ψ-exponential Stability for Non-linear Impulsive Differential Equations

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Abstract—In this paper, we shall present sufficient conditions for the ψ-exponential stability of a class of nonlinear impulsive differential equations. We use the Lyapunov method with functions that are not necessarily differentiable. In the last section, we give some examples to support our theoretical results.

Keywords—Exponential stability, globally exponential stability, impulsive differential equations, Lyapunov function, ψ-stability.

I. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. The impulsive system of differential equations is an adequate apparatus for the mathematical simulation of such processes and phenomena studied in biology, economics and technology etc. That is why, in recent years, the study of such systems has been very intensive [3,11]. One of the most investigated problems in stability analysis of such systems is exponential stability, since it has played an important role in many areas such as control theory, designs and applications of neural networks [7,8].

Lyapunov method and Lyapunov-Razumikhin technique have been successfully utilized in the investigation of asymptotic and exponential stability of impulsive differential systems [2,4,5,10].

Akinleye [9] introduced the notion of ψ-stability of degree k with respect to a function ψ ∈ C(ℝ+,ℝ+) increasing and differentiable on ℝ+ and such that ψ(t) ≥ 1 for t ≥ 0 and lim_{t→∞} ψ(t) = b, b ∈ [1,∞]. In [6], Morachalo introduced the notions of ψ-stability, ψ-uniform stability and ψ-asymptotic stability of trivial solution of the nonlinear system

\[ x = f(t,x), \]

asymptotically stable of trivial solution of the nonlinear system.

II. PRILIMINARIES

Let ℝn denote the Euclidean n-space. Elements of this space are denoted by \( x = (x_1, x_2, ..., x_n)^T \) and their norm is given by \( ||x|| = \max\{|x_1|, |x_2|, ..., |x_n|\} \). For n×n real matrices, we define the norm \( |A| = \max\|Ax\| \). Let \( ψ : ℝ+ → (0,∞), i = 1, 2, ..., n, \) where \( ℝ+ = [0,∞) \) be the continuous functions and let \( ψ = \text{diag}[ψ_1, ψ_2, ..., ψ_n] \).

Consider the impulsive differential system

\[
\frac{dx}{dt} = f(t,x), \quad t \neq t_k, \quad k = 1, 2, ..., n, \quad (1)
\]

\[
x(t_0 + 0) = x_0,
\]

where \( f : ℝ+ × ℝ^n → ℝ^n \) is a nonlinear function, \( I_k : ℝ^n → ℝ^n \) are continuous functions, \( 0 ≤ t_0 < t_1 < t_2 < ... < t_n < t \) are fixed moments of impulse effect and \( \Delta x = I_k(x) = x(t_k + 0) - x(t_k - 0) \).

Here we assume that functions \( f, I_k, k ∈ N \), satisfy all necessary conditions for the global existence and uniqueness of solution for all \( t ≥ t_0 \).

Definition 2.1: Let \( E ⊆ ℝ^n \) be an open set containing the origin. A function \( V : ℝ+ × E → ℝ+ \) is said to belong to class \( V_0 \) if

(i) \( V \) is continuous in each of the sets \( [t_{k-1}, t_k) × E \).

(ii) \( V(t,x) \) is locally Lipschitzian in all \( x ∈ E ⊆ ℝ^n \) and for all \( t ≥ t_0, V(t,0) = 0 \).

(iii) For each \( x ∈ E ⊆ ℝ^n \) and \( t ∈ [t_{k-1}, t_k), k ∈ N, \lim_{t→t_k⁻}(t_k⁻, x)V(t,y) = V(t_k, x) \).

Definition 2.2: Given a function \( V : ℝ+ × E → ℝ+ \), the upper right hand derivative of \( V \) with respect to system (1) is defined by

\[
D^+V(t,x) = \lim_{h→0^+} \sup_h \frac{1}{h}[V(t+h,x(t+h)) - V(t,x)]
\]

for \( (t,x) ∈ ℝ+ × E \).

Definition 2.3: The zero solution of system (1) is ψ-exponentially stable if any solution \( x(t_0,x_0) \) of (1) satisfies

\[
||ψ(t)||x(t_0,x_0) || ≤ \beta(||x_0||, t_0)e^{-\delta(t-t_0)}, \forall \quad t ∈ [t_{k-1}, t_k), \quad k = 1, 2, ..., n \) where \( \beta(h,t) : ℝ+ × ℝ+ → ℝ+ \) is a non-negative function increasing in \( h ∈ ℝ+, \) and \( δ \) is a positive constant.

If the function \( \beta(\cdot) \) in the above definition does not depend on \( t_0, \) the zero solution of (1) is called ψ-uniformly exponentially stable.

Definition 2.4: The zero solution of system (1) is said to be ψ-globally exponentially stable if there exist some constants \( δ > 0 \) and \( M ≥ 1 \) such that for any solution \( x(t,t_0,x_0) \)
of (1), we have \( \| \psi(t)x(t, t_0, x_0) \| \leq M e^{-\delta(t-t_0)} \), \( \forall t \in [t_{k-1}, t_k) \), \( k = 1, 2, \ldots, n \).

Definition 2.5: A function \( V(t, x) \in \mathcal{V}_0 \) is called a Piecewise continuous Lyapunov-\( \psi \)-function for (1) if \( V(t, x) \) is continuously differentiable in \([t_{k-1}, t_k) \), \( k = 1, 2, \ldots, n \) and there exist positive numbers \( \lambda_1, \lambda_2, \lambda_3, L, p, q, r, \delta \) such that

\[
\lambda_1 \| \psi(t)x(t) \|^p \leq V(t, x) \leq \lambda_2 \| \psi(t)x(t) \|^q, \quad \forall t \geq 0, x \in \mathbb{R}^n; \\
D^+ V(t, x) \leq -\lambda_3 \| \psi(t)x(t) \|^r + Le^{-\delta t}, \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n; \\
V(t_k, x(t_k)) \leq V(t_k^+, x(t_k^+)).
\]

Definition 2.6: A function \( V(t, x) \in \mathcal{V}_0 \) is called a generalized Piecewise continuous Lyapunov-\( \psi \)-function for (1) if there exist positive functions \( \lambda_1(t), \lambda_2(t), \lambda_3(t) \), where \( \lambda_1(t) \) is non-decreasing, and there exist positive numbers \( L, p, q, r, \delta \) such that

\[
\lambda_1(t) \| \psi(t)x(t) \|^p \leq V(t, x) \leq \lambda_2(t) \| \psi(t)x(t) \|^q, \quad \forall t \geq 0, x \in \mathbb{R}^n; \\
D^+ V(t, x) \leq -\lambda_3(t) \| \psi(t)x(t) \|^r + Le^{-\delta t}, \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n; \\
V(t_k, x(t_k)) \leq V(t_k^+, x(t_k^+)).
\]

In order to study exponential stability of (1), we need the following comparison principle. Consider a scalar impulsive differential system

\[
\dot{u} = g(t, u), \quad t \neq t_k, \\
\Delta u = G_k(u), \quad t = t_k, \quad k = 1, 2, \ldots, n, \\
u(t_0 + 0) = u_0 
\]

where \( g(t, u) \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+] \) and \( g(t, 0) = 0 \).

Lemma 2.1: Let \( u(t) \) be a maximal solution of above system. If a piecewise continuous function \( v(t) \) with \( v(t) = u_0 \) satisfy

\[
D^+ v(t) \leq g(t, u(t)), \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n, \\
v(t) - v(t_0) \leq \int_{t_0}^t g(s, u(s))ds, \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n.
\]

III. MAIN RESULTS

In this section, we shall present sufficient conditions for the \( \psi \)-exponential stability, \( \psi \)-uniformly exponential stability and \( \psi \)-globally exponential stability of (1) via proposed Piecewise continuous Lyapunov-\( \psi \)-function.

Theorem 3.1: The zero solution of system (1) is \( \psi \)-exponentially stable if it admits a generalized Piecewise continuous Lyapunov-\( \psi \)-function and the following two conditions hold:

\[
\delta > \inf_{t \in \mathbb{R}_+} \frac{\lambda_1(t)}{[\lambda_2(t)]^{r/q}} > 0, \quad t \in [t_{k-1}, t_k); \\
\exists \gamma > 0 \text{ such that } V(t, x) - [V(t, x)]^{r/q} \leq \gamma e^{-\delta t}. \quad (9)
\]

Proof. Let \( x(t) \) be the solution of (1) with \( x(t_0) = x_0 \), where \( t_0 \geq 0 \) is any initial time. Set

\[
Q(t, x(t)) = V(t, x(t))e^{M(t-t_0)}, \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n,
\]

where \( M = \inf_{t \in \mathbb{R}_+} \frac{\lambda_1(t)}{[\lambda_2(t)]^{r/q}} \). We see that \( M < \delta \) and

\[
D^+ Q(t, x) = D^+ V(t, x)e^{M(t-t_0)} + MV(t, x)e^{M(t-t_0)}.
\]

Taking (7) into account, for all \( t \geq t_0, t \neq t_k \), we get

\[
D^+ Q(t, x) \leq \left( -\lambda_3(t) \| \psi(t)x(t) \|^r + Le^{-\delta t} \right)e^{M(t-t_0)} + MV(t, x)e^{M(t-t_0)}, \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n.
\]

From (6) and since, by the assumption, \( \lambda_2(t) > 0 \), we have

\[
\| \psi(t)x(t) \|^q \geq \frac{V(t, x)}{\lambda_2(t)}e^{M(t-t_0)}, \\
\text{equivalently}
\]

\[
-\| \psi(t)x(t) \|^r \leq -\left( \frac{V(t, x)}{\lambda_2(t)} \right)^{r/q}.
\]

Therefore, we have

\[
D^+ Q(t, x) \leq \left\{ \frac{-\lambda_3(t) \| \psi(t)x(t) \|^r + Le^{-\delta t}}{\lambda_2(t)^{r/q}} + Le^{-\delta t} \right\}e^{M(t-t_0)} + MV(t, x)e^{M(t-t_0)}.
\]

Since

\[
\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} \geq M, \quad \forall t \in [t_{k-1}, t_k),
\]

and by the condition (10), we obtain

\[
D^+ Q(t, x) \leq M \left\{ V(t, x) - [V(t, x)]^{r/q} \right\}e^{M(t-t_0)} + Le^{-\delta t}e^{M(t-t_0)}
\]

\[
\leq \left( L + M\gamma \right)e^{-\delta(t-t_0)}e^{M(t-t_0)}.
\]

Therefore, \( D^+ Q(t, x) \leq \left( L + M\gamma \right)e^{(M-\delta)(t-t_0)}, \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n. \)

By Lemma 2.1 to the case

\[
\nu(t) = Q(t, x(t)), \quad g(t, u(t)) = (L + M\gamma)e^{(M-\delta)(t-t_0)},
\]

we obtain

\[
Q(t, x(t)) - Q(t_0, x_0) \leq \int_{t_0}^t (L + M\gamma)e^{(M-\delta)(s-t_0)}ds \quad t \neq t_k
\]

\[
= \frac{1}{M - \delta} \left( e^{(M-\delta)(t-t_0)} - 1 \right).
\]

Setting \( \delta_1 = -(M - \delta) \), by condition (9), we have \( \delta_1 > 0 \) and

\[
Q(t, x(t)) \leq Q(t_0, x_0) + \frac{L + M\gamma}{\delta_1} - \frac{L + M\gamma}{\delta_1}e^{(M-\delta)(t-t_0)} \leq Q(t_0, x_0) + \frac{L + M\gamma}{\delta_1}.
\]

Since \( Q(t_0, x_0) = V(t_0, x_0) \leq \lambda_2(t_0)\| \psi(t_0)x_0 \|^q \), we get

\[
Q(t, x(t)) \leq \lambda_2(t_0)\| \psi(t_0)x_0 \|^q + \frac{L + M\gamma}{\delta_1}.
\]
Letting
\[ \lambda_2(t_0)\|\psi(t_0)x_0\|^q + \frac{L + M\gamma}{\delta_1} = \beta(\|x_0\|, t_0) > 0, \]
we have
\[ Q(t, x(t)) \leq \beta(\|x_0\|, t_0), \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n. \]
(11)
Furthermore, from condition (6), it follows that
\[ \lambda_1(t)\|\psi(t)x(t)\|^p \leq V(t, x(t)), \]
\[ \|\psi(t)x(t)\| \leq \left\{ \frac{V(t, x(t))}{\lambda_1(t)} \right\}^{1/p}. \]
Since \( \lambda_1(t) \) is non-decreasing, \( \lambda_1(t_0) \geq \lambda_1(t_0) \), we have
\[ \|\psi(t)x(t)\| \leq \left\{ \frac{V(t, x(t))}{\lambda_1(t_0)} \right\}^{1/p}. \]
Substituting
\[ V(t, x) = \frac{Q(t, x)}{\lambda_1(t_0)} \]
into the last inequality, we obtain
\[ \|\psi(t)x(t)\| \leq \left\{ \frac{Q(t, x(t))}{\lambda_1(t_0)} \right\}^{1/p}. \]
(12)
Combining (11) and (12),
\[ \|\psi(t)x(t)\| \leq \left\{ \frac{\beta(\|x_0\|, t_0)}{\lambda_1(t_0)} \right\}^{1/p} = \left\{ \frac{\beta(\|x_0\|, t_0)}{\lambda_1(t_0)} \right\}^{1/p} e^{-\frac{M}{\lambda_1(t_0)}(t-t_0)}, \]
\[ \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n, \]
which shows that system (1) is \( \psi \)-exponentially stable and hence the Theorem.

If we consider \( \psi \) as scaler function independent of \( t \), then we get a sufficient condition for \( \psi \)-uniformly exponential stability as stated below:

**Theorem 3.2** Let \( \psi \) be a constant function independent of \( t \). Then the zero solution of system (1) is \( \psi \)-uniformly exponentially stable, if it admits a piecewise continuous Lyapunov-\( \psi \) function and the following two conditions hold:
\[ \frac{a}{\lambda_2} < \frac{\lambda_1}{\lambda_2} \]
\[ \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n. \]
(13)

Proof. The proof is in the same line of the proof of Theorem 3.1, so omitted.

**Theorem 3.3** The zero solution of system (1) is \( \psi \)-globally exponentially stable, if it admits a piecewise continuous Lyapunov-\( \psi \) function with \( p = q = r \) and \( \delta \), with \( \delta > \frac{\lambda_1}{\lambda_2} \).

Proof. Let
\[ Q(t, x) = V(t, x)e^{\lambda_1(t)/\lambda_2}, \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, \ldots, n. \]
(16)

Then from (3), (4) and (16),
\[ D^+Q(t, x) = D^+V(t, x)e^{\lambda_1(t)/\lambda_2} + \frac{\lambda_3}{\lambda_2}V(t, x)e^{\lambda_1(t)/\lambda_2}, \quad t \neq t_k \]
\[ \leq (-\lambda_3\|\psi(t)x(t)\|^p + Le^{-\delta t})e^{\lambda_1(t)/\lambda_2} + \frac{\lambda_3}{\lambda_2}Q(t, x) \]
\[ \leq (-\frac{\lambda_3}{\lambda_2}V(t, x) + Le^{-\delta t})e^{\lambda_1(t)/\lambda_2} + \frac{\lambda_3}{\lambda_2}Q(t, x) \]
\[ = Le^{(-\delta + \frac{\lambda_3}{\lambda_2})t}, \quad t \neq t_k, \]
(17)
where \( \beta = \delta - \frac{\lambda_3}{\lambda_2} \).

Integrating both sides (17) from \( t_0 \) to \( t \), \( t \neq t_k \), we get
\[ Q(t, x) - Q(t_0, x_0) \leq \beta^{-1}\left[ Le^{-\beta(t-t_0)} - e^{-\beta t} \right] \]
\[ \leq \beta^{-1}L(\delta - \frac{\lambda_3}{\lambda_2})^{-1}. \]
(18)

Therefore
\[ Q(t, x) \leq Q(t_0, x_0) + L(\delta - \frac{\lambda_3}{\lambda_2})^{-1}a, \]
where \( a = Q(t_0, x_0) + L(\delta - \frac{\lambda_3}{\lambda_2})^{-1} \).
From (3), (16) and (18) we have,
\[ \|\psi(t)x(t)\| \leq \left\{ \frac{Q(t, x(t))}{\lambda_1(t_0)} \right\}^{1/p} \]
\[ = \left\{ \frac{Q(t, x(t))}{\lambda_1(t_0)} \right\}^{1/p} e^{-\frac{M}{\lambda_1(t_0)}(t-t_0)}, \]
\[ \forall t \in [t_{k-1}, t_k), \]
\[ \left\{ \frac{Q(t, x(t))}{\lambda_1(t_0)} \right\}^{1/p} e^{-\frac{M}{\lambda_1(t_0)}(t-t_0)} \leq M_+e^{-\eta(t-t_0)}, \quad \forall t \in [t_{k-1}, t_k), \]
where \( M = (a/\lambda_3)^{1/p} \) and \( \eta = \lambda_3/(\lambda_3p) \).
This completes the proof.

**IV. EXAMPLES**

In this section, we give some examples to support our results in above section.

**Example 4.1** Consider an impulsive differential equation
\[ \dot{x} = \frac{1}{6}e^{x^2} + x + e^{-3t/2} \cos t, \quad t \neq k\pi/4, k = 1, 2, \ldots, \]
\[ \Delta x = -1/2, \quad t = k\pi/4. \]
(19)
Let \( \psi(t) = e^t \) and a piecewise continuous Lyapunov-\( \psi \) function \( V(t, x) \) \( \forall_0 \) with \( E = \{x : x \leq 1\} \) given by
\[ V(t, x) = e^{-t^2/2}x^6. \]
Then (6) holds for \( p = q = 6, \lambda_1(t) = e^{-3t/2}, \lambda_2(t) = e^{-6t} \).
Now
\[ \dot{V}(t, x) = -\frac{1}{2}e^{-t^2/2}x^6 - e^{-t^2/2}x^{28/5} + 6e^{-2t}x^5 \cos t \]
\[ \leq -e^{-t^2/2}x^{28/5} + 6e^{-2t}, \quad t \neq k\pi/4. \]
It follows that conditions (7) holds for \( \lambda_3(t) = e^{-5t/10}, L = 6, r = \frac{20}{3}, \delta = 2. \)
Now inf \( \lambda_2(t)/(\lambda_2(t))^{1/r} = \inf \lambda^2/2 = 1 < 2 = \delta \)
and
\[ V(t, x) - V(t, x)^{1/r} = e^{-t^2/2}x^6 - e^{-7t/15}x^{28/5} \leq 0 \leq e^{-2t}, \quad t \neq k\pi/4. \]
Hence by Theorem 3.1, the zero solution of (19) is \(\psi\)-exponentially stable.

**Example 4.2** Consider impulsive differential equation

\[
\dot{x} = -\frac{1}{3}x^3 + x e^{-2t}, \quad t \neq k, \quad k = 1, 2, \ldots, \tag{20}
\]

\[
\Delta x = -1, \quad t = k.
\]

Let \(\psi(t) = \frac{1}{t}\) and a piecewise continuous Lyapunov-\(\psi\) function \(V(t, x) \in \mathcal{V}_0\) with \(\mathcal{E} = \{x : |x| \leq 1\}\) is

\[
V(t, x) = \begin{cases} 
  x^3 & \text{for } x \geq 0 \\
  -x^3 & \text{for } x < 0
\end{cases}
\]

Now

\[
\dot{V}(t, x) = \begin{cases} 
  -x^3 + 3x^2 e^{-2t} & \text{for } x \geq 0 \\
  x^2 - 3x^2 e^{-2t} & \text{for } x < 0
\end{cases}
\]

\[
=- |x|^2 + 3 |x|^3 e^{-2t} \leq - |x|^2 + 3 e^{-2t}
\]

It follows that conditions (3) and (4) holds for \(\lambda_1 = 1, \lambda_2 = 16, \lambda_3 = 2^{11}, p = q = 3, L = 3, r = 9\) and \(\delta = 2\).

Now \(\lambda_2 / \lambda_3 < 2 < \delta\) and

\[
V(t, x) - [V(t, x)]^{r/q} \leq |x|^{3/2} \left(|x|^{3/2} - 1\right) \leq e^{-2t}, \quad t \neq k.
\]

Hence the zero solution of (20) is \(\psi\)-uniformly exponentially stable.

**Example 4.3** Consider impulsive differential equation

\[
\dot{x} = -x^2 + x^2 e^{-3t}, \quad t \neq k, \quad k = 1, 2, \ldots, \tag{21}
\]

\[
\Delta x = -2, \quad t = k.
\]

Let \(\psi(t) = 1\) and a piecewise continuous Lyapunov-\(\psi\) function \(V(t, x) \in \mathcal{V}_0\) with \(\mathcal{D} = \{x : |x| \leq 1\}\) given by

\[
V(t, x) = |x|^{1/2} = \begin{cases} 
  x^{1/2} & \text{for } x \geq 0 \\
  -x^{1/2} & \text{for } x < 0
\end{cases}
\]

Now

\[
\dot{V}(t, x) = \begin{cases} 
  -\frac{1}{2}x^0 + \frac{3}{2}x^{3/2} e^{-3t} & \text{for } x \geq 0 \\
  \frac{1}{2}x^0 - \frac{3}{2}x^{3/2} e^{-3t} & \text{for } x < 0
\end{cases}
\]

\[
= -\frac{1}{2} |x|^0 + \frac{3}{2} |x|^{3/2} e^{-3t} \leq -\frac{1}{2} |x|^2 + \frac{1}{2} e^{-3t}, \quad t \neq k.
\]

It follows that conditions (3) and (4) holds for \(\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1/2, p = q = r = 1/2, L = 1/2, \delta = 3\).

Now \(\lambda_2 / \lambda_3 = 1/2 < 3 = \delta\).

Hence the zero solution of (21) is \(\psi\)-globally exponentially stable.

**REFERENCES**


