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Periodic solutions of recurrent neural networks with distributed delays and impulses on time scales

Yaping Ren and Yongkun Li

Abstract—In this paper, by using the continuation theorem of coincidence degree theory, $M-$matrix theory and constructing some suitable Lyapunov functions, some sufficient conditions are obtained for the existence and global exponential stability of periodic solutions of recurrent neural networks with distributed delays and impulses on time scales. Without assuming the boundedness of the activation functions g_j, h_j, these results are less restrictive than those given in the earlier references.

Keywords—Recurrent neural networks; Global exponential stability; Periodic solutions; Distributed delays; Impulses; Time scales.

I. Introduction

It is well known that the recurrent neural networks are very general and the architectures of recurrent neural networks can take many different forms, such as Hopfield neural networks, cellular neural networks and BAM neural networks. The fundamental feature of a recurrent neural network is that the network contains at least one feed-back connection, so activation can flow around in a loop. The networks are able to do temporal processing and learn sequences (i.e. perform sequence recognition, sequence reproduction, and temporal association/prediction). The recurrent neural networks have been successfully applied to signal and image processing, pattern recognition and optimization. Hence, the problem of the stability of the recurrent neural networks have been intensive studied by numerous authors in recent years, see [1-10] and the references cited therein.

However, most authors assumed the boundedness of the activation functions. For examples, in Ref.[1] the authors studied the global robust stability of the following delayed recurrent neural network:

$$\begin{cases} \dot{x}_i(t) = -c_i(t)x_i(t) + \sum\limits_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) \\ \qquad\qquad + \sum\limits_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau)) + u_i,\ i = 1, 2, \dots, n, \end{cases}$$

where the activation functions satisfy the following assumption:

(A_1) $f_i(x_i)(i = 1, 2, \dots, n)$ is bounded and monotonically nondecreasing on $\mathbb{R}$.

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In Ref.[6], the authors got some new delay-dependent criterion for the stability of recurrent neural networks with time-varying delay as following

$$\begin{cases} \dot{u}_i(t) = -a_iu_i(t) + \sum\limits_{j=1}^{n} w_{ij}(t)\bar{g}_j(u_j(t)) \\ \qquad\qquad + \sum\limits_{j=1}^{n} w_{ij}^1(t)\bar{f}_j(u_j(t-\tau(t))) + U_i, i = 1, 2, \dots, n, \end{cases}$$

where $\bar{g}_j(u_j(t))$ and $\bar{f}_j(u_j(t))$ are the bounded activation functions.

To the best of our knowledge, few results are available on the existence and exponential stability of periodic solutions for the recurrent neural networks with impulses and without assuming the boundedness of the activation functions, while the existence of periodic solutions plays an important role in characterizing the behavior of nonlinear differential equations. In [11], Li and Lu used the continuation theorem of coincidence degree theory and Lyapunov functions to study the existence and global exponential stability of periodic solutions for the following neural network with impulses:

$$\begin{cases} \dot{x}_i(t) = -a_i(t)x_i(t) + \sum\limits_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) \\ \qquad\qquad +J_i(t),\ t > 0, t \neq t_k, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = -\gamma_{ik}x_i(t_k), \\ \qquad\qquad i = 1, 2, \dots, m,\ k = 1, 2, \dots, \end{cases}$$

where the activation functions are assumed to have the following property:

(H_3) There exist positive constant $M_i > 0$ such that $|f_i(x)| \leq M_i$ for $i = 1, 2, \dots, m$, $x \in \mathbb{R}$.

In this paper, these restrictions on the activation functions are removed.

In fact, both continuous and discrete systems are very important in implementing and applications. It is well known that the theory of time scales has received a lot of attention which was introduced by Stefan Hilger in order to unify continuous and discrete analysis. Therefore, it is meaningful to study dynamic system on time scales which can unify the differential and difference system, see [12-23] and references therein.

In this paper we apply the continuation theorem of coincidence degree theory and constructing some suitable Lyapunov functions to study the existence and global exponential stability of periodic solutions solutions of recurrent neural networks with distributed delays and impulses on time scales, without assuming the boundedness of the activation functions g_j, h_j, in the form of

$$\begin{cases} x_i^{\Delta}(t) &= -a_i(t)x_i(t) + \sum\limits_{j=1}^{n} d_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ & \quad + \sum\limits_{j=1}^{n} b_{ij}(t)\int_0^{\infty} k_{ij}(\theta)h_j(x_j(t-\theta))\Delta\theta \\ & \quad + r_i(t), i=1,2,\ldots,n,\ t\in\mathbb{T}^+,\ t\neq t_k, \\ \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k)), \\ & \quad i=1,2,\ldots,n, k\in\mathbb{N} \end{cases} \tag{1}$$

with the initial values

$$x_i(s) = \phi_i(s),\ s\in(-\infty,0]_{\mathbb{T}},$$

where $\mathbb{T}$ is an ω-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$, for each interval L of $\mathbb{R}$ we denote by $L_{\mathbb{T}} = L\cap\mathbb{T}$, $L^+ = L\cap[0,\infty)$, n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the i-th unit at the time t, $a_i(t)$ represents the amplification function, $I_i(t)$ denotes the external input at time t, $\tau_{ij}(t)\geq 0$ is the transmission delay and is an ω-periodic function, g_i and h_i are the activation functions which describe the manner in which the neurons respond to each other, $D = (d_{ij}(t))$ is the delayed feedback matrix which represents the strengthen of the neuron interconnection within the network with bounded delay parameter $\tau_{ij}(t)$, $t-\tau_{ij}(t)\in(-\infty,0]_{\mathbb{T}}$ and $t-\theta\in[0,\infty)_{\mathbb{T}}$, k_{ij} is the delayed feedback matrix, $r_i(t)$ denotes the external input at time t, $\phi_i(\cdot)$ denotes continuous function defined on $(-\infty,0]_{\mathbb{T}}$, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $x_i(t_k^+)$ and $x_i(t_k^-)(i=1,2,\ldots,n)$ represent the right and left limit of $x_i(t_k)$ in the sense of time scales, $\{t_l\}$ is a sequence of real numbers such that $0<t_1<t_2<\ldots<t_l\to\infty$ as $l\to\infty$. There exists a positive integer q such that $t_{l+q} = t_l+\omega$, $I_{i(k+q)} = -I_{ik}$, $l\in\mathbb{Z}$, $i=1,2,\ldots,n$. Without lossing generality, we also assume that $[0,\omega)_{\mathbb{T}}\cap\{t_l : l\in\mathbb{Z}\} = \{t_1,t_2,\ldots,t_q\}$.

Throughout this paper, we assume that:

(H_1) a_i, b_{ij}, d_{ij}, $r_i\in C(\mathbb{T}^+,\mathbb{R})$ are ω-periodic functions and there exist positive numbers $\underline{a}_i$, $\bar{a}_i$, such that $\underline{a}_i\leq a_i(\cdot)\leq \bar{a}_i$, $i,j,=1,\ldots,n$.

(H_2) The activation functions $g=(g_1,\ldots,g_n)^T\in C(\mathbb{R},\mathbb{R})$ and $h=(h_1,\ldots,h_n)^T\in C(\mathbb{R},\mathbb{R})$ are Lipschitz functions, that is, there exist positive numbers α_i, β_i such that $|h_i(x)-h_i(y)|\leq\alpha_i|x-y|$, $|g_i(x)-g_i(y)|\leq\beta_i|x-y|$, $x,y\in\mathbb{R}$, $i=1,\ldots,n$.

(H_3) For $i,j=1,\ldots,n$, the delay kernels $k_{ij}:\mathbb{T}^+\to\mathbb{R}$ are continuous and the integral $\int_0^{\infty}k_{ij}(\theta)\Delta\theta\leq K$.

(H_4) $I_{ik}\in C(\mathbb{R},\mathbb{R})$ and there exist positive numbers I_{ik}^M such that $|I_{ik}(x)-I_{ik}(y)|\leq I_{ik}^M|x-y|$, $x,y\in\mathbb{R}$, $i=1,2,\ldots,n$, $k\in\mathbb{N}$.

(H_5) For $i=1,2,\ldots,n, k\in\mathbb{N}$, the impulsive operators $I_{ik}(x_i(t))$ satisfy

$$I_{ik}(x_i(t_k)) = -\gamma_{ik}x_i(t_k),\ 0\leq\gamma_{ik}\leq 2.$$

For the sake of convenience, we denote

$$\bar{u}_i = \max_{t\in[0,\omega]_{\mathbb{T}}}|u_i(t)|,\quad \|u\|_2 = \left(\int_0^{\omega}|u(t)|^2\Delta t\right)^{\frac{1}{2}},$$

where u is an ω-periodic function, and

$$||x_i||_0 = \max_{t\in[0,\omega]_{\mathbb{T}}}|x_i(t)|.$$

And we can choose constant $\bar{r}$, and τ such that

$$\bar{r} = \sup_{t\in\mathbb{T}^+}\max_{1\leq i\leq n}|r_i(t)|,\quad \tau = \sup_{t\in\mathbb{T}^+}\max_{1\leq i,j\leq n}\{\tau_{ij}(t)\}.$$

The paper is organized as follows: In Section 2, we make some preparations. In Section 3, by using the continuation theorem of coincidence degree theory, we obtain the existence of periodic solutions of (1) without assuming the boundedness of the activation functions g_j, h_j, these results are less restrictive than those given in the earlier references. In Section 4, by constructing some suitable Lyapunov functions, we study the global exponential stability of the periodic solutions of (1). In Section 5, an example is provided to demonstrate the results obtained in the previous sections. The conclusions are drawn in Section 6.

II. Preliminaries

In this section, we will cite some definitions and lemmas which will be used in the proofs of our main results.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ and the graininess $\mu:\mathbb{T}\to\mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s\in\mathbb{T}: s>t\},\quad \rho(t) = \sup\{s\in\mathbb{T}: s<t\}$$

and

$$\mu(t) = \sigma(t)-t.$$

A point $t\in\mathbb{T}$ is called left-dense if $t>\inf\mathbb{T}$ and $\rho(t)=t$, left-scattered if $\rho(t)<t$, right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, and right-scattered if $\sigma(t)>t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Let $\omega\in\mathbb{R}$, $\omega>0$, $\mathbb{T}$ is an ω-periodic time scale if $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ such that $t+\omega\in\mathbb{T}$ and $\mu(t) = \mu(t+\omega)$ whenever $t\in\mathbb{T}$.

A function $f:\mathbb{T}\to\mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f:\mathbb{T}\to\mathbb{R}$ will be denoted by $C(\mathbb{T})$.

For $y:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon>0$, there exists a neighborhood U of t such that

$$\left|[y(\sigma(t))-y(s)]-y^{\Delta}(t)[\sigma(t)-s]\right| < \varepsilon|\sigma(t)-s|,$$

for all $s\in U$.

If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous. If $y^{\Delta}(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t)-Y(a).$$

Definition 1. *Function f=$(f_1,\dots,f_n)$ is a lipschitz if it satisfies $|f_i(x)-f_i(y)|\le l_i|x-y|$, $i=1,\dots,n$ for any $x,y\in\mathbb{T}$.*

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Definition 2. *[22] If $a\in\mathbb{T},\sup\mathbb{T}=\infty$, and f is rd-continuous on $[a,\infty)$, then we define the improper integral by*

$$\int_a^\infty f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t,$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Definition 3. *[23] For each $t\in\mathbb{T}$, let N be a neighborhood of t, then, for $V\in C_{rd}[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}^+]$, define $D^+V^\Delta(t,x(t))$ to mean that, given $\varepsilon>0$, there exists a right neighborhood $N_\varepsilon\subset N$ of t such that*

$$\frac{[V(\sigma(t),x(\sigma(t)))-V(s,x(\sigma(t)))-\mu(t,s)f(t,x(t))]}{\mu(t,s)}$$
$$<D^+V^\Delta(t,x(t))+\varepsilon,$$

for each $s\in N_\varepsilon$, $s>t$, where $\mu(t,s)\equiv\sigma(t)-s$. If t is rd and $V(t,x(t))$ is continuous at t, this reduce to

$$D^+V^\Delta(t,x(t))=\frac{V(\sigma(t),x(\sigma(t)))-V(t,x(\sigma(t)))}{\sigma(t)-t}.$$

Definition 4. *[24] We say that a time scale $\mathbb{T}$ is a periodic if there exists $p>0$, such that if $t\in\mathbb{T}$ then $t\pm p\in\mathbb{T}$. For $\mathbb{T}\neq\mathbb{R}$, the smallest positive p is called the period of the time scale.*

A function $r:\mathbb{T}\to R$ is called regressive if

$$1+\mu(t)r(t)\neq 0,$$

for all $t\in\mathbb{T}^k$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s)=\exp\left\{\int_s^t\xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\},\ \text{ for } s,t\in\mathbb{T},$$

with the cylinder transformation

$$\xi_h(z)=\begin{cases}\dfrac{\operatorname{Log}(1+hz)}{h} & \text{if } h\neq 0,\\ z & \text{if } h=0.\end{cases}$$

Let $p,q:\mathbb{T}\to\mathbb{R}$ be two regressive functions, we define

$$p\oplus q:=p+q+\mu pq,\ \ p\ominus q:=p\oplus(\ominus q),\ \ \ominus p:=\frac{p}{1+\mu p}.$$

Then the generalized exponential function has the following properties.

Lemma 1. *[20] Assume that $p,\ q:\mathbb{T}\to\mathbb{R}$ are two regressive functions, then*

(i) $e_0(t,s)\equiv 1$ *and* $e_p(t,t)\equiv 1$;
(ii) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$;
(iii) $e_p(t,\sigma(s))=\frac{e_p(t,s)}{1+\mu(s)p(s)}$;
(iv) $\frac{1}{e_p(t,s)}=e_{\ominus p}(t,s)$;
(v) $e_p(t,s)=\frac{1}{e_p(s,t)}=e_{\ominus p}(s,t)$;
(vi) $e_p(t,s)e_p(s,r)=e_p(t,r)$;
(vii) $e_p(t,s)e_q(t,s)=e_{p\oplus q}(t,s)$;
$(viii)$ $\frac{e_p(t,s)}{e_q(t,s)}=e_{p\ominus q}(t,s)$.

Lemma 2. *[25] Let $t_1,t_2\in[0,\omega]_\mathbb{T}$. If $x:\mathbb{T}\to\mathbb{R}$ is ω-periodic, then*

$$x(t)\le x(t_1)+\int_0^\omega|x^\Delta(s)|\Delta s,\ x(t)\ge x(t_2)-\int_0^\omega|x^\Delta(s)|\Delta s.$$

Lemma 3. *[?] (Cauchy-Schwarz inequality on time scale) Let $a,b\in\mathbb{T}$. For rd-continuous functions $f,g:[a,b]\to\mathbb{R}$ we have*

$$\int_a^b|f(t)g(t)|\Delta t\le\left(\int_a^b|f(t)|^2\Delta t\right)^{\frac12}\left(\int_a^b|g(t)|^2\Delta t\right)^{\frac12}.$$

We know that if $\theta,\tau\in\mathbb{T}$ are constants, then $e_\theta(t,-\tau)$ is an ω-periodic function.

Definition 5. *Let $x^*(t)=(x_1^*(t),x_2^*(t),...,x_n^*(t))^T$ is a periodic solution of system (1) with initial value $\phi(t)=(\phi_1^*(t),\phi_2^*(t),...,\phi_n^*(t))^T$. If there exist constants $\lambda>0$ and $M>1$ such that for every solution $x(t)=(x_1(t),x_2(t),...,x_n(t))^T$ of system (1) with any initial value $\phi(t)=(\phi_1(t),\phi_2(t),...,\phi_n(t))^T\in C([-\infty,0]_\mathbb{T},\mathbb{R}^n)$,*

$$|x_i(t)-x_i^*(t)|\le M\|\phi-\phi^*\|e_{-\lambda}(t,\delta),t\in\mathbb{T}^+,\ i=1,2,\dots,n,$$

where

$$\|\phi-\phi^*\|=\sup_{s\in(-\infty,0]_\mathbb{T}}\max_{0\le i\le n}|\phi_i(s)-\phi_i^*(s)|,\quad\delta\in(-\infty,0]_\mathbb{T}.$$

Then $x^*(t)$ is said to be globally exponentially stable.

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 4. *[27] Let $\mathbb{X},\mathbb{Y}$ be two Banach spaces, $\Omega\subset\mathbb{X}$ be open bounded and symmetric with $0\in\Omega$. Suppose that $L:D(L)\subset\mathbb{X}\to\mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L)\cap\overline{\Omega}\neq\emptyset$ and $N:\overline{\Omega}:\to\mathbb{Y}$ is L-compact. Further, we assume that*

(H) *$Lx-Nx\neq\lambda(-Lx-N(-x))$ for all $x\in D(L)\cap\partial\Omega$, $\lambda\in(0,1]$.*

Then equation $Lx=Nx$ has at least one solution on $D(L)\cap\partial\Omega$.

Definition 6. *A real $n\times n$ matrix $A=(a_{ij})$ is said to be a non-singular M-matrix if $a_{ij}\le 0,i\neq j,\ i,j=1,\dots,n$ and all successive principal minors of A are positive.*

III. Existence of periodic solutions

In this section, by using Lemma 4, we will study the existence of at least one periodic solution of (1).

Theorem 1. *Assume that the assumptions $(H_1)-(H_5)$ are satisfied and*

(H_6) *$E=(e_{ij})_{n\times n}$ is a nonsingular M-matrix, where*

$$e_{ij}=\begin{cases}\Phi_i-\Psi_i,\ i=j,\\ \Psi_i,\ i\neq j,\end{cases}$$

$$\Phi_i = \Big[\underline{a}_i\omega - \sum_{k=1}^{q} I_{ik}^M - \underline{a}_i\omega^2\bar{a}_i - \underline{a}_i\omega\sum_{k=1}^{q} I_{ik}^M\Big],$$

$$\Psi_i = \Big\{\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega + \sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j + \underline{a}_i\omega\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega + \underline{a}_i\omega\sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j\Big\},$$

for $i, j = 1, 2, \dots, n$. *Then system* (1) *has at least one* ω*-periodic solution.*

Proof: Let $C^k[0,\omega; t_1, t_2, \dots, t_q]_{\mathbb{T}} = \{x : [0,\omega]_{\mathbb{T}} \to \mathbb{R}^n | x^k(t)|$ is a piecewise continuous map with first-class discontinuous points in $[0,\omega]_{\mathbb{T}} \cap \{t_l : l \in \mathbb{N}\}\}$ and at each discontinuous point it is continuous on the left$\}$, $k = 1, 2$. Take

$$\mathbb{X} = \{x \in C[0,\omega; t_1, t_2, \dots, t_q]_{\mathbb{T}} : x(t+\omega) = x(t), t \in [0,\omega]_{\mathbb{T}}\}$$

and

$$\mathbb{Y} = \mathbb{X} \times \mathbb{R}^{n\times(q+1)},$$

be two Banach spaces with the norms

$$\|x\|_{\mathbb{X}} = \sum_{i=1}^{n}\|x_i\|_0,\ \|z\|_{\mathbb{Y}} = \|x\|_{\mathbb{X}} + \|y\|,\ x \in \mathbb{X},\ y \in \mathbb{R}^{n\times q},$$

in which $\|x_i\|_0 = \max_{t\in[0,\omega]_{\mathbb{T}}} |x_i(t)|,\ i = 1, 2, \dots, n,\ ||.||$ is any norm of $\mathbb{R}^{n\times q}$. Set

$$L : \mathrm{Dom}\,L \cap \mathbb{X} \to \mathbb{Y},\ x \to (x^{\Delta}, \Delta x(t_1), \dots, \Delta x(t_q)),$$

where $\mathrm{Dom}\,L = \{x \in C^1[0,\omega; t_1, t_2, \dots, t_q]_{\mathbb{T}} : x(t+\omega) = x(t),\ t \in [0,\omega]_{\mathbb{T}}\}$, and $N : \mathbb{X} \to \mathbb{Y}$,

$$Nx = \left(\begin{pmatrix} A_1(t) \\ \vdots \\ A_n(t)\end{pmatrix}, \begin{pmatrix} \Delta x_1(t_1) \\ \vdots \\ \Delta x_n(t_1)\end{pmatrix}, \begin{pmatrix} \Delta x_1(t_2) \\ \vdots \\ \Delta x_n(t_2)\end{pmatrix}, \dots, \begin{pmatrix} \Delta x_1(t_q) \\ \vdots \\ \Delta x_n(t_q)\end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}\right),$$

where

$$A_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} d_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} k_{ij}(\theta)h_j(x_j(t-\theta))\Delta\theta + r_i(t),$$
$$i = 1, 2, \dots, n.$$

Obviously,

$$\mathrm{Ker}\,L = \{x \in \mathbb{X} | x = h \in R^n\},$$

$$\mathrm{Im}\,L = \Big\{z = (f, C_1, C_2, \dots, C_q, d) \in \mathbb{Y} \Big| \int_0^{\omega} f(s)\Delta s + \sum_{k=1}^{q} C_k + d = 0\Big\} = \mathbb{X} \times R^{n\times(q+1)} \times \{0\},$$

and then

$$\dim \mathrm{Ker}\,L = \mathrm{codim}\,\mathrm{Im}\,L = n.$$

So, $\mathrm{Im}\,L$ is closed in $\mathbb{Y}$, L is a Fredholm mapping of index zero. Define the continuous operators $P : \mathbb{X} \longrightarrow \mathrm{Ker}\ L$ and the averaging projector $Q : \mathbb{Y} \longrightarrow \mathbb{Y}$ as

$$Px = \frac{1}{\omega}\int_0^{\omega} x(t)\Delta t,\ x \in \mathbb{X},$$

$$Qz = Q(f, C_1, C_2, \dots, C_q, d) = \left(\frac{1}{\omega}\Big[\int_0^{\omega} f(s)\Delta s + \sum_{k=1}^{q} C_k + d\Big], 0, \dots, 0\right),\ z \in \mathbb{Y}.$$

It is not difficult to show that P and Q are continuous projectors and satisfy

$$\mathrm{Im}\ P = \mathrm{Ker}\,L,\ \mathrm{Im}\ L = \mathrm{Ker}\ Q = \mathrm{Im}\ (I - Q).$$

Further, let $L_P^{-1} : \mathrm{Im\,L} \longrightarrow \mathrm{Dom} \cap \mathrm{Ker\,P}$ the inverse of $L|_{\mathrm{Dom}\cap\mathrm{Ker\,P}}$, we have

$$(L_P^{-1})(t) = \int_0^t f(s)\Delta s + \sum_{t>t_k} C_k - \frac{1}{\omega}\int_0^{\omega}\int_0^t f(s)\Delta s\Delta t - \sum_{k=1}^{q} C_k,$$

for all $1 \le i \le q$. Thus, the expression of QNx is

$$\left(\begin{pmatrix} \frac{1}{\omega}\int_0^{\omega} A_1(t)\Delta t + \frac{1}{\omega}\sum_{k=1}^{q} I_{1k}(x_1(t_k)) \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega} A_n(t)\Delta t + \frac{1}{\omega}\sum_{k=1}^{q} I_{nk}(x_n(t_k))\end{pmatrix}, 0, \dots, 0\right),$$

and then

$$K_P(I-Q)Nx = \begin{pmatrix} \frac{1}{\omega}\int_0^{t} A_1(s)\Delta s + \frac{1}{\omega}\sum_{t>t_k} I_{1k}(x_1(t_k)) \\ \vdots \\ \frac{1}{\omega}\int_0^{t} A_n(s)\Delta s + \frac{1}{\omega}\sum_{t>t_k} I_{nk}(x_n(t_k))\end{pmatrix} - \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega}\int_0^{t} A_1(s)\Delta s\Delta t \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega}\int_0^{t} A_n(s)\Delta s\Delta t\end{pmatrix} - \begin{pmatrix} \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^{\omega}\int_0^{t} A_1(s)\Delta s\Delta t \\ \vdots \\ \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^{\omega}\int_0^{t} A_n(s)\Delta s\Delta t\end{pmatrix} - \begin{pmatrix} \sum_{k=1}^{q} I_{1k}(x_1(t_k)) \\ \vdots \\ \sum_{k=1}^{q} I_{nk}(x_n(t_k))\end{pmatrix}.$$

Clearly, QN and $K_P(I-Q)N$ are both continuous. Similar to [12], it is easy to show that $QN(\overline{\Omega})$, $K_P(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$.

Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 4, we need to find an appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x)), \lambda \in (0,1]$, we have

$$\begin{cases} x_i^{\Delta}(t) &= \dfrac{1}{1+\lambda}B_i(t,x) - \dfrac{\lambda}{1+\lambda}B_i(t,-x), \\ & t \in \mathbb{T}^+,\ t \neq t_k,\ i = 1,2,\ldots,n, \\ \Delta x_i(t_k) &= \dfrac{1}{1+\lambda}I_{ik}(x_i(t_k)) - \dfrac{\lambda}{1+\lambda}I_{ik}(-x_i(t_k)), \\ & i = 1,2,\ldots,n,\ k \in \mathbb{N}, \end{cases} \tag{2}$$

where

$$B_i(t,x) = -a_i(t)x_i(t) + \sum_{j=1}^{n} d_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} k_{ij}(\theta)h_j(x_j(t-\theta))\Delta\theta + r_i(t)$$

and

$$B_i(t,-x) = -a_i(t)(-x_i(t)) + \sum_{j=1}^{n} d_{ij}(t)g_j(-x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} k_{ij}(\theta)h_j(-x_j(t-\theta))\Delta\theta + r_i(t),$$

for $i = 1,2,\ldots,n$.

Suppose that $x = (x_1, x_2, \ldots, x_n)^T$ is a solution of system (2) for a certain $\lambda \in (0,1]$, set $t_0 = t_0^+ = 0$, $t_{q+1} = \omega$, we obtain

$$\begin{aligned}
&\int_0^{\omega} |x_i^{\Delta}(t)|\Delta t \\
&= \sum_{k=1}^{q+1}\int_{t_{k-1}^+}^{t_k} |x_i^{\Delta}(t)|\Delta t + \sum_{k=1}^{q} |I_{ik}(x_i(t_k))| \\
&\le \int_0^{\omega}\left|\frac{1}{1+\lambda}B_i(t,x) - \frac{\lambda}{1+\lambda}B_i(t,-x)\right|\Delta t \\
&\quad + \sum_{k=1}^{q}\left|\frac{1}{1+\lambda}I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda}I_{ik}(-x_i(t_k))\right| \\
&\le \left[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right]\int_0^{\omega}\max\{|B_i(t,x)|, |B_i(t,-x)|\}\Delta t \\
&\quad + \left[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right] \\
&\quad \times \sum_{k=1}^{q}\max\{|I_{ik}(x_i(t_k))|, |I_{ik}(-x_i(t_k))|\} \\
&\le \bar{a}_i\int_0^{\omega}|x_i(t)|\Delta t + \sum_{j=1}^{n}\int_0^{\omega}\bar{d}_{ij}|g_j(x_j(t-\tau_{ij}(t))) \\
&\quad - g_j(0)|\Delta t + \sum_{j=1}^{n}\bar{b}_{ij}\int_0^{\omega}\int_0^{\infty}k_{ij}(\theta)|h_j(x_j(t-\theta)) \\
&\quad - h_j(0)|\Delta\theta\Delta t + \bar{r}_i\omega + \max\left\{\sum_{k=1}^{q}|I_{ik}(x_i(t_k)) - I_{ik}(0)|,\right. \\
&\quad \left.\sum_{k=1}^{q}|I_{ik}(-x_i(t_k)) - I_{ik}(0)|\right\} + \sum_{k=1}^{q}|I_{ik}(0)| \\
&\quad + \sum_{j=1}^{n}\int_0^{\omega}\bar{d}_{ij}|g_j(0)|\Delta t \\
&\quad + \sum_{j=1}^{n}\bar{b}_{ij}\int_0^{\omega}\int_0^{\infty}k_{ij}(\theta)|h_j(0)|\Delta\theta\Delta t \\
&\le \bar{a}_i\omega^{\frac{1}{2}}\|x_i\|_2 + \omega^{\frac{1}{2}}\sum_{j=1}^{n}(\bar{d}_{ij}\beta_j)\|x_j\|_2 + \omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)| \\
&\quad + \sum_{j=1}^{n}\bar{b}_{ij}\int_0^{\infty}k_{ij}(\theta)\alpha_j\omega^{\frac{1}{2}}\|x_j\|_2\Delta\theta \\
&\quad + \sum_{j=1}^{n}\bar{b}_{ij}\omega\int_0^{\infty}k_{ij}(\theta)|h_j(0)|\Delta\theta \\
&\quad + \omega\bar{r}_i + \|x_i\|_0\sum_{k=1}^{q}I_{ik}^M + \sum_{k=1}^{q}|I_{ik}(0)| \\
&\le \bar{a}_i\omega^{\frac{1}{2}}\|x_i\|_2 + \omega^{\frac{1}{2}}\sum_{j=1}^{n}(\bar{d}_{ij}\beta_j)\|x_j\|_2 + \omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)| \\
&\quad + \sum_{j=1}^{n}\bar{b}_{ij}K\alpha_j\omega^{\frac{1}{2}}\|x_j\|_2 + \sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)| \\
&\quad + \omega\bar{r}_i + \|x_i\|_0\sum_{k=1}^{q}I_{ik}^M + \sum_{k=1}^{q}|I_{ik}(0)| := \Pi_i.
\end{aligned}$$

Integrating both sides of (2) from 0 to ω, we have

$$\begin{aligned}
&\left|\int_0^{\omega}\left[\frac{a_i(t)x_i(t)}{1+\lambda} - \frac{\lambda a_i(t)(-x_i(t))}{1+\lambda}\right]\Delta t\right| \\
&= \left|\int_0^{\omega}\left[\frac{a_i(t)x_i(t)}{1+\lambda} + \frac{\lambda a_i(t)x_i(t)}{1+\lambda}\right]\Delta t\right| \\
&= \left|\int_0^{\omega}a_i(t)x_i(t)\Delta t\right| \\
&= \left|\frac{1}{1+\lambda}\int_0^{\omega}\left[\sum_{j=1}^{n}d_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))\right.\right. \\
&\quad \left. + \sum_{j=1}^{n}b_{ij}(t)\int_0^{\infty}k_{ij}(\theta)h_j(x_j(t-\theta))\Delta\theta + r_i(t)\right]\Delta t \\
&\quad + \frac{\lambda}{1+\lambda}\int_0^{\omega}\left[-\sum_{j=1}^{n}d_{ij}(t)g_j(-x_j(t-\tau_{ij}(t)))\right. \\
&\quad - \sum_{j=1}^{n}b_{ij}(t)\int_0^{\infty}k_{ij}(\theta)h_j(-x_j(t-\theta))\Delta\theta \\
&\quad \left. - r_i(t)\right]\Delta t + \frac{1}{1+\lambda}\sum_{k=1}^{q}I_{ik}(x_i(t_k)) \\
&\quad \left. - \frac{\lambda}{1+\lambda}\sum_{k=1}^{q}I_{ik}(-x_i(t_k))\right| \\
&\le \left[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\right]\int_0^{\omega}\max\left\{\left|\sum_{j=1}^{n}d_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))\right.\right. \\
&\quad \left. + \sum_{j=1}^{n}b_{ij}(t)\int_0^{\infty}k_{ij}(\theta)h_j(x_j(t-\theta))\Delta\theta + r_i(t)\right|, \\
&\quad \left| - \sum_{j=1}^{n}d_{ij}(t)g_j(-x_j(t-\tau_{ij}(t)))\right.
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} k_{ij}(\theta)h_j(-x_j(t-\theta))\Delta\theta - r_i(t)\Big|\Big\}\Delta t\\
&+\left[\frac{1}{1+\lambda}+\frac{\lambda}{1+\lambda}\right]\max\Big\{\sum_{k=1}^{q}|I_{ik}(x_i(t_k))|,\\
&\sum_{k=1}^{q}|I_{ik}(-x_i(t_k))|\Big\}\\
\le &\sum_{j=1}^{n}\int_0^{\omega}\bar{d}_{ij}|g_j(x_j(t-\tau_{ij}(t)))-g_j(0)|\Delta t\\
&+\sum_{j=1}^{n}\int_0^{\omega}\bar{d}_{ij}|g_j(0)|\Delta t\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\int_0^{\omega}\int_0^{\infty}k_{ij}(\theta)|h_j(x_j(t-\theta))-h_j(0)|\Delta\theta\Delta t\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\int_0^{\omega}\int_0^{\infty}k_{ij}(\theta)|h_j(0)|\Delta\theta\Delta t\\
&+\bar{r}_i\omega+\max\Big\{\sum_{k=1}^{q}|I_{ik}(x_i(t_k))-I_{ik}(0)|,\\
&\sum_{k=1}^{q}|I_{ik}(-x_i(t_k))-I_{ik}(0)|\Big\}+\sum_{k=1}^{q}|I_{ik}(0)|\\
\le &\ \omega^{\frac{1}{2}}\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\|x_j\|_2+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\omega^{\frac{1}{2}}K\alpha_j\|x_j\|_2+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|\\
&+\omega\bar{r}_i+\|x_i\|_0\sum_{k=1}^{q}I_{ik}^{M}+\sum_{k=1}^{q}|I_{ik}(0)|:=\Omega_i.
\end{aligned}
$$

From Lemma 2, for any $\zeta_i,\eta_i\in[0,\omega]_{\mathbb{T}}$, $i=1,2,\dots,n$, we have

$$
\begin{aligned}
\int_0^{\omega}a_i(t)x_i(t)\Delta t \le &\int_0^{\omega}a_i(t)x_i(\xi_i)\Delta t\\
&+\int_0^{\omega}a_i(t)\left(\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\right)\Delta t \quad (3)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{\omega}a_i(t)x_i(t)\Delta t \ge &\int_0^{\omega}a_i(t)x_i(\eta_i)\Delta t\\
&-\int_0^{\omega}a_i(t)\left(\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\right)\Delta t. \quad (4)
\end{aligned}
$$

Diving by $\int_0^{\omega}a_i(x_i(t))\Delta t$ on the two sides of (3) and (4) respectively, we have for $i=1,2,\dots,n$,

$$
x_i(\zeta_i)\ge\left[\frac{1}{\int_0^{\omega}a_i(t)\Delta t}\int_0^{\omega}a_i(t)x_i(t)\Delta t-\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\right], \quad (5)
$$

and

$$
x_i(\eta_i)\le\left[\frac{1}{\int_0^{\omega}a_i(t)\Delta t}\int_0^{\omega}a_i(t)x_i(t)\Delta t+\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\right]. \quad (6)
$$

Let $\bar{t}_i,\underline{t}_i\in[0,\omega]_{\mathbb{T}}$, such that $x_i(\bar{t}_i)=\max_{t\in[0,\omega]_{\mathbb{T}}}x_i(t)$, $x_i(\underline{t}_i)=\min_{t\in[0,\omega]_{\mathbb{T}}}x_i(t)$, $i=1,2,\dots,n$, by the arbitrariness of ζ_i,η_i, in view of (5) and (6), we obtain

$$
\begin{aligned}
x_i(\underline{t}_i)\ &\ge\ \left[\frac{1}{\int_0^{\omega}a_i(t)\Delta t}\int_0^{\omega}a_i(t)x_i(t)\Delta t-\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\right]\\
&\ge\ -\frac{1}{\underline{a}_i\omega}\left|\int_0^{\omega}a_i(t)x_i(t)\Delta t\right|-\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\\
&\ge\ -\frac{1}{\underline{a}_i\omega}\Omega_i-\Pi_i
\end{aligned}
$$

and

$$
\begin{aligned}
x_i(\bar{t}_i)&\le\left[\frac{1}{\int_0^{\omega}a_i(t)\Delta t}\int_0^{\omega}a_i(t)x_i(t)\Delta t+\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\right]\\
&\le\frac{1}{\omega\underline{a}_i}\left|\int_0^{\omega}a_i(t)x_i(t)\Delta t\right|+\int_0^{\omega}|x_i^{\Delta}(t)|\Delta t\\
&\le\frac{1}{\omega\underline{a}_i}\Omega_i+\Pi_i,\ i=1,2,\dots,n.
\end{aligned}
$$

So, we can obtain that

$$
\begin{aligned}
\|x_i\|_0 &= \max_{t\in[0,\omega]_{\mathbb{T}}}|x_i(t)|\\
&\le\frac{1}{\omega\underline{a}_i}\Omega_i+\Pi_i,\ i=1,2,\dots,n. \quad (7)
\end{aligned}
$$

In addition, we have that

$$
\begin{aligned}
\|x_i\|_2 &= \left(\int_0^{\omega}|x_i(s)|^2\Delta s\right)^{\frac{1}{2}}\\
&\le\omega^{\frac{1}{2}}\max_{t\in[0,\omega]_{\mathbb{T}}}|x_i(t)|=\sqrt{\omega}||x_i||_0,
\end{aligned}
$$

where $i=1,2,\dots,n$. By (7), we have

$$
\begin{aligned}
\underline{a}_i\omega\|x_i\|_0\ \le\ &\Omega_i+\Pi_i\omega\underline{a}_i\\
=\ &\omega^{\frac{1}{2}}\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\|x_j\|_2+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\omega^{\frac{1}{2}}K\alpha_j\|x_j\|_2+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|\\
&+\omega\bar{r}_i+\|x_i\|_0\sum_{k=1}^{q}I_{ik}^{M}+\sum_{k=1}^{q}|I_{ik}(0)|+\\
&+\underline{a}_i\omega\Big[\bar{a}_i\omega^{\frac{1}{2}}\|x_i\|_2+\omega^{\frac{1}{2}}\sum_{j=1}^{n}(\bar{d}_{ij}\beta_j)\|x_j\|_2\\
&+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|+\sum_{j=1}^{n}\bar{b}_{ij}K\alpha_j\omega^{\frac{1}{2}}\|x_j\|_2\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|+\omega\bar{r}_i\\
&+\|x_i\|_0\sum_{k=1}^{q}I_{ik}^{M}+\sum_{k=1}^{q}|I_{ik}(0)|\Big]\\
\le\ &\omega^{\frac{1}{2}}\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega^{\frac{1}{2}}||x_j||_0+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\omega^{\frac{1}{2}}K\alpha_j\omega^{\frac{1}{2}}||x_j||_0
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|+\omega\bar{r}_i+\|x_i\|_0\sum_{k=1}^{q}I_{ik}^M\\
&+\sum_{k=1}^{q}|I_{ik}(0)|+\underline{a}_i\omega\Big[\bar{a}_i\omega^{\frac{1}{2}}\omega^{\frac{1}{2}}||x_i||_0\\
&+\omega^{\frac{1}{2}}\sum_{j=1}^{n}(\bar{d}_{ij}\beta_j)\omega^{\frac{1}{2}}||x_j||_0+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}K\alpha_j\omega^{\frac{1}{2}}\omega^{\frac{1}{2}}||x_j||_0+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|\\
&+\omega\bar{r}_i+\|x_i\|_0\sum_{k=1}^{q}I_{ik}^M+\sum_{k=1}^{q}|I_{ik}(0)|\Big]\\
=\ &||x_i||_0\Big\{\sum_{k=1}^{q}I_{ik}^M+\underline{a}_i\omega^2\bar{a}_i+\underline{a}_i\omega\sum_{k=1}^{q}I_{ik}^M\Big\}\\
&+||x_j||_0\Big\{\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega+\sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j\\
&+\underline{a}_i\omega\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega+\underline{a}_i\omega\sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j\Big\}\\
&+\Big[\omega\bar{r}_i+\sum_{k=1}^{q}|I_{ik}(0)|+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|\Big]\\
&+\underline{a}_i\omega\Big[\omega\bar{r}_i+\sum_{k=1}^{q}|I_{ik}(0)|\\
&+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|\Big]\\
=\ &||x_i||_0\Big\{\sum_{k=1}^{q}I_{ik}^M+\underline{a}_i\omega^2\bar{a}_i+\underline{a}_i\omega\sum_{k=1}^{q}I_{ik}^M\Big\}\\
&+||x_j||_0\Big\{\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega+\sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j\\
&+\underline{a}_i\omega\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega+\underline{a}_i\omega\sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j\Big\}\\
&+(1+\underline{a}_i\omega)\Big[\omega\bar{r}_i+\sum_{k=1}^{q}|I_{ik}(0)|\\
&+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|\Big].
\end{aligned}
$$

That is

$$
\begin{aligned}
&\|x_i\|_0\Big[\underline{a}_i\omega-\sum_{k=1}^{q}I_{ik}^M-\underline{a}_i\omega^2\bar{a}_i-\underline{a}_i\omega\sum_{k=1}^{q}I_{ik}^M\Big]\\
&-||x_j||_0\Big\{\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega+\sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j\\
&+\underline{a}_i\omega\sum_{j=1}^{n}\bar{d}_{ij}\beta_j\omega+\underline{a}_i\omega\sum_{j=1}^{n}\bar{b}_{ij}\omega K\alpha_j\Big\}\\
&\le(1+\underline{a}_i\omega)\Big[\omega\bar{r}_i+\sum_{k=1}^{q}|I_{ik}(0)|+\omega\sum_{j=1}^{n}\bar{d}_{ij}|g_j(0)|\\
&\quad+\sum_{j=1}^{n}\bar{b}_{ij}\omega K|h_j(0)|\Big]:=D_i, \qquad (8)
\end{aligned}
$$

where $i=1,2,\dots,n$. Denote $\|x\|_0=(\|x_1\|_0,\|x_2\|_0,\dots,\|x_n\|_0)^T$ and $D=(D_1,D_2,\dots,D_n)^T$.

Then (8) can be rewritten in the matrix form

$$E\|x\|_0\le D.$$

From the conditions of Theorem 1, E is a nonsingular M matrix, hence

$$\|x\|_0\le E^{-1}D:=(N_1,N_2,\dots,N_n)^T.$$

Let $N=\sum_{i=1}^{n}N_i+N_0$, where N_0 is any positive constant. It is clear that N is independent of λ. Take $N_{\mathbb{X}}=\{x\in\mathbb{X}\big|\|x\|_{\mathbb{X}}<N\}$. Obviously, $N_{\mathbb{X}}$ satisfies all the requirement in Lemma 4 and the condition (H) satisfied. In view of all the discussion above, we conclude from Lemma 4 that system (1) has at least one ω-periodic solution. This completes the proof. ∎

IV. Global exponential stability of periodic solutions

Suppose that $x^*(t)=(x_1^*(t),x_2^*(t),\dots,x_n^*(t))^T$ is an ω-periodic solution of system (1). In this section, we will construct some suitable Lyapunov functions to study the global exponential stability of this periodic solution.

To prove the global exponential stability, we assume that

(H_7) there exist constants $\eta>0,\lambda>0$ and $\xi_i>0, i=1,2,\dots,n$, such that for all $t\in\mathbb{T}^+$, there holds

$$
\begin{aligned}
&(\lambda-a_i(t))\xi_i+\sum_{j=1}^{n}|d_{ij}(t)|\beta_j\xi_je_\lambda(t,t-\tau_{ij}(t))\\
&+\sum_{j=1}^{n}|b_{ij}(t)|\int_0^\infty|k_{ij}(\theta)|\alpha_j\xi_je_\lambda(t,t-\theta)\Delta\theta<-\eta<0.
\end{aligned}
$$

Theorem 2. *Assume that $(H_1)-(H_7)$ hold. Let $x^*(t)=(x_1^*(t),x_2^*(t),...,x_n^*(t))^T$ is an ω-periodic solution of system (1) with initial value $\phi^*(t)=(\phi_1^*(t),\phi_2^*(t),...,\phi_n^*(t))$, for every solution $x(t)=(x_1(t),x_2(t),...,x_n(t))^T$ of the system (1) with any initial value $\phi(t)=(\phi_1(t),\phi_2(t),...,\phi_n(t))$. Then there exist constants $\lambda>0$ and $M>1$ such that*

$$|x_i(t)-x_i^*(t)|\le M\|\phi-\phi^*\|e_{-\lambda}(t,\delta), t\in\mathbb{T}^+,\ i=1,2,\dots,n.$$

That is, system (1) has a unique ω-periodic solution which is globally exponentially stable.

Proof: Let $y_i(t)=(x_i(t)-x_i^*(t))$, from (1) we have

$$
\begin{cases}
\quad(x_i(t)-x_i^*(t))^\Delta\\
=-a_i(t)[x_i(t)-x_i^*(t)]+\sum_{j=1}^{n}d_{ij}(t)[g_j(y_j(t-\tau_{ij}(t))\\
\quad+x_j^*(t-\tau_{ij}(t)))-g_j(x_j^*(t-\tau_{ij}(t)))]\\
\quad+\sum_{j=1}^{n}b_{ij}(t)\int_0^\infty k_{ij}(\theta)[h_j(y_j(t-\theta)+x_j^*(t-\theta))\\
\quad-h_j(x_j^*(t-\theta))]\Delta\theta, t\in\mathbb{T}^+, t\ne t_k,\ i=1,2,\dots,n,\\
\quad\Delta(x_i(t_k)-x_i(t_k^*))\\
=I_{ik}(x_i(t_k)-x_i(t_k^*)),\quad i=1,2,\dots,n,\ k\in\mathbb{N},
\end{cases}
$$

where $i = 1, 2, \ldots, n$. And from (H_5), we have

$$\begin{aligned} & |x_i^+(t_k) - x_i^+(t_k^*)| \\ =\ & |1-\gamma_{ik}||x_i(t_k) - x_i(t_k^*)| \\ \leq\ & |x_i(t_k) - x_i(t_k^*)|,\ i = 1, 2, \ldots, n. \end{aligned}$$

Now let us consider the Lyapunov functional

$$V_i(t) = |y_i(t)|e_\lambda(t, \delta),\ \delta \in (-\infty, 0]_{\mathbb{T}}\ i = 1, 2, \ldots, n. \quad (9)$$

For $t \in \mathbb{T}^+, t \neq t_k,\ i = 1, 2, \ldots, n$, the upper right Dini derivative D^+V_1 along the solutions of (1) is

$$\begin{aligned} & D^+(V_i(t))^\Delta \\ =\ & D^+|y_i(t)|^\Delta e_\lambda(t,\delta) + \lambda|y_i(t)|e_\lambda(t,\delta) \\ =\ & -a_i(t)|y_i(t)|e_\lambda(t,\delta) + \lambda|y_i(t)|e_\lambda(t,\delta) \\ & +\sum_{j=1}^n \bigg| d_{ij}(t)[g_j(y_j(t-\tau_{ij}(t)) + x_j^*(t-\tau_{ij}(t))) \\ & -g_j(x_j^*(t-\tau_{ij}(t)))]\bigg| e_\lambda(t,\delta) \\ & +\sum_{j=1}^n \bigg| b_{ij}(t)\int_0^\infty k_{ij}(\theta)[h_j(y_j(t-\theta) + x_j^*(t-\theta)) \\ & -h_j(x_j^*(t-\theta))]\Delta\theta \bigg| e_\lambda(t,\delta) \\ \leq\ & (\lambda - a_i(t))|y_i(t)|e_\lambda(t,\delta) \\ & +\sum_{j=1}^n |d_{ij}(t)|\beta_j|y_j(t-\tau_{ij}(t))|e_\lambda(t,\delta) \\ & +\sum_{j=1}^n |b_{ij}(t)|\int_0^\infty |k_{ij}(\theta)|\alpha_j|y_j(t-\theta)|e_\lambda(t,\delta)\Delta\theta, \quad (10) \end{aligned}$$

where $i = 1, 2, \ldots, n$.

Also, for $i = 1, 2, \ldots, n, t \in \mathbb{T}^+$,

$$\begin{aligned} V_i(t_k^+) &= |y_i(t_k^+)|e_\lambda(t_k^+, \delta) \\ &= |x_i(t_k^+) - x_i^*(t_k^+)|e_\lambda(t_k^+, \delta) \\ &\leq |x_i(t_k) - x_i^*(t_k)|e_\lambda(t_k, \delta) = V_i(t_k). \end{aligned}$$

Let $m > 1$ denote an arbitrary real number such that

$$m\xi_i > ||\phi - \phi^*|| = \sup_{s\in(-\infty,0]_{\mathbb{T}}} \max_{0\leq i\leq n} |\phi_i(s) - \phi_i^*(s)| > 0.$$

It follows from (9) that

$$V_i(t) = |y_i(t)|e_\lambda(t,\delta) < m\xi_i,\ t \in (-\infty, 0]_{\mathbb{T}},\ i = 1, 2, \ldots, n.$$

We claim that

$$V_i(t) = |y_i(t)|e_\lambda(t,\delta) < m\xi_i,\ t \in \mathbb{T}^+,\ i = 1, 2, \ldots, n. \quad (11)$$

Otherwise, there exist $i \in \{1, 2, \ldots, n\}$ and $t_i > 0$ such that

$$V_i(t_i) = m\xi_i, V_j(t) < m\xi_j, t \in (-\infty, t_i)_{\mathbb{T}}, j = 1, 2, \ldots, n. \quad (12)$$

That is

$$V_i(t_i) - m\xi_i = 0,\ V_j(t) - m\xi_j < 0,\ t \in (-\infty, t_i)_{\mathbb{T}},$$

where $j = 1, 2, \ldots, n$. From Lemma 1, together with (10) and (12), we obtain

$$\begin{aligned} 0\ \leq\ & D^+(V_i(t_i) - m\xi_i)^\Delta \\ \leq\ & D^+(V_i(t_i))^\Delta \\ \leq\ & (\lambda - a_i(t_i))|y_i(t_i)|e_\lambda(t_i,\delta) \\ & +\sum_{j=1}^n |d_{ij}(t_i)|\beta_j|y_j(t_i-\tau_{ij}(s_i))|e_\lambda(t_i,\delta) \\ & +\sum_{j=1}^n |b_{ij}(t_i)|\int_0^\infty |k_{ij}(\theta)|\alpha_j|y_j(t_i-\theta)|\Delta\theta e_\lambda(t_i,\delta) \\ =\ & (\lambda - a_i(t_i))|y_i(t_i)|e_\lambda(t_i,\delta) \\ & +\sum_{j=1}^n |d_{ij}(s_i)|\beta_j|y_j(t_i-\tau_{ij}(t_i))| \\ & \times e_\lambda(t_i, t_i-\tau_{ij}(t_i))e_\lambda(t_i-\tau_{ij}(t_i),\delta) \\ & +\sum_{j=1}^n |b_{ij}(t_i)|\int_0^\infty |k_{ij}(\theta)|\alpha_j|y_j(t_i-\theta)| \\ & \times e_\lambda(t_i, t_i-\theta)e_\lambda(t_i-\theta,\delta)\Delta\theta \\ \leq\ & \bigg[(\lambda - a_i(t_i))\xi_i \\ & +\sum_{j=1}^n |d_{ij}(t_i)|\beta_j e_\lambda(t_i, t_i-\tau_{ij}(t_i))\xi_j \\ & +\sum_{j=1}^n |b_{ij}(t_i)|\int_0^\infty |k_{ij}(\theta)|\alpha_j\xi_j e_\lambda(t_i, t_i-\theta)\Delta\theta\bigg] m. \end{aligned}$$

Thus

$$\begin{aligned} 0\ \leq\ & (\lambda - a_i(t_i))\xi_i + \sum_{j=1}^n |d_{ij}(t_i)|\beta_j e_\lambda(t_i, t_i-\tau_{ij}(t_i))\xi_j \\ & +\sum_{j=1}^n |b_{ij}(t_i)|\int_0^\infty |k_{ij}(\theta)|\alpha_j\xi_j e_\lambda(t_i, t_i-\theta)\Delta\theta, \end{aligned}$$

which contradicts (H_7). Hence, (11) holds, Letting $M > 1$ such that

$$\max_{0\leq i\leq n}\{m\xi_i\} \leq M||\phi - \phi^*||,\ \ i = 1, 2, \ldots, n. \quad (13)$$

In view of (11) and (13), we get

$$\begin{aligned} |x_(t) - x_i^*(t)| &= |y_i(t)| \leq \max_{0\leq i\leq n}\{m\xi_i\}e_{-\lambda}(t,\delta) \\ &\leq M||\phi - \phi^*||e_{-\lambda}(t,\delta), \end{aligned}$$

where $i = 1, 2, \ldots, n$, $t \in \mathbb{T}^+$. This completes the proof. ■

V. An example

In this section, we give an example to demonstrate the results obtained in previous sections. Consider the following recurrent neural network with continuously distributed delays and impulses on time scales:

$$\begin{cases} \dot{x}_1(t) &= -x_1(t) + \frac{1}{24}(\cos t)g_1(x_1(t-22)) \\ &\quad +\frac{1}{108}(\cos t)g_2(x_2(t-11)) \\ &\quad +\frac{1}{24}(\sin t)\int_0^\infty |\sin u|e^{-u}h_1(x_1(t-u))du \\ &\quad +\frac{1}{108}(\sin t)\int_0^\infty |\cos u|e^{-u}h_2(x_2(t-u))du \\ &\quad +r_1(t), t\in\mathbb{T}^+, t\neq t_k, \\ \Delta x_1(t_k) &= x_1(t_k^+)-x_1(t_k^-)=\frac{1}{24}(\cos t), \quad k\in\mathbb{N}, \\ \dot{x}_2(t) &= -x_2(t) + \frac{1}{12}(\cos t)g_1(x_1(t-6)) \\ &\quad +\frac{1}{24}(\cos t)g_2(x_2(t-8)) \\ &\quad +\frac{1}{12}(\sin t)\int_0^\infty |\cos u|e^{-u}h_1(x_1(t-u))du \\ &\quad +\frac{1}{24}(\sin t)\int_0^\infty |\sin u|e^{-u}h_1(x_1(t-u))du \\ &\quad +r_2(t), t\in\mathbb{T}^+, t\neq t_k, \\ \Delta x_2(t_k) &= x_2(t_k^+)-x_2(t_k^-)=\frac{1}{24}(\sin t), \quad k\in\mathbb{N}, \end{cases} \tag{14}$$

where $g_1(x) = g_2(x) = \sin x$, $h_1(x) = h_2(x) = x$, $r_1(t) = \frac{1}{8}\cos t$, $r_2(t) = \frac{1}{8}\sin t$.

Note that $a_1 = a_2 = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $d_{11}^+ = b_{11}^+ = \frac{1}{8}$, $d_{12}^+ = b_{12}^+ = \frac{1}{36}$, $d_{21}^+ = b_{21}^+ = \frac{1}{4}$, $d_{22}^+ = b_{22}^+ = \frac{1}{8}$, $\int_0^\infty k_{ij}(\theta)\Delta\theta \leq 1$, where $i,j = 1,2$. $d_{ij}^+ = \sup_{t\in\mathbb{T}^+} d_{ij}(t)$ and $b_{ij}^+ = \sup_{t\in\mathbb{T}^+} b_{ij}(t)$. Then, we get

$$E = \begin{pmatrix} \frac{1}{4} & \frac{1}{18} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

From the theory of M-matrix in [28], thus (H_6) is satisfied. The system (14) has at least one 2π-periodic solutions. We can choose constants $\eta = \frac{1}{4}$ and $\xi_i = 1$, $i = 1,2$, such that for all $t \in \mathbb{T}^+$, there holds

$$\begin{aligned} &-a_i\xi_i + \sum_{j=1}^2 d_{ij}^+\beta_j\xi_j + \sum_{j=1}^2 b_{ij}^+\int_0^\infty |k_{ij}(\theta)|\alpha_j\Delta\theta\xi_j < \\ &-a_i\xi_i + \sum_{j=1}^2 d_{ij}^+\beta_j\xi_j + \sum_{j=1}^2 b_{ij}^+\alpha_j\xi_j := \Omega_i < -\eta < 0, \end{aligned}$$

where $i = 1,2$, $\Omega_1 = -\frac{97}{108} < -\frac{1}{4} = -\eta$ and $\Omega_1 = -\frac{3}{4} < -\frac{1}{4} = -\eta$.

Thus (H_7) is satisfied. From Theorem 1 and Theorem 2, we know that system (14) has at least one 2π-periodic solutions and it is globally exponentially stable.

VI. Conclusion

Using the time scale calculus theory, coincidence degree theory and the Liapunov functional method, some sufficient conditions are obtained to ensure the existence and the global exponential stability of periodic solutions for recurrent neural networks with impulses and distributed delays on time scales. The results obtained in this paper possess highly important significance and are easily checked in practice. In addition, the method in this paper can be applied to other neural networks such as the BAM and DCNNs systems and so on.

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