A Hamiltonian Decomposition of 5-star

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Abstract—Star graphs are Cayley graphs of symmetric groups of permutations, with transpositions as the generating sets. A star graph is a preferred interconnection network topology to a hypercube for its ability to connect a greater number of nodes with lower degree. However, an attractive property of the hypercube is that it has a Hamiltonian decomposition, i.e. its edges can be partitioned into disjoint Hamiltonian cycles, and therefore a simple routing can be found in the case of an edge failure. The existence of Hamiltonian cycles in Cayley graphs has been known for some time. So far, there are no published results on the stronger condition of the existence of Hamiltonian decompositions. In this paper, we give a construction of a Hamiltonian decomposition of the star graph 5-star of degree 4, by defining an automorphism for 5-star and a Hamiltonian cycle which is edge-disjoint with its image under the automorphism.

Keywords—interconnection networks; paths and cycles; graphs and groups.

I. INTRODUCTION

NETWORKED computer systems have basic requirements such as fast communication and fault tolerance, which are met by an appropriate choice of interconnection topology. The star graph has been proposed as an interconnection network topology that is better than the hypercube for its ability to connect a greater number of nodes with lower degree [1]. On the other hand, an attractive property of the hypercube is that its edges can be partitioned into disjoint Hamiltonian cycles [2]. The presence of edge-disjoint Hamiltonian cycles is desirable for interconnection networks for various reasons. Fault tolerance is easier to achieve as a simple routing can be found in the case of an edge failure. Efficiency can also be improved. An example is the case of all-to-all broadcasting in multiprocessor systems, where a node can send to or receive from all its neighbours in unit time, as messages can be broken down into smaller messages and sent along edge-disjoint Hamiltonian cycles. Edge-disjoint Hamiltonian cycles have been investigated in various interconnection topologies, for example in deBruijn networks [3] and tori [4]. They have also been studied in star graphs and lower bounds for the number of pairwise edge-disjoint Hamiltonian cycles have been given in [5]. However, there has been no significant progress on the optimum case of edge-disjoint cycles, where all the edges in the network topology are partitioned into Hamiltonian cycles, beyond the case of the hypercube.

Star graphs are Cayley graphs of symmetric groups of permutations of finitely many elements and certain restricted sequences of the first element with one of the other elements. A star graph is of the form:

\[ \rho = (a_1 \cdots a_n) \]

\[ \rho = (a_1 \cdots a_{\rho(j-1)} a_\rho(j) a_{\rho(j+1)} \cdots a_\rho(n)) \]

II. PRELIMINARIES

We give the basic definitions of star graphs, Hamiltonian cycles and automorphisms.

Definition 1: The n-star graph \( S_n \) is the simple undirected regular graph of degree n-1 whose vertices \( V(S_n) \) are sequences of n elements \( \{a_1, \ldots, a_n\} \)

\[ V(S_n) = \{a_1 \cdots a_n : \rho \text{ is a permutation of } \{1, \ldots, n\}\} \]

and whose edges \( E(S_n) \) correspond to swapping the positions of the first element with one of the other n-1 elements, i.e. \( e = E(S_n) \) is of the form:

\[ e = (a_1 \cdots a_{\rho(j-1)} a_\rho(j) a_{\rho(j+1)} \cdots a_\rho(n)) \]

We define the distance between two distinct elements to be:

\[ \delta(a_i, a_j) = \min \{|i-j|, n-|i-j|\} \]

Clearly \( \delta(a_i, a_j) = \delta(a_j, a_i) \). The length of the edge \( e \) above, \( \lambda(e) \), is defined to be \( \delta(a_\rho(j), a_\rho(j)) \).

Definition 2: A Hamiltonian cycle in \( S_n \) is a pair of sequences \( (v, e) \) of vertices

\[ v = v_1 \cdots v_{n+1} \]

and edges

\[ e = e_1 \cdots e_n \]

such that:

(i) \( e_i = (v_i, v_{i+1}) \in E(S_n) \) (\( 1 \leq i \leq n \)),

(ii) \( \{v_1, \ldots, v_{n+1}\} = V(S_n) \),

(iii) \( v_1 = v_{n+1} \).

Thus, a Hamiltonian cycle follows a path along edges visiting each vertex exactly once before returning to the first vertex. A Hamiltonian decomposition of \( S_{2k+1} \) where \( k \geq 1 \) consists of k Hamiltonian cycles that are edge-disjoint, i.e. no two Hamiltonian cycles have a common edge.
Definition 3: Let \((V, E)\) be a graph, where \(V\) is a set of vertices and \(E \subseteq V \times V\) a set of edges. Then, a mapping \(\Phi: V \to V\) is an automorphism iff:

(i) \(\Phi\) is bijective

(ii) for all \(v_1, v_2 \in V\),

\((v_1, v_2) \in E\) implies \((\Phi(v_1), \Phi(v_2)) \in E\)

III. AN AUTOMORPHISM

The following lemma gives a basic class of graph automorphisms preserving Hamiltonian cycles.

Lemma 4: Let \(\phi: \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}\) be a bijection. Then:

(i) \(\Phi: V(St_n) \to V(St_n), \) given by \(\Phi(a(i)) = \phi(a(i))\) is an automorphism of the graph \(St_n\)

(ii) if \(v = v_1 \cdot \cdots \cdot v_{n+1}, \) \(e = (v_1, v_2) \cdots (v_{n+1}, v_{n+1})\) and \(v', e'\) is a Hamiltonian cycle in \(St_n\), then \(\Phi(v', e') = (\Phi(v_1), \Phi(v_2)) \cdots (\Phi(v_{n+1}), \Phi(v_1))\) is also a Hamiltonian cycle.

Proof: We check that \(\Phi\) is an automorphism. If \(v_1 \neq v_2 \in V(St_n), \) say \(v_1 = (a_{p(1)}, \ldots, a_{p(i-1)}, a_{p(i)}), \) \(v_2 = (a_{p(1)}, \ldots, a_{p(i-1)}, a_{p(i)}), \)

where \(p(i) \neq p'(i)\) for some \(1 \leq i \leq n\), then \(\phi(a_{p(i)}) \neq \phi(a_{p'(i)})\) as \(\phi\) is injective, and \(\Phi(v_1)\) and \(\Phi(v_2)\) differ on their respective \(i\)-th elements \(\phi(a_{p(i)})\) and \(\phi(a_{p'(i)})\).

Thus, \(\Phi\) is injective. It is surjective as, given \(b_1 \ldots b_n \in V(St_n), \) by surjectivity of \(\phi\) we can choose \(a_{p(1)}, \ldots, a_{p(n)}\) such that \(\phi(a_{p(i)}) = b_i, \) \(\phi(a_{p(i)}) = b_i\) and therefore \(\Phi(a_{p(1)}, \ldots, a_{p(n)}) = b_1 \cdots b_n. \) To show that Definition 3(ii) holds, let \((v_1, v_2) \in E(St_n).\) By Definition 1, for some permutation \(\rho\) and \(1 \leq i \leq n, \)

\[ e = (a_{p(1)}, \ldots, a_{p(i-1)}, a_{p(i)} a_{p(i+1)}, \ldots, a_{p(n)}) \]

Then,

\[ \Phi(v_1) = \phi(a_{p(1)}) \cdots \phi(a_{p(i-1)}) \phi(a_{p(i)}) \cdots \phi(a_{p(n)}) \]

\[ \Phi(v_2) = \phi(a_{p(1)}) \cdots \phi(a_{p(i-1)}) \phi(a_{p(i)}) \cdots \phi(a_{p(n)}) \]

As \(\phi: \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}\) is a bijection, there is a permutation \(\sigma\) of \(\{1, \ldots, n\}\) such that:

\[ \phi(a_j) = a_{\sigma(j)} \] for \(1 \leq j \leq n\)

Therefore,

\[ \Phi(v_1) = a_{\sigma p(1)} a_{\sigma p(i-1)} a_{\sigma p(i-1)} a_{\sigma p(i+1)} \ldots a_{\sigma p(n)} \]

\[ \Phi(v_2) = a_{\sigma p(i)} a_{\sigma p(i-1)} a_{\sigma p(i-1)} a_{\sigma p(i+1)} \ldots a_{\sigma p(n)} \]

and so \((\Phi(v_1), \Phi(v_2))\) satisfies (1) with \(\sigma\) in place of \(\rho\) and thus \((\Phi(v_1), \Phi(v_2)) \in E.\) This completes the check of (i) of this lemma, that \(\Phi\) is an automorphism.

To prove (ii) of this lemma, we check that (i), (ii) and (iii) of Definition 2 are satisfied. Put \((\rho', \rho') = \Phi(\rho, \rho)\) so that:

\[ \rho' = \Phi(v_1) \cdots \Phi(v_{n+1}) \]

\[ \rho' = (\Phi(v_1), \Phi(v_2)) \cdots (\Phi(v_{n+1}), \Phi(v_{n+1})) \]

For (i) of Definition 2, let \((\Phi(v_i), \Phi(v_{i+1})) \in e'\) where \(1 \leq i \leq n!\). As \((\rho', \rho')\) is a Hamiltonian cycle in \(St_n, \) we have that \((v_i, v_{i+1})\) is an edge in \(E(St_n).\) By (i) of this lemma, \(\Phi\) is an automorphism and so \((\Phi(v_i), \Phi(v_{i+1}))\) is an edge in \(E(St_n)\) as required. For Definition 2(ii), as \(\Phi\) is an automorphism, it maps the set of all vertices \(\{v_1, \ldots, v_{n+1}\}\) onto itself. Thus \(V(St_n) = \{v_1, \ldots, v_{n+1}\} = \{\Phi(v_1), \ldots, \Phi(v_{n+1})\}\). For Definition 2(iii), we note that \((\rho', \rho')\) is a Hamiltonian cycle in \(St_n,\)

\(v_1 = v_{n+1}\) and therefore \(\Phi(v_1) = \Phi(v_{n+1}).\)

IV. CONSTRUCTION OF A HAMILTONIAN CYCLE

A Hamiltonian cycle for \(St_n\) is constructed by partitioning the vertices of \(St_5\) into 6 pairwise disjoint cycles \(C_1, \ldots, C_6,\) and then producing a 7th cycle \(C_7\) that meets each of the other cycles at exactly two vertices and a common edge. It is clear that the union of the edges in the 7 cycles, excluding edges that \(C_7\) has in common with any of the other 6 cycles, is then a Hamiltonian cycle; we denote it by \(C.\) Below, in Lemma 7, we define a cycle \(C_1\) from which 5 further cycles \(C_2, C_3, C_4, C_5\) and \(C_6\) are generated by the 5 length-preserving automorphisms of the following lemma.

Lemma 6: The 5 maps given by:

\[ \Psi_2(a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)}) = a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)} \]

\[ \Psi_3(a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)}) = a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)} \]

\[ \Psi_4(a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)}) = a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)} \]

\[ \Psi_5(a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)}) = a_{p(1)} a_{p(2)} a_{p(3)} a_{p(4)} a_{p(5)} \]
\[
\Psi_3(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}) = \\
a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)} = \\
\Psi_3(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}) = \\
a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i)
\]

where \( a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)} \) is any given point in \( S_{T_5} \) with corresponding permutation \( \rho \), are automorphisms of \( S_{T_5} \) which preserve cycles and lengths of edges, i.e.

\[
\lambda(v_1, v_2) = \lambda(\Psi_3(v_1), \Psi_3(v_2)) \quad (v_1, v_2 \in V(S_{T_5})), \quad 2 \leq i \leq 6
\]

(The 5 maps correspond to the 5 possible alternative orders of the last 3 positions in a vertex.)

**Proof:** We check that the lemma holds for \( \Psi_3 \) - a very similar check can be performed for \( \Psi_2, \Psi_4, \Psi_5 \) and \( \Psi_6 \). Suppose that \( a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i) \) and \( a_{\rho'(1)} a_{\rho'(2)} a_{\rho'(3)} a_{\rho'(4)} a_{\rho'(5)} \) \( i \in S_{T_5} \) differ, i.e. \( \rho \) and \( \rho' \) differ. Then, clearly,

\[
\Psi_3(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}) = \\
a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i) \neq \\
\rho_{\rho'(2)} a_{\rho'(3)} a_{\rho'(4)} a_{\rho'(5)} a_{\rho(3)}(i) = \\
\Psi_3(a_{\rho'(1)} a_{\rho'(2)} a_{\rho'(3)} a_{\rho'(4)} a_{\rho'(5)}(i))
\]

Thus, \( \Psi_3 \) is injective. If \( a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i) \) is any vertex in \( S_{T_5} \), then \( \Psi_3(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i)) = a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i) \). Thus, \( \Psi_3 \) is surjective. To show that \( \Psi_3 \) is an automorphism, suppose that \( v_1 = a_{\rho(1)} ... a_{\rho(i)} ..., v_2 = a_{\rho(i)} ... a_{\rho(1)} ..., \) so that \( (v_1, v_2) \in E(S_{T_5}) \), where \( i = 2, 3, 4 \) or 5. In the case \( i = 2 \),

\[
(v_1, v_2) = \\
(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i), a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i)) = \\
(\Psi_3(v_1), \Psi_3(v_2)) = \\
E(S_{T_5})
\]

In the case \( i = 3 \),

\[
(v_1, v_2) = \\
(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i), a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i)) = \\
(\Psi_3(v_1), \Psi_3(v_2)) = \\
E(S_{T_5})
\]

In the case \( i = 4 \),

\[
(v_1, v_2) = \\
(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i), a_{\rho(4)} a_{\rho(5)}(i)) = \\
(\Psi_3(v_1), \Psi_3(v_2)) = \\
E(S_{T_5})
\]

In the case \( i = 5 \),

\[
(v_1, v_2) = \\
(a_{\rho(1)} a_{\rho(2)} a_{\rho(3)} a_{\rho(4)} a_{\rho(5)}(i), a_{\rho(5)}(i)) = \\
(\Psi_3(v_1), \Psi_3(v_2)) = \\
E(S_{T_5})
\]

Thus, \( \Psi_3 \) is an automorphism and therefore also preserves cycles by an argument similar to that in Lemma 4(ii). We note from the above cases that, given an edge of the form:

\[
(a_{\rho(1)} ... a_{\rho(i)} ..., a_{\rho(i)} ... a_{\rho(1)} ... ) \in E(S_{T_5}),
\]

we have that

\[
(\Psi_3(a_{\rho(1)} ... a_{\rho(i)} ... ), \Psi_3(a_{\rho(i)} ... a_{\rho(1)} ... ))
\]

is still of the form (2), albeit \( a_{\rho(i)} \) occurs in a different position in \( \Psi_3(a_{\rho(1)} ... a_{\rho(i)} ... ) \) and \( a_{\rho(i)} \) occurs in a different position in \( \Psi_3(a_{\rho(i)} ... a_{\rho(1)} ... ) \). It follows, by the definition of the length of edges, that \( \Psi_3 \) preserves lengths of edges.

**Lemma 7:** The six cycles \( C_1, C_2, C_3, C_4, C_5, C_6 \) formed by starting at vertices \( abedc, abedc, abedc, abedc, abedc, abedc, abedc \) respectively, and progressing along edges of length 1 until cycles are completed, partition \( S_{T_5} \) into 6 disjoint cycles.

**Proof:** We list the 20 vertices of \( C_1 \):

\[
abedc, bacde, cbaed, dabec, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc,
\]

As, we have that

\[
\Psi_2(abedc) = abedc, \Psi_3(abedc) = abedc, \Psi_4(abedc) = abedc,
\]

\[
\Psi_5(abedc) = abedc, \text{ and } \Psi_6(abedc) = abedc,
\]

it follows by Lemma 6 that the other cycles also contain 20 vertices. The only vertex with \( ab \) in the first two positions in \( C_1 \) is \( abedc \). By Lemma 6, as \( \Psi_2, \Psi_3, \Psi_4, \Psi_5 \) and \( \Psi_6 \) only reorder the last 3 elements of a vertex, the only vertices with \( ab \) in the first two positions in \( C_2, C_3, C_4, C_5, \) and \( C_6 \) are \( abedc, abedc, abedc, abedc, abedc \) respectively. Thus, each of \( abedc, abedc, abedc, abedc, abedc \) can only occur in one of the cycles, and it follows that the \( C_i \)'s are pairwise disjoint and account for the \( 6 \times 20 \) vertices of \( S_{T_5} \).

**Lemma 8:** The cycle \( C_7 \) given by:

\[
bcade, bcada, bdcda, bdace, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc, abedc
\]

meets each \( C_i \) (\( 1 \leq i \leq 6 \)) at exactly two vertices and a common edge.

**Proof:** We have that:

\[
(bcade, abedc) \in E(C_1)
\]

by Lemma 7,

\[
(bcade, dbacea) = (\Psi_2(bcdea), \Psi_2(dbeaca)) \in E(C_2)
\]
by Lemmas 7 and 6, 
\[ (bdace, adbec) = (\Psi_3(bdace), \Psi_3(adbec)) \in E(C_3) \]
by Lemmas 7 and 6, 
\[ (cdbae, dcbae) = (\Psi_4(dcbae), \Psi_4(dcbae)) \in E(C_4) \]
by Lemmas 7 and 6, 
\[ (bdace, adbec) = (\Psi_5(bdace), \Psi_5(acdbe)) \in E(C_5) \]
by Lemmas 7 and 6, 
\[ (cdbea, dcaeb) = (\Psi_6(cdaeb), \Psi_6(dcaeb)) \in E(C_6) \]
and by Lemmas 7 and 6,
\[ \Phi \]
\[ (bdace, adbec) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i) \]
\[ (\Phi_3(cdabe), \Phi_3(dabe)) = \]
\[ (cdbea, dcaeb) \in E(C_7) - \bigcup_{i=1}^{6} E(C_i) \]

**Theorem 10:** The Hamiltonian cycles \( C \) and \( \Phi_5(C) \) are edge-disjoint.

**Proof:** Let \( v \in V(C) - V(C_7) \), \( v \in V(C_i) \) say, where 1 \( \leq i \leq 6 \). Then, there exist \( u_1, u_2 \in V(C_i) \) such that the edges incident at \( v \) in \( C_i \), \( (u_1, v) \) and \( (v, u_2) \), belong to \( C_i \) and so, by the definition of \( C_i \) in Lemma 7, 
\[ \lambda(u_1, v) = \lambda(v, u_2) = 1 \] (5)

Consider the edges \( (v_1, v) \) and \( (v_2, v) \) incident at \( v \) in \( \Phi_5(C) \). As \( v \notin V(C_7) \), by Lemma 9(i) \( v = \Phi_5(v') \) for some \( v' \in V(C) - V(C_7) \), say \( v' \in C_j \) where 1 \( \leq j \leq 6 \). Then, there exist edges \( (v'_1, v'), (v'_2, v'_2) \) in \( C_j \) such that \( \Phi_5(v'_1) = v_1 \) and \( \Phi_5(v'_2) = v_2 \). By the definition of \( C_j \) in Lemma 7, 
\[ \lambda(v'_1, v') = \lambda(v'_2, v'_2) = 1 \] (6)
By (6) and Lemma 5,
\[ \lambda(\Phi_5(v'_1), \Phi_5(v')) = \lambda(\Phi_5(v'_2), \Phi_5(v'_2)) = 2 \]
i.e. 
\[ \lambda(v_1, v) = \lambda(v, v_2) = 2 \] (7)
By (5) and (7), different edges are incident at the vertex \( v \in C - C_7 \) in \( C \) and \( \Phi_5(C) \). Hence, we have shown that an edge in \( C \), which has a vertex not in \( C_7 \), cannot belong to \( \Phi_5(C) \). It follows that an edge common to both \( C \) and \( \Phi_5(C) \) must be an edge in \( C_7 \). So, let \( e \) be an edge of \( C \) belonging to \( C_7 \). For 1 \( \leq i \leq 6 \), \( e \) cannot be an edge in \( C_i \) as \( C \) does not contain edges common to \( C_7 \) and \( C_i \). By Lemma 9(ii), \( \Phi_5 \) maps an edge in \( C_7 \) and some \( C_i \), where 1 \( \leq i \leq 6 \) to \( e \). Thus, \( \Phi_5 \) maps an edge not in \( C \) to \( e \). Therefore \( e \notin \Phi_5(C) \). We conclude that \( C \) and \( \Phi_5(C) \) have no common edges. ■

VI. Conclusions

We have given a Hamiltonian decomposition of 5-star, based on a graph automorphism relating the two Hamiltonian cycles. Our further work will investigate properties of similar automorphisms in higher degree star graphs to determine whether they can be used to establish or refute the existence of Hamiltonian decompositions there.

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REFERENCES


