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Two-dimensional solitary wave solution to the quadratic nonlinear Schrödinger equation

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Abstract—The solitary wave solution of the quadratic nonlinear Schrdinger equation is determined by the iterative method called Petviashvili method. This solution is also used for the initial condition for the time evolution to study the stability analysis. The spectral method is applied for the time evolution.

Keywords—soliton, iterative method, spectral method, plasma

I. Introduction

Solitary waves are well known as the solution of the nonlinear wave equations. These kind waves retain their shape as a result of the balance between nonlinearity and dispersion. In one-dimensional case, there are two major types of the nonlinear wave equations for produce the solitary wave, one is the Korteweg-de Vries (KdV) equation,

$$n_t + nn_x + n_{xxx} = 0,$$

in which the subscripts x and t denote differentiation with respect to space and time, respectively, which originally derived for water, (see e.g. [1] for further details). For plasmas, this governs weakly nonlinear ion-acoustic waves in plasma when the electrons have a Maxwellian distribution [2]. In that case n is the electrostatic potential. Another type is the cubic nonlinear Schrödinger equation (cNLS),

$$\mathrm{i}\phi_t + \phi_{xx} + |\phi|^2\phi = 0,$$

which is the most famous equation for solitons travel along fiber optics [3], [4]. A connection between the KdV and cNLS equations can be made within the context of Madelung's fluid [5]. This means that one can determine the solitary wave solution of the cNLS by solving the KdV equation. On the other hands, it is the transformation from cNLS to KdV by using this relationship,

$$\phi(x,t) = \sqrt{n(x,t)}\mathrm{e}^{\mathrm{i}\Theta(x,t)},$$

where $n = |\phi|^2$ is the Madelung's fluid density. We are interested in the quadratic nonlinear Schrödinger equation (qNLS),

$$\mathrm{i}\phi_t + \phi_{xx} + |\phi|\phi = 0,$$

because after we applied the Madelung's fluid, we then have the Schamel equation [6], [7],

$$n_t + n^{1/2}n_x + n_{xxx} = 0.$$

This equation also governs weakly nonlinear ion-acoustic wave in plasma but some of electrons are trapped on ion-acoustic waves, while others have not. The solitary wave solution for 1-D case is not too difficult to obtain, which is

$$\phi(x - 2V_0t) = 6\eta^2 \operatorname{sech}^2 \eta(x - 2V_0t)\mathrm{e}^{\mathrm{i}\left(xV_0 - 2\left(V_0^2 - 2\eta^2\right)t\right)}.$$

where V_0 denotes a wave speed and η is a constant. The Madelung's method can also apply to higher dimensional nonlinear Schrödinger equation but some KdV-like equations are not integrable. To determine solitary wave solutions for the higher dimension, one can choose one of numerical methods to solve the problem. In the next section we use Petviashvili method [8] applied to 2D qNLS,

$$\mathrm{i}\phi_t + \phi_{xx} + \phi_{yy} + |\phi|\phi = 0. \tag{1}$$

for the solitary wave solution. We next apply the spectral method to study time evolution of the solution. We then go to the conclusion for this work.

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II. Petviashvili method

This method has been used for obtaining a ground state solution of the nonlinear Schrödinger equation with the power-law potential [8] where Pelinovsky and Stepanyants [9] found the convergence conditions for homogenous equations. To determine the solution, one considers

$$\phi(x, y, t) = u(x, y)\mathrm{e}^{\mathrm{i}\mu t} \tag{2}$$

where $u(x, y) > 0$ and μ is the propagation constant. After substitute (2) into (1), we then get

$$-\mu u + u_{xx} + u_{yy} + u^2 = 0.$$

We introduce a new variable,

$$Mu = (\mu - \partial_{xx} - \partial_{yy})u = u^2. \tag{3}$$

For determining the steady solution, we have to calculate $u(x, y)$ from (3) by iteration, namely,

$$u_{i+1} = M^{-1}u_i^2, \tag{4}$$

where i denotes a number of iteration from zero to any integer number. The result of (4) gives zero or infinity. The key idea of Petviashvili method is to find the stabilized factor which maintains the result of the iteration without diverge to infinity or to zero. Then,

$$u_{i+1} = S_i^{\gamma} M^{-1}u_i^2,$$

in which γ is a constant and this stabilizing factor can be obtained by inner product

$$S_i = \frac{< u_i, Mu_i >}{< u_i, u_i^2 >}$$

To calculate γ, we assume that the exact solution $u(x, y) = O(1)$. After some iterations, the solution becomes $u_i(x, y) = O(\epsilon)$, where $\epsilon \leq 1$ or $\epsilon \geq 1$. The order of magnitude of the stabilizing factor can be represented as

$$S_i = O(\epsilon^{-1})$$

The order of magnitude of $u(x, y)$ becomes

$$u_{i+1} = O(\epsilon^{-\gamma+2}) \tag{5}$$

This means that $\gamma = 2$. We next first try the Gaussian function as the initial condition

$$u_0(x, y) = 5\mathrm{e}^{-(x^2+y^2)}$$

in which $L_x = L_y = [-5, 5]$ and $N_x = N_y = 128$. We choose $\mu = 2$ for this examination. Fig. 1 shows how the initial condition approaches to the steady state solution for each step of iteration. Fig. 1(f) presents the solitary wave solution of the qNLS. Fig. 2 shows the error for each step of iteration, at

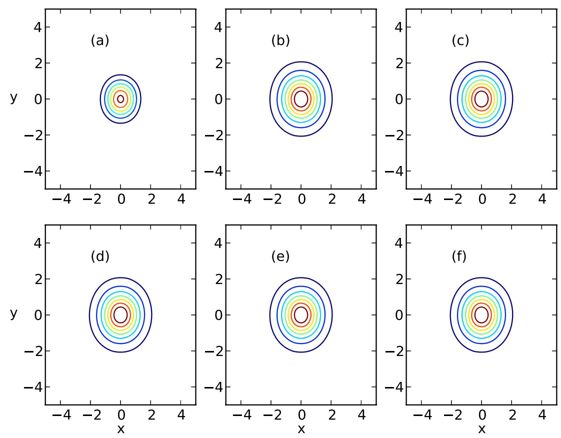

Fig. 1. results for each iterations : (a) $i = 0$, (b) $i = 10$, (c) $i = 20$, (d) $i = 30$, (e) $i = 35$, (f) $i = 56$.

$i = 56$, the error is around $9.9 * 10^{-11}$. We now change the initial condition to

$$u_0(x, y) = 5\,\mathrm{sech}^2(\sqrt{x^2 + y^2})\cos(x)\sin(y)$$

This profile is shown in Fig. 3 and the result from the Petviashvili method is shown in Fig. 4. The error again shows in Fig. 5. To check the stability of u_{i+1}, we next apply the spectral method for the time evolution of this solitary wave solution.

III. Evolution of Solitary Waves

To study the evolution of the perturbed soliton, we express the qNLS equation in the form

$$\frac{d\phi}{dt} = -\mathrm{i}\left(F^{-1}\left(\left(\xi^2 + \chi^2\right)F(\phi)\right) - |\phi|\phi\right)$$

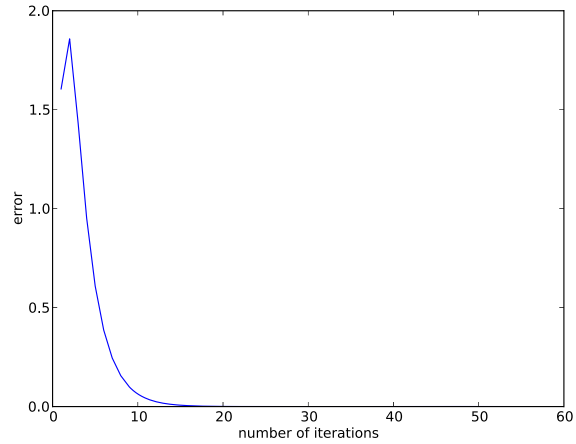

Fig. 2. shows error for each iteration

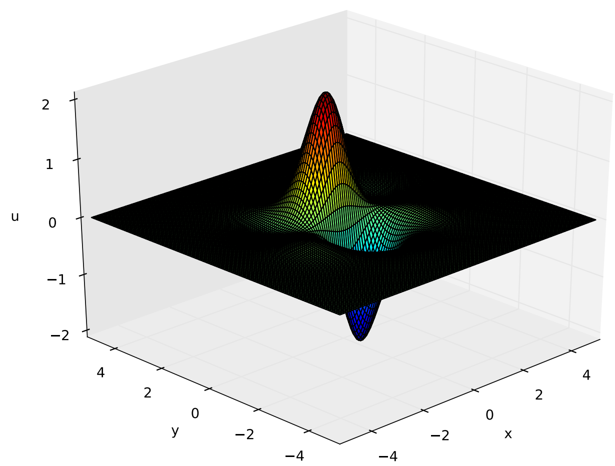

Fig. 3. the initial profile of $5\,\mathrm{sech}^2(\sqrt{x^2 + y^2})\cos(x)\sin(y)$

where F and F^{-1} denote the Fourier and inverse Fourier transforms, respectively. These are obtained in the numerical scheme using the discrete Fourier transform

$$[F(\phi)]_{p,q} = \sum_{l=0}^{N_x-1}\sum_{m=0}^{N_y-1}\phi_{l,m}e^{i(\xi_p x_l + \chi_q y_m)}$$

where N_x and N_y are the number of mesh points in the x and y directions, $(x_l, y_m) = (lL_x/N_x, mL_y/N_y)$, L_x and L_y are the lengths of the domain in the x and y directions, and $(\xi_p, \chi_q) = 2\pi(p/L_x, q/L_y)$ for $p = 0, \ldots, N_x - 1$ and $q = 0, \ldots, N_y - 1$. The Runge-Kutta method [10] was used for the time derivative. We next used the final step of iteration as the initial condition. If we have a truly solitary wave solution, we will see nothing for the time evolution because of the steady state solution. The result is shown in Fig 6.

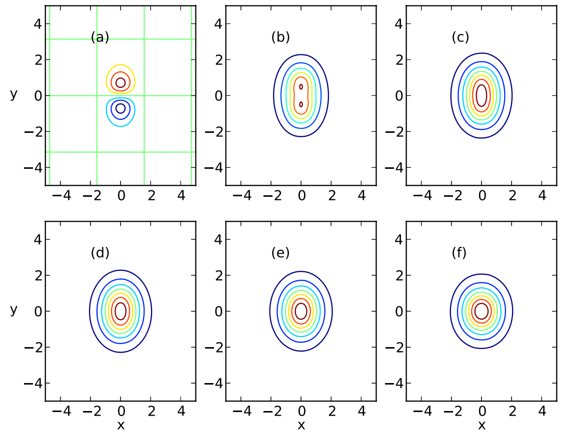

Fig. 4. results for each iterations : (a) $i = 0$, (b) $i = 1$, (c) $i = 2$, (d) $i = 3$, (e) $i = 4$, (f) $i = 58$.

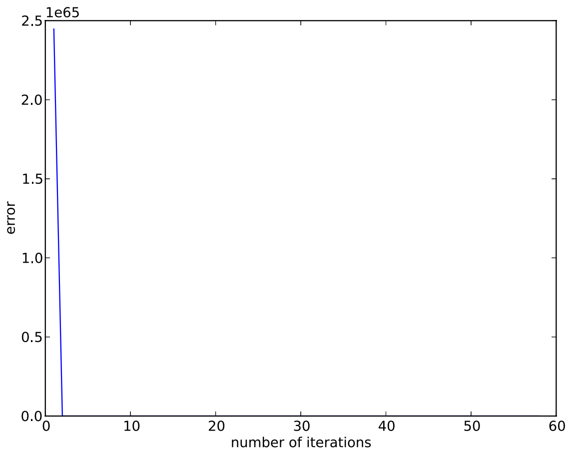

Fig. 5. shows error for each iteration

IV. Conclusions

The solitary wave solution of the qNLS can be determined by the Petviashvili method. The time evolution is shown that the results are correct. This is the one example of the application of the computational methods to calculate the solitary wave solution of the nonlinear wave equations. This is also a good start to study some phenomena of the vortex solitons or excited states of others nonlinear wave equations.

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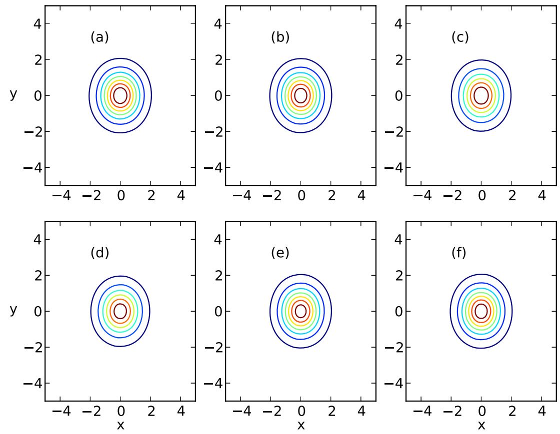

Fig. 6. time evolution of the solitary wave solution: (a) $t = 0$, (b) $t = 2$, (c) $t = 4$, (d) $t = 6$, (e) $t = 8$, (f) $t = 10$

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