the second part containing θ_j^t iteratively from (13). We note that the parts are not related; we can maximize independently.

These two steps are iterated as necessary until the saturation value of the expected complete log-likelihood. At the saturation value, the M-step performs the set of parameters θ, otherwise we repeat the E-step for the next iteration.

Although the iteration increases the marginal log-likelihood function, the EM algorithm uses a random restart approach to avoiding a local maximum of the observed data log-likelihood function.

VII. Numerical Results and Model Validation

We start to look at how long a call conversation time into emergency ambulance incoming call services in the PSN.

In a practical matter of models, it is always very difficult to fit the spikes and tails of frequencies. We observe clearly that the CHT pdf has extreme spikes around 56 and 66 seconds and long tails around more than 93 seconds, which shows why the CHT is not accurately modeled by Lognormal, shifted Lognormal, Exponential, and Weibull distributions, while a mixture of lognormal distributions can be used to approximate accurately the empirical data with the random spikes and long tails as a best fitted model.

In the EM performance, we performed 50-321 iterations to find reasonable fits.

From equation (8), the probability density function of CHTs of "103" is expressed by a mixture of lognormal distributions which has five lognormal components as follows:

$$\phi_\theta(x)=\frac{1}{x\sqrt{2\pi}}[6.302066\,exp-\frac{(Lnx-4.160792)^2}{0.000281}$$

$$+0.038295exp-\frac{(Lnx-4.005497)^2}{0.034232}$$

For the CHTs of other emergency call types, we express a mixture model similar as the above expression of "103" ambulance (Fig 3). The CHT distributions for emergency police ("102"), fire ("105"), and hazards ("101") are fitted quite well by five, five, and four lognormal distributions, respectively, in each case. The Fig. 3 through Fig. 6 shows the graphs for each data set.

As shown in these figures, the observations also prove that a mixture lognormal model may be considered for the fitting of the average CHT data which may have incomplete history in future research.

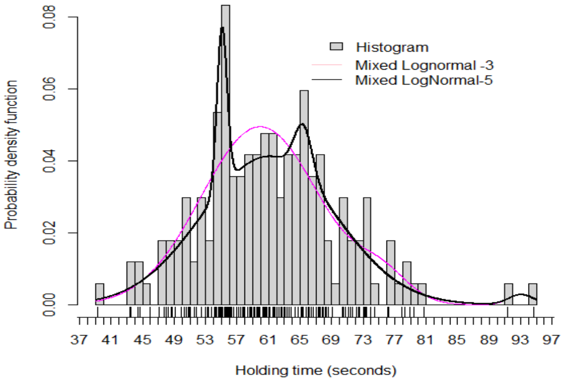

Fig. 3 Histogram of "103" data and its fitting pdf curves

This mixed lognormal model uses the average emergency conversation times for setting the model MLE parameters, so that the CHTs of simulated calls of emergency ambulance, police, fire, and hazard incoming calls are computed by the average values for the chosen model.

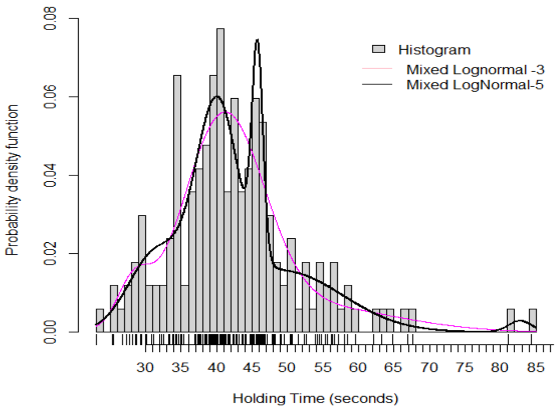

Fig. 4 Histogram of "102" data and its fitting pdf curves

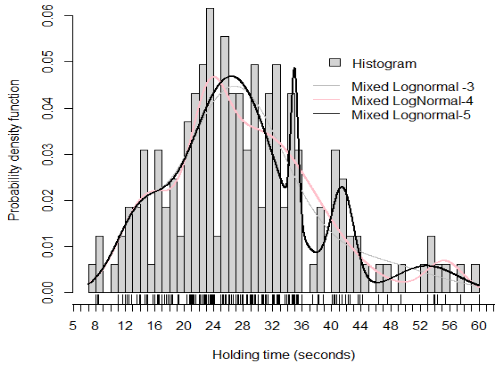

Fig. 5 Histogram of "101" data and its fitting pdf curves

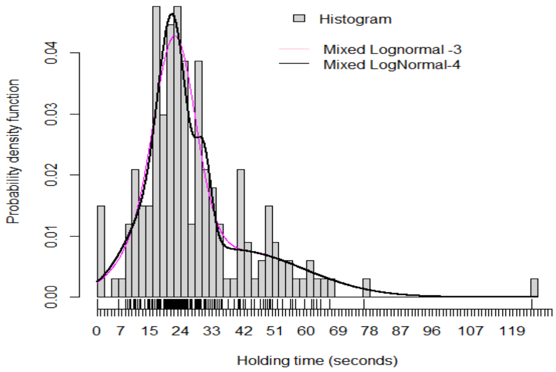

Fig. 6 Histogram of "105" data and its fitting pdf curves

As described in the previous section, we should have a desired equation for the chosen CHT model by a mixture of k random variables of Gaussian densities on a logarithmic time scale:

$$\phi_\theta(x)=\frac{\pi_k}{x\gamma_k\sqrt{2\pi}}\exp-\frac{(Lnx_i-\mu_k)^2}{2\gamma_k^2} \tag{19}$$

TABLE II
PARAMETERS OF FITTED MIXTURE LOGNORMAL DISTRIBUTIONS

CHTs	"102"	"103"	"105"	"101"
D.max	0.2172569	0.0374377	0.0364377	0.025939
ϵ	0.00013	0.00023	0.00023	0.00016
μ1	3.590257	4.009660	2.787204	2.841544
μ2	3.823747	4.160792	3.105966	2.840751
μ3	3.701574	4.156612	3.508355	3.131199
μ4	3.968549	4.181588	3.333218	3.703959
μ5	3.928348	4.005497	4.013681	N/A
γ1	0.184237	0.011857	0.342812	0.272628
γ2	0.017876	0.1788397	0.057409	0.029038
γ3	0.050927	0.102826	0.195627	0.200754
γ4	0.089731	0.017986	0.239888	0.355409
γ5	0.253997	0.130829	0.0414164	N/A
π1	0.421289	0.0747236	0.237278	0.136657
π2	0.115688	0.178327	0.060063	0.042747
π3	0.220727	0.410079	0.315353	0.465886
π4	0.100068	0.0441521	0.351031	0.354708
π5	0.142225	0.292717	0.0362751	N/A

According (19), we would be able to obtain mathematical models that can accurately capture the spikes and heavy tails of call conversation time distribution of modern networks.

By comparing the parameter results in Table I and II, we show that moving from Gamma family distributions to a mixture of Lognormals leads us to more accurate result and less error. As stated in the section III, although values of distance D.max in the Table I represent a successful outcome of experiments that the empirical CHT data may come from the Gamma distribution family, with respect to the all types of Emergency incoming call holding times.

However, when simulations of a mixture model are carried out, a mixture of lognormal distributions represents significant improvements to hold the spikes and as well to estimate tails of frequencies.

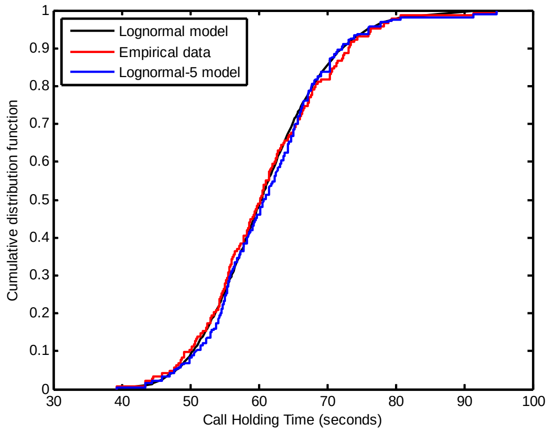

Fig. 7 P-P plot of "103" data and its fitting cdf curves

As a model validation, both pdf and cdf were depicted. Whereas pdf plots the probability of occurrence of the random variable under study, the cdf plots the probability that the random variable will not exceed specific values.

The cdf results of police, fire, hazard's CHTs are similar to that of the ambulance's CHT, which is depicted in Fig. 7. As the illustration in this figure shows the relative head, body, and tail frequencies of the real data are fitted quite well by a mixed lognormals compared to values of Lognormal model.

In Table II, the parameters (all μ, γ) of lognormal components with their D.max values are listed through the K-S test and ϵ of fitting a mixture of lognormal distributions. The values of D.max are 0.0217, 0.0374, 0.0364 and 0.0259 for the police, ambulance, fire, and hazards CHTs, respectively.

The values of D.max = 0.0374377 and $\varepsilon = 2.3 * 10^{-4}$ for the ambulance CHT. The weights of component lognormal distributions are 0.0747236, 0.178327, 0.410079, 0.0441521, and 0.292717 respectively.

It is clear that the smaller Dmax distance represents that the derived mixed lognormal distribution to model the spikes and tails is better than Gamma family distributions.

Looking at D.max factors for each model by K-S, A-D and Chi-Sq tests represents that the model of a mixture of lognormals has the lowest distance D.max and error ε, then it can be chosen as the best call holding time model in a public safety network.

VIII. Conclusion

The results contribute to the existing literature in two important ways: First, a mixture of lognormal distributions provides reasonable fits for the incoming CHTs of PSNs due to spikes of higher probability and long tails of lower probability in a data set. Second, the use of finite mixture lognormal modeling provides a best example of a statistically suitable method for modeling the distribution function when the long tails and peak spikes are randomly and frequently occurred in the frequencies, when we do a peak period performance analysis for PSN. We emphasize the EM algorithm is a powerful method to model the CHTs for IP based PSNs. A mixture of lognormal distributions with explicit analytical expressions is described for this model. This result is important as it provides a useful mathematical tool in an explicit manner for a mixture of lognormal distributions.

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