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On Detour Spectra of Some Graphs

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Abstract—The Detour matrix (DD) of a graph has for its (i , j) entry the length of the longest path between vertices i and j. The DD-eigenvalues of a connected graph G are the eigenvalues for its detour matrix, and they form the DD-spectrum of G. The DD-energy E_{DD} of the graph G is the sum of the absolute values of its DD-eigenvalues. Two connected graphs are said to be DD- equienergetic if they have equal DD-energies. In this paper, the DD- spectra of a variety of graphs and their DD-energies are calculated.

Keywords—Detour eigenvalue (of a graph), detour spectrum(of a graph), detour energy(of a graph), detour - equienergetic graphs.

I. Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix[4]. In what follows we denote the ordinary eigenvalues of the graph G by λ_i, i = 1,2, . . . ,n and the respective spectrum by spec(G). The detour matrix DD = DD(G) of G is defined so that its (i,j) - entry is equal to the length of longest path between vertices i and j. The eigenvalues of the DD(G) are said to be the DD-eigenvalues of G and form the DD-spectrum of G, denoted by $spec_{DD}(G)$. Since the detour matrix is symmetric, all its eigenvalues μ_i, i = 1,2, . . . ,n are real and can be labeled so that $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$. If $\mu_{i_1} \geq \mu_{i_2} \geq ... \geq \mu_{i_g}$ are the distinct DD- eigenvalues, then the DD- spectrum can be written as

$spec_{DD}(G) = \begin{bmatrix} \mu_{i_1} & \mu_{i_2} & . & . & . & \mu_{i_g} \\ m_1 & m_2 & . & . & . & m_g \end{bmatrix}$ where m_j indicates the algebraic multiplicity of the eigenvalue μ_{ij}. Of course, $m_1 + m_2 + ... + m_g = n$. Two graphs G and H for which $spec_{DD}(G) = spec_{DD}(H)$ are said to be DD- cospectral. Otherwise, they are non-DD-cospectral. The DD-energy,E_{DD}, of G is defined as $E_{DD} = \sum_{i=1}^{n} |\mu_i|$. Two graphs with equal DD-energy are said to be DD-equienergetic. DD- cospectral graphs are evidently DD-equienergetic. Therefore, in what follows we focus our attention on DD- equienergetic non-DD-cospectral graphs. The concept of detour matrix was introduced in graph theory by F. Harary[6] for describing the connectivity in directed graphs. The detour matrix was then extensively studied in[8,9,10]. In the subsequent section we derive a Hoffman-type relation for the detour matrices of complete graphs, complete bipartite graphs and cycles. By means of it, the detour spectra of some graphs and their energies are obtained. The following results are used in the subsequent sections:

Result 1[4]. Let G be a graph with adjacency matrix A and spec(G) = $\{\lambda_1, \lambda_2, ..., \lambda_n\}$. Then $detA = \prod_{i=1}^{n} \lambda_i$. In addition ,for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(x)$ and hence $detP(A) = \prod_{i=1}^{n} \ P(\lambda_i)$.

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Result 2[5]. Let A = $\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ be a 2×2 block symmetric matrix. Then the eigenvalue of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.

Result 3 [4]. Let M, N, P and Q be matrices, and let M be invertible. Let S = $\begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. Then $detS = detMdet(Q - PM^{-1}N)$. Besides, if M and P commute, then $detS = det(MQ - PN)$.

Result 4 ([4])$spec(K_{n,n}) =$. $\begin{bmatrix} n & 0 & -n \\ 1 & 2n-2 & 1 \end{bmatrix}$

Result 5 ([7]) Let M be a real symmetric irreducible square matrix of order n in which each row sum is equal to a constant k. Then there exists a polynomial Q(x) such that Q(M) = J,where J is the all one square matrix whose order as that of M.

Result 6 ([7]). Let D be the distance matrix of a connected distance regular graph G. Then D is irreducible and there exists a polynomial P(x) such that P(D) = J. In this case $P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_2)...(x-\lambda_g)}{(k-\lambda_2)(k-\lambda_3)...(k-\lambda_g)}$ where k is the unique sum of each row which is also the greatest simple eigenvalue of D, whereas $\lambda_2, \lambda_3, ..., \lambda_g$ are the other distinct eigenvalues of D. In [7], it is shown that the distance spectrum of double graph of any simple graph,Cartesian product of distance regular graph G with K_2 and lexicographic product of any simple graph G with K_2 depends on the distance spectrum of G. However for detour spectrum this varies from graphs to graphs.

Definition 1[7]. Let G be a graph with vertex set$V(G) = \{v_1, v_2, ..., v_n\}$. Take another copy of G with the vertices labeled by $\{u_1, u_2, ..., u_n\}$ where u_i corresponds to v_i for each i. Make u_i adjacent to all the vertices in $N(v_i)$ in G, for each i. The resulting graph, denoted by D_2G,is called the double graph of G.

Definition 2[4].Let G be a graph. Attach a pendant vertex to each vertex of G. The resulting graph, denoted by $G \circ K_1$, is called the corona of G with K_1.

In this paper we first derive a detour matrices of complete graphs and complete bipartite graphs. By means of it the detour spectra of complete graph and complete bipartite are obtained. The largest eigenvalue of cycle of length n is also obtained. All graphs considered in this paper are simple and we follow[2,4] for other graph theoretic terminologies.

II. Detour spectrum of some graphs

Theorem 2.1. If G is the complete graph of order n, then the detour energy of G is $E_{DD}(G) = 2(n-1)^2$.

Proof. As the detour distance between any two disjoint vertices is $n-1$,it follows that DD(G)=(n-1)(J-I), where J is the all one square matrix whose order is same as that of the detour matrix of G. Hence the DD- spectrum of G is $\begin{bmatrix} (n-1)^2 & -(n-1) \\ 1 & n-1 \end{bmatrix}$ and consequently $E_{DD}(G) = 2(n-1)^2$.

Theorem 2.2.
$Spec_{DD}(K_{n,n}) = \begin{bmatrix} 4n^2-5n+2 & -(2n-1) & C \\ 1 & 2n-2 & 1 \end{bmatrix}$ where $C = -(3n-2)$

Proof. The Theorem follows from the fact that $DD(K_{n,n})$ = (2n - 2) (J -I) +A, where A is the adjacency matrix of $K_{n,n}$ and from Result 4.

Corollary 2.3.$E_{DD}(K_{n,n}) = 8n^2 - 10n + 4$.

Theorem 2.4. If G is a cycle of length n, then the largest eigenvalue of detour matrix of C_n is $\frac{3n^2-4n+1}{4}$if n is odd and $\frac{3n^2-4n}{4}$if n is even.

Proof. Let n be odd, the entries in the first row are $0, (n-1), (n-2), ..., \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}+1, ..., (n-2), (n-1)$whose sum is $2\left(\frac{n+1}{2} + \frac{n+1}{2} + 1 + ... + (n-2) + (n-1)\right)$ which is equal to $\frac{3n^2-4n+1}{4}$. Let n be even, The entries in the first row are $0, (n-1), (n-2), ..., \frac{n}{2}+1, \frac{n}{2}, \frac{n}{2}+1, ..., (n-2), (n-1)$ whose sum is$\frac{3n^2-4n}{4}$. All other entries of other rows are formed cyclically, and hence the result.

III. The Detour Spectrum of Double Graph of Some Graphs

Theorem 3.1. If G is the complete graph of order n with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\}$then $spec_{DD}(DD_2(G)) =$

$$\begin{bmatrix} 4n^2-4n+1 & -(2n-1) & -(2n-1) \\ 1 & n-1 & n \end{bmatrix}$$

Proof. The detour matrix $DD_2(G)$ of is of the form

$$\begin{bmatrix} DD+n(J-I) & DD+nJ+(n-1)I \\ DD+nJ+(n-1)I & DD+n(J-I) \end{bmatrix}$$

Using Result 2,we get $spec_{DD}(DD_2(G)) =$

$$\begin{bmatrix} 2(k+n^2)-1 & 2\mu_i-1 & -(2n-1) \\ 1 & 1 & n \end{bmatrix}, \quad i = 2, ..., n$$

and the theorem follows from Theorem 2.1.

Corollary 3.2. $E_{DD}(D_2(K_n)) = 2(4n^2-4n+1)$.

Theorem 3.3. If G is $K_{n,n}$ with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_{2n}\}$,then $spec_{DD}(DD_2(G)) =$

$$\begin{bmatrix} 2(8n^2-5n+1) & -2(3n-1) & -(2n-1) \\ 1 & 1 & 4n-2 \end{bmatrix}$$

Proof. The detour matrix of $DD_2(G)$ is of the form $\begin{bmatrix} DD+2n(J-I) & DD+2nJ+(2n-2)I \\ DD+2nJ+(2n-2)I & DD+2n(J-I) \end{bmatrix}$

Using Result 2,
we get $spec_{DD}(DD_2(G)) =$

$$\begin{bmatrix} 2(k+4n^2-1) & 2(\mu_i-1) & -2(2n-1) \\ 1 & 1 & 2n \end{bmatrix}, i = 2, ..., 2n$$

The theorem now follows from Theorem 2.2.

Corollary 3.4. $E_{DD}(D_2(K_{n,n})) = 8n^2 - 10n + 4$.

Theorem 3.5. If G is a cycle of length n with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\}$,then $spec_{DD}(DD_2(G)) =$
$\begin{bmatrix} 2(k+n^2-1) & 2\mu_i-2 & -2(n-1) \\ 1 & 1 & n \end{bmatrix}, i = 2, ..., n.$

Proof.The Theorem follows from the fact that the detour matrix of $DD_2(G)$ has of the form $\begin{bmatrix} DD+n(J-I) & DD+nJ+(n-2)I \\ DD+nJ+(n-2)I & DD+n(J-I) \end{bmatrix}$ and from Result 2.

IV. The Detour Spectrum of the Corona of G and K_1

Theorem 4.1. Let G be a connected detour regular graph with detour regularity k. If $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\}$,then $spec_{DD}(G \circ K_1)$ consists of numbers $n+k-1+\sqrt{(n+k)^2+(n-1)^2}$,$n+k-1-\sqrt{(n+k)^2+(n-1)^2}$, $\mu_i - 1 + \sqrt{\mu_i^2+1}$,$\mu_i - 1 - \sqrt{\mu_i^2+1}$, i = 2,3,...,n .

Proof. In the corona $G \circ K_1$, we observe that new pendent vertices do not contribute any additional length in finding either the shortest path (or) the longest path of any two vertices $G \circ K_1$. Hence the detour matrix of the corona $G \circ K_1$is same as its distance matrix. Hence we get the required result from [7].

V. The Detour Spectrum of Cartesian Product of Some Graphs with K_2

Theorem 5.1. If G is the complete graph of order n with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\}$,then $spec_{DD}(G \times K_2) =$

$$\begin{bmatrix} 4n^2-4n+1 & -(2n-1) & -(2n-1) \\ 1 & n-1 & n \end{bmatrix}$$

Proof.The detour matrix of $G \times K_2$ is of the form$\begin{bmatrix} DD+n(J-I) & DD+nJ+(n-1)I \\ DD+nJ+(n-1)I & DD+n(J-I) \end{bmatrix}$
Applying Result 2, we get $spec_{DD}(G \times K_2) =$
$\begin{bmatrix} 2(k+n^2)-1 & 2\mu_i-1 & -(2n-1) \\ 1 & 1 & n \end{bmatrix}, \quad i = 2, ..., n$
and applying Theorem 2.1, we get the result.

Corollary 5.2. $E_{DD}(G \times K_2) = (8n^2 - 8n + 2)$.

Theorem 5.3. If G is $K_{n,n}$ with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_{2n}\}$, then $spec_{DD}(G \times K_2) =$

$$\begin{bmatrix} 2(8n^2-5n+1) & -2(3n-1) & -2(2n-1) \\ 1 & 1 & 4n-2 \end{bmatrix}$$

Proof. The detour matrix of $G \times K_2$ is of the form$\begin{bmatrix} DD+2n(J-I) & DD+2nJ+(2n-2)I \\ DD+2nJ+(2n-2)I & DD+2n(J-I) \end{bmatrix}$and by Result 2, $spec_{DD}(G \times K_2) =$
$\begin{bmatrix} 2(k+4n^2-1) & 2(\mu_i-1) & -2(n-1) \\ 1 & 1 & 2n \end{bmatrix}, i = 2, ..., 2n.$

The theorem is then immediate from Theorem 2.2.

Corollary 5.4. $E_{DD}(G \times K_2) = 8n^2 - 10n + 4)$.

Theorem 5.5. If G is the cycle of length n with detour spectrum $\{\mu_1 = r_1, \mu_2, ..., \mu_n\}$ and ordinary spectrum $\{\lambda_1 = r_1^1, \lambda_2, ..., \lambda_n\}$,then $spec_{DD}(C_n \times K_2) = \begin{bmatrix} 2r_1 + (2n+1)n - 2 - 2r_1^1 & 2\mu_i - 2 - 2\lambda_i & A & B \\ 1 & 1 & 1 & 1 \end{bmatrix}$ where $A = 2 - 3n + 2r_1^1$ and $B = 2(1-n) + 2\lambda_i$, $i = 2, 3, ..., n$.

Proof.The Theorem follows from the fact that the detour matrix of $DD(C_n \times K_2)$ has of the form $\begin{bmatrix} DD + n(J-I) & DD + (n-1)J + nI + 2\overline{A} \\ DD + (n-1)J + nI + 2\overline{A} & DD + n(J-I) \end{bmatrix}$ where $\overline{A} = J - I - A$ and from Result 2.

VI. The detour spectrum of Lexicographic product of some graphs with K_2

Theorem 6.1. If G is the complete graph of order n with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\}$ then $spec_{DD}(G[K_2]) =$

$$\begin{bmatrix} 4n^2 - 4n + 1 & -(2n-1) & -(2n-1) \\ 1 & n-1 & n \end{bmatrix}$$

Proof.The detour matrix of $G[K_2]$ is of the form$\begin{bmatrix} DD + n(J-I) & DD + nJ + (n-1)I \\ DD + nJ + (n-1)I & DD + n(J-I) \end{bmatrix}$

Using Result 2, we get $spec_{DD}(G[K_2]) = \begin{bmatrix} 2(k+n^2) - 1 & 2\mu_i - 1 & -(2n-1) \\ 1 & 1 & n \end{bmatrix}$, $i = 2, ..., n$

The theorem follows from Theorem 2.1.

Corollary 6.2.$E_{DD}(G[K_2]) = (8n^2 - 8n + 2)$.

Theorem 6.3. If G is $K_{n,n}$ with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_{2n}\}$, then $spec_{DD}(G[K_2]) =$

$$\begin{bmatrix} 16n^2 - 8n + 1 & -(4n-1) \\ 1 & 4n-1 \end{bmatrix}$$

Proof. Since the detour matrix of $G[K_2]$ is same as the detour matrix of K_{4n}, the spectrum of $G[K_2]$ is same as spectrum of K_{4n}. Now the theorem follows from Theorem 2.1.

Corollary 6.4. $E_{DD}(G[K_2]) = 32n^2 - 16n + 2$.

Theorem 6.5. If G is a cycle of length n with detour spectrum $spec_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\}$ then $spec_{DD}(G[K_2]) =$

$$\begin{bmatrix} 4n^2 - 4n + 1 & -(2n-1) \\ 1 & 4n-1 \end{bmatrix}$$

Proof. Since the detour matrix of $G[K_2]$is same as the detour matrix of K_{2n}, the spectrum of $G[K_2]$is same as spectrum of K_{2n}. Now the theorem follows from Theorem 2.1.

Corollary 6.6. $E_{DD}(G[K_2]) = 8n^2 - 8n + 2$.

VII. The extended double cover of regular graphs

In [1], N. Alon introduced the concept of extended double cover graph of a graph as follows: Let G be a graph on the vertex set$\{v_1, v_2, ..., v_n\}$. Define a bipartite graph H with $V(H) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$in which $v_i$is adjacent to u_i for each $i = 1, 2, ..., n$ and v_i is adjacent to $v_j$in G. The graph H is known as the extended double cover graph (EDC - graph) of G. The ordinary spectrum of H has been determined in [3] and the distance spectrum of EDC of a regular graph of diameter 2 has been determined in [7]. In this section we obtain the detour spectrum of the EDC - graph of a r- regular graph on n vertices.

Theorem 7.1. Let G be a r- regular graph on n vertices. Then the DD- spectrum of the EDC- graph of G is given by $\begin{bmatrix} 4n^2 - 5n + 2 & 2 - 3n & 2 - 2n \\ 1 & 1 & 2n-2 \end{bmatrix}$

Proof. For any regular graph G with vertices $\{v_1, v_2, ..., v_n\}$ and $\{u_1, u_2, ..., u_n\}$ the new vertices to form the extended double cover H, the detour path from v_i to u_j is $2n-1$ and the detour path from v_i to v_j or u_i to u_j is $2n-2$ for all i and j. This shows that $DD(H) = \begin{bmatrix} (2n-2)(J-I) & (2n-1)J \\ (2n-1)J & (2n-2)(J-I) \end{bmatrix}$. Using Result 2 we get the required result.

Remark 1. In their paper [7], Indulal et al have shown that the D- spectrum of the EDC- graph of any r - regular graph G of diameter 2 depends on r and the spectrum of G. However, this is not the case for detour spectrum of such graphs. Surprisingly the detour spectrum of EDC- graphs of any r- regular graph are free of r and the DD- spectrum of the original graph. For example, the detour spectrum of EDC- graphs of C_n , K_n and circulant graphs are $\begin{bmatrix} 4n^2 - 5n + 2 & 2 - 3n & 2 - 2n \\ 1 & 1 & 2n-2 \end{bmatrix}$,whereas for detour spectrum of $EDC(C_n \nabla C_n)$is $DD(EDC(C_n \nabla C_n)) = \begin{bmatrix} 8n^2 - 2n & 0 & 2(4n^2 - 5n + 1) & -(8n-2) \\ 1 & 2n-1 & 1 & 2n-1 \end{bmatrix}$, since $DD(EDC(C_n \nabla C_n)) = \begin{bmatrix} (4n-1)(J-I) & (4n-1)J \\ (4n-1)J & (4n-1)(J-I) \end{bmatrix}$

Corollary 7.2. $E_{DD}(EDC(C_n \nabla C_n)) = 24n^2 - 14n + 2$, in particular, $E_{DD}(EDC(C_3 \nabla C_3)) = 176$.

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