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Generalized module homomorphisms of triangular matrix rings of order three

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Abstract—Let T, U and V be rings with identity and M be a unitary (T, U)-bimodule, N be a unitary (U, V)-bimodule, D be a unitary (T, V)-bimodule . We characterize homomorphisms and isomorphisms of the generalized matrix ring $\Gamma = \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix}$.

Keywords—generalized module homomorphism, ring homomorphism, isomorphisms, triangular matrix rings.

I. Introduction

THROUGHOUT the paper all rings are assumed to have identity and all modules are unitary. The additive map $\delta : R \longrightarrow R$ is called a derivation, if for each $a, b \in R, \delta(ab) = a\delta(b)+\delta(a)b$. For an element $x \in R$, the mapping I_x, given by $I_x(a) = ax - xa$, for each $a \in R$, is called an inner derivation of R. Derivations of the algebra of triangular matrices and some class of their subalgebras have been the object of active research for a long time [1, 2, 3, 4,5]. Coelho and Milies provided in [2] a description of the derivations in Tn(R), the upper triangular matrices over R. They proved that every derivation is the sum of an inner derivation and another one induced from R. Jondrup in [5] gave a new proof of this result. A similar result for full matrix rings appears in [3],and the special case where R is an algebra over a field, with char$(R) \neq 2, 3$and $n > 2$, is given in [1]. The case of upper triangular matrix rings over a simple algebra finite dimensional over its center appears in [3].

A large class of ring extensions which have a generalized triangular matrix representations is investigated by Birkenmeier et al. in [6]. A description of homomorphisms and derivations of generalized matrix rings $T := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ assuming no restrictions on R, S, M, other than the existence of the identity element in [7]. Analysts have studied these derivations in the context of algebras on certain normed spaces. Many widely studied algebras, including upper triangular matrix algebras, nest algebras and triangular Banach algebras, may be viewed as triangular algebras.

In this paper, we generalized the result of [7],and give a description of homomorphisms and isomorphisms of generalized matrix rings of three, denoted by $\Gamma := \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix}$ assuming no restrictions on T, U, V and M, N, D , other than the existence of the identity element, which devoted to determine the derivations of generalized matrix rings of order three.

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II. Generalized module homomorphisms

In order to describe homomorphisms of the generalized matrix rings, firstly we introduce the generalized matrix rings of order three [8]. Let $T, U, V,$ be rings, M an (T, U)-bimodule,N an (U, V)-bimodule, D an (T, V)-bimodule and $\eta : M \otimes N \longrightarrow D$ is a (T, V)-bimodule homomorphism. We defined $\Gamma := \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix}$. For any

$$\begin{pmatrix} t_1 & m_1 & d_1 \\ 0 & u_1 & n_1 \\ 0 & 0 & v_1 \end{pmatrix}, \begin{pmatrix} t_2 & m_2 & d_2 \\ 0 & u_2 & n_2 \\ 0 & 0 & v_2 \end{pmatrix} \in \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix},$$

we define the addition and multiplication operations as follows:

$$\begin{pmatrix} t_1 & m_1 & d_1 \\ 0 & u_1 & n_1 \\ 0 & 0 & v_1 \end{pmatrix} + \begin{pmatrix} t_2 & m_2 & d_2 \\ 0 & u_2 & n_2 \\ 0 & 0 & v_2 \end{pmatrix} =$$

$$\begin{pmatrix} t_1+t_2 & m_1+m_2 & d_1+d_2 \\ 0 & u_1+u_2 & n_1+n_2 \\ 0 & 0 & v_1+v_2 \end{pmatrix};$$

$$\begin{pmatrix} t_1 & m_1 & d_1 \\ 0 & u_1 & n_1 \\ 0 & 0 & v_1 \end{pmatrix} \begin{pmatrix} t_2 & m_2 & d_2 \\ 0 & u_2 & n_2 \\ 0 & 0 & v_2 \end{pmatrix} =$$

$$\begin{pmatrix} t_1t_2 & t_1m_2+m_1u_2 & t_1d_2+\eta(m_1\otimes n_2)+d_1v_2 \\ 0 & u_1u_2 & u_1n_2+n_1v_2 \\ 0 & 0 & v_1v_2 \end{pmatrix}.$$

It is clear that Γ is a ring based on the addition and multiplication operations above.

Next we study the notion of generalized module homomorphisms.

Definitions 2.1 Let T, T', U, U', V, V' be rings, M an (T, U)-bimodule, M' an (T', U')-bimodule; N an (U, V)-bimodule, N' an (U', V')-bimodule; D an (T, V)-bimodule , D' an (T', V')-bimodule , $\eta : M \otimes N \longrightarrow D$ is a (T, V)-bimodule homomorphism . Assume that $\varphi_1 : T \longrightarrow T', \varphi_2 : U \longrightarrow U'$ and $\varphi_3 : V \longrightarrow V'$ be ring homomorphisms. Then an additive mapping $f = (f_1, f_2, f_3)$ is called a generalized module homomorphism related to $(\varphi_1, \varphi_2, \varphi_3)$, if

$f_1 : M \longrightarrow M', f_2 : N \longrightarrow N'$and $f_3 : D \longrightarrow D'$ are module homomorphisms such that

$$\eta(f_1(m) \otimes f_2(n)) = f_3(\eta(m \otimes n)),$$

$$f_1(tm) = \varphi_1(t)f_1(m), f_1(mu) = f_1(m)\varphi_2(u);$$

$$f_2(un) = \varphi_2(u)f_2(n), f_2(nv) = f_2(n)\varphi_3(v);$$

$$f_3(td) = \varphi_1(t)f_3(d), f_3(dv) = f_3(d)\varphi_3(v)$$

for each $t \in T, u \in U, v \in V, m \in M, n \in N$ and $d \in D$.

Lemma 2.2 Let M an (T, U)-bimodule, M' an (T', U')-bimodule; N an (U, V)-bimodule, N' an (U', V')-bimodule; D an (T, V)-bimodule , D' an (T', V')-bimodule and $f = (f_1, f_2, f_3)$ is a generalized module homomorphism related to $(\varphi_1, \varphi_2, \varphi_3)$. Then the mapping

$$\psi : \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix} \longrightarrow \begin{pmatrix} T' & M' & D' \\ 0 & U' & N' \\ 0 & 0 & V' \end{pmatrix},$$

given by

$$\psi\left[\begin{pmatrix} t & m & d \\ 0 & u & n \\ 0 & 0 & v \end{pmatrix}\right] = \begin{pmatrix} \varphi_1(t) & f_1(m) & f_3(d) \\ 0 & \varphi_2(u) & f_2(n) \\ 0 & 0 & \varphi_3(v) \end{pmatrix}$$

is a ring homomorphisms.

Proof Clearly ψ is additive. We have

$$\psi\left[\begin{pmatrix} t_1 & m_1 & d_1 \\ 0 & u_1 & n_1 \\ 0 & 0 & v_1 \end{pmatrix}\begin{pmatrix} t_2 & m_2 & d_2 \\ 0 & u_2 & n_2 \\ 0 & 0 & v_2 \end{pmatrix}\right] =$$

$$\psi\left[\begin{pmatrix} t_1t_2 & t_1m_2 + m_1u_2 & t_1d_2 \\ & & +\eta(m_1 \otimes n_2) + d_1v_2 \\ 0 & u_1u_2 & u_1n_2 + n_1v_2 \\ 0 & 0 & v_1v_2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} \varphi_1(t_1t_2) & f_1(t_1m_2 + m_1u_2) & f_3(t_1d_2) \\ & & +f_3(\eta(m_1 \otimes n_2)) \\ & & +f_3(d_1v_2) \\ 0 & \varphi_2(u_1u_2) & f_2(u_1n_2 + n_1v_2) \\ 0 & 0 & \varphi_3(v_1v_2) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_1(t_1) & f_1(m_1) & f_3(d_1) \\ 0 & \varphi_2(u_1) & f_2(n_1) \\ 0 & 0 & \varphi_3(v_1) \end{pmatrix}$$

$$\begin{pmatrix} \varphi_1(t_2) & f_2(m_2) & f_3(d_2) \\ 0 & \varphi_2(u_2) & f_2(n_2) \\ 0 & 0 & \varphi_3(v_2) \end{pmatrix}$$

$$= \psi\left[\begin{pmatrix} t_1 & m_1 & d_1 \\ 0 & u_1 & n_1 \\ 0 & 0 & v_1 \end{pmatrix}\right]\psi\left[\begin{pmatrix} t_2 & m_2 & d_2 \\ 0 & u_2 & n_2 \\ 0 & 0 & v_2 \end{pmatrix}\right]$$

Theorem 2.3. Let T, T', U, U', V, V' be rings, and M an (T, U)-bimodule, M' an (T', U')-bimodule; N an (U, V)-bimodule, N' an (U', V')-bimodule; D an (T, V)-bimodule , D' an (T', V')-bimodule. If

$$\psi : \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix} \longrightarrow \begin{pmatrix} T' & M' & D' \\ 0 & U' & N' \\ 0 & 0 & V' \end{pmatrix}$$

is a mapping. Then the followings are equivalent:

$$(1)\psi\left[\begin{pmatrix} t & m & d \\ 0 & u & n \\ 0 & 0 & v \end{pmatrix}\right] = \begin{pmatrix} \varphi_1(t) & f_1(m) & f_3(d) \\ 0 & \varphi_2(u) & f_2(n) \\ 0 & 0 & \varphi_3(v) \end{pmatrix}$$

,where $\varphi_1 : T \longrightarrow T', \varphi_2 : U \longrightarrow U', \varphi_3 : V \longrightarrow V'$ are ring homomorphisms and $f = (f_1, f_2, f_3)$ is a generalized module homomorphism related to $(\varphi_1, \varphi_2, \varphi_3)$;

(2)ψ is a ring homomorphisms such that $\psi(TE_{11}) \subseteq T'E_{11}, \psi(UE_{22}) \subseteq U'E_{22}$ and $\psi(VE_{33}) \subseteq V'E_{33}$.

Proof $(1) \Longrightarrow (2)$ is clearly follows from Lemma 2.2.

$(2) \Longrightarrow (1)$ The mapping $\varphi_1 : T \longrightarrow T', \varphi_2 : U \longrightarrow U', \varphi_3 : V \longrightarrow V'$ defined by $\psi(tE_{11}) = \varphi_1(t)E_{11}, \psi(uE_{22}) = \varphi_2(u)E_{22}$ and $\psi(vE_{33}) = \varphi_3(v)E_{33}$ for each $t \in T, u \in U$ and $v \in V$. By considering the effect of ψ on $\begin{pmatrix} t_1 + t_2 & 0 & 0 \\ 0 & u_1 + u_2 & 0 \\ 0 & 0 & v_1 + v_2 \end{pmatrix}$, we see that $\varphi_1, \varphi_2, \varphi_3$ are additive and

$$\psi\left[\begin{pmatrix} t_1t_2 & 0 & 0 \\ 0 & u_1u_2 & 0 \\ 0 & 0 & v_1v_2 \end{pmatrix}\right]$$

$$= \psi\left[\begin{pmatrix} t_1 & 0 & 0 \\ 0 & u_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}\right]\psi\left[\begin{pmatrix} t_2 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & v_2 \end{pmatrix}\right]$$

So we have

$$\begin{pmatrix} \varphi_1(t_1t_2) & 0 & 0 \\ 0 & \varphi_2(u_1u_2) & 0 \\ 0 & 0 & \varphi_3(v_1v_2) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_1(t_1)\varphi_1(t_2) & 0 & 0 \\ 0 & \varphi_2(u_1)\varphi_2(u_2) & 0 \\ 0 & 0 & \varphi_3(v_1)\varphi_3(v_2) \end{pmatrix}.$$

Hence we have $\varphi_1(t_1t_2) = \varphi_1(t_1)\varphi_1(t_2), \varphi_2(u_1u_2) = \varphi_2(u_1)\varphi_2(u_2)$ $\varphi_3(v_1v_2) = \varphi_3(v_1)\varphi_3(v_2)$ and $\varphi_1, \varphi_2, \varphi_3$ are ring homomorphisms.

Now assume that

$$\psi(mE_{12}) = \begin{pmatrix} \alpha_1(m) & f_1(m) & g_1(m) \\ 0 & \beta_1(m) & h_1(m) \\ 0 & 0 & \gamma_1(m) \end{pmatrix}$$

for some $\alpha_1 : M \longrightarrow T', \beta_1 : M \longrightarrow U', \gamma_1 : M \longrightarrow V', f_1 : M \longrightarrow M', g_1 : M \longrightarrow D', h_1 : M \longrightarrow N'$. Then , for each $m \in M$, we have , $\psi(mE_{12}) = \psi(E_{11}mE_{12}) = \varphi_1(1)E_{11}\psi(mE_{12})$. So, $\begin{pmatrix} \alpha_1(m) & f_1(m) & g_1(m) \\ 0 & \beta_1(m) & h_1(m) \\ 0 & 0 & \gamma_1(m) \end{pmatrix} = \begin{pmatrix} \varphi_1(1)\alpha_1(m) & \varphi_1(1)f_1(m) & \varphi_1(1)g_1(m) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and hence $\beta_1(m) = 0, \gamma_1(m) = 0, h_1(m) = 0$. So $\psi(mE_{12}) = \psi(mE_{12}E_{22}) = \psi(mE_{12})\varphi_2(1)E_{22}$. Thus, we have $\begin{pmatrix} \alpha_1(m) & f_1(m) & g_1(m) \\ 0 & \beta_1(m) & h_1(m) \\ 0 & 0 & \gamma_1(m) \end{pmatrix} = \begin{pmatrix} 0 & f_1(m)\varphi_2(1) & 0 \\ 0 & \beta_1(m)\varphi_2(1) & 0 \\ 0 & \gamma_1(m)\varphi_2(1) & 0 \end{pmatrix}$ and so $g_1(m) = 0$. Therefore $\psi(mE_{12}) = f_1(m)E_{12}$, we have $\psi(tmE_{12}) = \psi(tE_{11})\psi(mE_{12})$ and hence $f_1(tm)E_{12} =$

$\varphi_1(t)f_1(m)E_{12}$. Thus $f_1(tm) = \varphi_1(t)f_1(m)$. Similarly $f_1(mu) = f_1(m)\varphi_2(u)$.

Next ,we assume that $\psi(nE_{23}) = \begin{pmatrix} \alpha_2(n) & h_2(n) & g_2(n) \\ 0 & \beta_2(n) & f_2(n) \\ 0 & 0 & \gamma_2(n) \end{pmatrix}$ for some $\alpha_2 : N \longrightarrow T', \beta_2 : N \longrightarrow U', \gamma_2 : N \longrightarrow V', h_2 : N \longrightarrow M', g_2 : N \longrightarrow D', f_2 : N \longrightarrow N'$. By the same method ,we proof that $\beta_2 = \gamma_2 = \alpha_2 = g_2 = h_2 = 0$ and $f_2(un) = \varphi_2(u)f_2(n)$ and $f_2(nv) = f_2(n)\varphi_3(v)$ for some $u \in U, v \in V$.

At last , assume that

$$\psi(dE_{13}) = \begin{pmatrix} \alpha_3(d) & h_3(d) & f_3(d) \\ 0 & \beta_3(d) & g_3(d) \\ 0 & 0 & \gamma_3(d) \end{pmatrix}$$

for some $\alpha_3 : D \longrightarrow T', \beta_2 : D \longrightarrow U', \gamma_3 : D \longrightarrow V', h_3 : D \longrightarrow U', f_3 : D \longrightarrow D', g_3 : D \longrightarrow N'$. Then for each $d \in D$, we have

$$\psi(dE_{13}) = \psi(E_{11}dE_{13}) = \varphi_1(1)E_{11}\psi(dE_{13}).$$

So

$$\begin{pmatrix} \alpha_3(d) & h_3(d) & f_3(d) \\ 0 & \beta_3(d) & g_3(d) \\ 0 & 0 & \gamma_3(d) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi_1(1)\alpha_3(d) & \varphi_1(1)h_3(d) & \varphi_1(1)f_3(d) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and hence $\beta_3 = g_3 = \gamma_3 = 0$.So

$$\psi(dE_{13}) = \psi(dE_{13}E_{33}) = \psi(dE_{13})\varphi_3(1)E_{33}.$$

Thus we have

$$\begin{pmatrix} \alpha_3(d) & h_3(d) & f_3(d) \\ 0 & \beta_3(d) & g_3(d) \\ 0 & 0 & \gamma_3(d) \end{pmatrix} = \begin{pmatrix} 0 & 0 & f_3(d)\varphi_1(1) \\ 0 & 0 & g_3(d)\varphi_1(1) \\ 0 & 0 & \gamma_3(d)\varphi_1(1) \end{pmatrix},$$

and so $\alpha_3 = h_3 = 0$. Therefore, $\psi(dE_{13}) = f_3(d)E_{13}$. We have $\psi(tdE_{13}) = \psi(tE_{11})\psi(dE_{13})$, and hence $f_3(td) = \varphi_1(t)f_3(d)$ for each $t \in T$.

Similarly, we have $f_3(dv) = f_3(d)\varphi_3(v)$ for each $v \in V$. Since ψ is a ring homomorphism , for each $m \in M, n \in N$, $\psi(mE_{12}nE_{23}) = \psi(mE_{12})\psi(nE_{23})$, and we know that $\psi(mE_{12}nE_{23})$

$$= \psi\left[\begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \psi\left[\begin{pmatrix} 0 & 0 & \eta(m \otimes n) \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix}\right] = f_3(\eta(m \otimes n))E_{13},$$

$\psi(mE_{12})\psi(nE_{23})$

$$= \psi\left[\begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\psi\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & f_1(m) & 0 \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f_2(n) \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \eta(f_1(m) \otimes f_2(n)) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \eta(f_1(m) \otimes f_2(n))E_{13}$$

Therefore $f_3(\eta(m \otimes n)) = \eta(f_1(m) \otimes f_2(n))$. Therefore, we have $\psi : \begin{pmatrix} t & m & d \\ 0 & u & n \\ 0 & 0 & v \end{pmatrix} \longrightarrow \begin{pmatrix} \varphi_1(t) & f_1(m) & f_3(d) \\ 0 & \varphi_2(u) & f_2(n) \\ 0 & 0 & \varphi_2(v) \end{pmatrix}$ and that $\varphi_1, \varphi_2, f_1, f_2, f_3$ satisfy the required condition.

Proposition 2.4 If

$$\begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix} and \begin{pmatrix} T' & M' & D' \\ 0 & U' & N' \\ 0 & 0 & V' \end{pmatrix}$$

have the identify elements and

$$\psi : \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix} \longrightarrow \begin{pmatrix} T' & M' & D' \\ 0 & U' & N' \\ 0 & 0 & V' \end{pmatrix}$$

is a ring homomorphism such that $\psi(E_{11}) = E_{11}, \psi(E_{22}) = E_{22}, \psi(E_{33}) = E_{33}$. Then ψ satisfies the condition (1) and (2) of Theorem 2.3.

Proof. Let $\psi(tE_{11}) = \begin{pmatrix} \alpha_1(t) & g_1(t) & h_1(t) \\ 0 & \beta_1(t) & \mu_1(t) \\ 0 & 0 & \gamma_1(t) \end{pmatrix}$ for some $\alpha_1 : T \longrightarrow T', \beta_1 : T \longrightarrow U', \gamma_1 : T \longrightarrow V', h_1 : T \longrightarrow D', g_1 : T \longrightarrow M', \mu_1 : T \longrightarrow N'$. We have $\psi(tE_{11})\psi(E_{11})$. So $\begin{pmatrix} \alpha_1(t) & g_1(t) & h_1(t) \\ 0 & \beta_1(t) & \mu_1(t) \\ 0 & 0 & \gamma_1(t) \end{pmatrix}$

$$= \begin{pmatrix} \alpha_1(t) & g_1(t) & h_1(t) \\ 0 & \beta_1(t) & \mu_1(t) \\ 0 & 0 & \gamma_1(t) \end{pmatrix} E_{11} = \alpha_1(t)E_{11}.$$

Hence

$$g_1 = h_1 = \beta_1 = \mu_1 = \gamma_1 = 0.$$

So $\psi(tE_{11}) = \alpha_1(t)E_{11}$, and $\psi(tE_{11}) \subseteq T'$.

Similarly, we have

$$\psi(uE_{22}) = \begin{pmatrix} \alpha_2(u) & g_2(u) & h_2(u) \\ 0 & \beta_2(u) & \mu_2(u) \\ 0 & 0 & \gamma_2(u) \end{pmatrix}$$

for some $\alpha_2 : U \longrightarrow T', \beta_2 : U \longrightarrow U', \gamma_2 : U \longrightarrow V', h_2 : U \longrightarrow D', g_2 : U \longrightarrow M', \mu_2 : U \longrightarrow N'$.

But, $\psi(uE_{22}) = \psi(E_{22})\psi(uE_{22})$, and

$$\begin{pmatrix} \alpha_2(u) & g_2(u) & h_2(u) \\ 0 & \beta_2(u) & \mu_2(u) \\ 0 & 0 & \gamma_2(u) \end{pmatrix}$$

$$= E_{22}\begin{pmatrix} \alpha_2(u) & g_2(u) & h_2(u) \\ 0 & \beta_2(u) & \mu_2(u) \\ 0 & 0 & \gamma_2(u) \end{pmatrix} = \alpha_2(u)E_{22}.$$

So,

$$g_2 = h_2 = \beta_2 = \mu_2 = \gamma_2 = 0.$$

Thus, $\psi(uE_{22}) = \beta_2(u)E_{22}$ and hence $\psi(uE_{22}) \subseteq U'E_{22}$.

At last, we assume that $\psi(vE_{33}) = \begin{pmatrix} \alpha_3(v) & g_3(v) & h_3(v) \\ 0 & \beta_3(v) & \mu_3(v) \\ 0 & 0 & \gamma_3(v) \end{pmatrix}$ for each $v \in V$,where

$\alpha_3 : V \longrightarrow T', \beta_3 : V \longrightarrow U', \gamma_3 : V \longrightarrow V', h_3 : V \longrightarrow D', g_3 : V \longrightarrow M', \mu_3 : V \longrightarrow N'$. By the same method , we have $\psi(vE_{33}) = \gamma_3(v)E_{33}$ and hence $\psi(vE_{33}) \subseteq V'E_{33}$. Therefore ψ satisfies the condition (2) of Theorem 2.3.

Example 2.5 The converse of Proposition 2.4 is not true, in general. Let M be a unitary (T, U)-bimodule,N be a unitary (U, V)-bimodule and D be a unitary (T, V)-bimodule . Then, we make M, N, D unitary $T \times U \times V$-bimodules by defining $(t, u, v)m := tm, m(t, u, v) := mu; (t, u, v)n = un, n(t, u, v) = nv, (t, u, v)d = td$ and $d(t, u, v) = dv$, respectively, for each $m \in M, n \in N, d \in D, t \in T, u \in U$ and $v \in V$. Define $\varphi_1 : T \longrightarrow T \times U \times V, \varphi_2 : U \longrightarrow T \times U \times V, \varphi_2 : U \longrightarrow T \times U \times V$, given by $\varphi_1(t) = (t, 0, 0), \varphi_2(u) = (0, u, 0)$ and $\varphi_3(v) = (0, 0, v)$ for each $t \in T, v \in V$ and $u \in U$. Then, φ_1, φ_2 and φ_3 are ring homomorphisms. Let $f_1 \in \mathrm{Hom}({}_TM_U, {}_TM_U)$,$f_2 \in \mathrm{Hom}({}_UN_V, {}_UN_V)$ and $f_3 \in \mathrm{Hom}({}_TD_V, {}_TD_V)$, such that $\eta(f_1(m) \otimes f_2(n)) = f_3(\eta(m \otimes n))$ for each $m \in M, n \in N$. Now we see that $f = (f_1, f_2, f_3)$ is a generalized module homomorphism related to $(\varphi_1, \varphi_2, \varphi_3)$. Since

$$f_1(tm) = tf_1(m) = (t, 0, 0)f_1(m) = \varphi_1(t)f_1(m),$$

$$f_1(mu) = f_1(m)u = f_1(m)(0, u, 0) = f_1(m)\varphi_2(u);$$

$$f_2(un) = uf_1(n) = (0, u, 0)f_2(n) = \varphi_2(u)f_2(n),$$

$$f_2(nv) = f_2(n)v = f_2(n)(0, 0, v) = f_2(n)\varphi_3(v);$$

$$f_3(td) = tf_3(d) = (t, 0, 0)f_3(d) = \varphi_1(t)f_3(d),$$

$$f_3(dv) = f_3(d)v = f_3(d)(0, 0, v) = f_3(d)\varphi_3(v).$$

Thus, the mapping

$$\psi : \begin{pmatrix} T & M & D \\ 0 & U & N \\ 0 & 0 & V \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} T \times U \times V & M & D \\ 0 & T \times U \times V & N \\ 0 & 0 & T \times U \times V \end{pmatrix}$$

given by

$$\psi : \begin{pmatrix} t & m & d \\ 0 & u & n \\ 0 & 0 & v \end{pmatrix} \longrightarrow \begin{pmatrix} \varphi_1(t) & f_1(m) & f_3(d) \\ 0 & \varphi_2(u) & f_2(n) \\ 0 & 0 & \varphi_3(v) \end{pmatrix}$$

is ring homomorphisms and we have $\psi(E_{11}) = \varphi_1(1)E_{11} = (1, 0, 0)E_{11}, \psi(E_{22}) = \varphi_2(1)E_{22} = (0, 1, 0)E_{22}, \psi(E_{33}) = \varphi_3(1)E_{33} = (0, 0, 1)E_{33}$, Note that $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are not the identity elements of $T \times U \times V$.

Proposition 2.6 Let T, T', U, U', V, V' be rings, and M an (T, U)-bimodule, M' an (T', U')-bimodule; N an (U, V)-bimodule, N' an (U', V')-bimodule; D an (T, V)-bimodule , D' an (T', V')-bimodule. Let $\varphi_1 : T \longrightarrow T', \varphi_2 : U \longrightarrow U', \varphi_3 : V \longrightarrow V'$ are ring isomorphisms and $f = (f_1, f_2, f_3)$ be a generalized module homomorphism related to $(\varphi_1, \varphi_2, \varphi_3)$. Then the mapping defined in Lemma 2.2 is a ring isomorphism.

Proof. By Lemma 2.2, ψ is a ring homomorphism, we have $\psi\left[\begin{pmatrix} t & m & d \\ 0 & u & n \\ 0 & 0 & v \end{pmatrix}\right] = 0$ and so $\varphi_1(t) = \varphi_2(u) = \varphi_3(v) = f_1(m) = f_2(n) = f_3(d) = 0$.Thus $\begin{pmatrix} t & m & d \\ 0 & u & n \\ 0 & 0 & v \end{pmatrix} = 0$ and hence ψ is injective.

If $\begin{pmatrix} t' & m' & d' \\ 0 & u' & n' \\ 0 & 0 & v' \end{pmatrix} \in \begin{pmatrix} T' & M' & D' \\ 0 & U' & N' \\ 0 & 0 & V' \end{pmatrix}$ and $\varphi_1, \varphi_2, \varphi_3, f_1, f_2, f_3$ are surjective, then there exist $t \in T, m \in M, d \in D, u \in U, n \in N, v \in V$ such that $\varphi_1(t) = t', \varphi_2(u) = u', \varphi_3(v) = v', f_1(m) = m', f_2(n) = n', f_3(d) = d'$. So we have

$$\psi\left[\begin{pmatrix} t & m & d \\ 0 & u & n \\ 0 & 0 & v \end{pmatrix}\right] = \begin{pmatrix} \varphi_1(t) & f_1(m) & f_3(d) \\ 0 & \varphi_2(u) & f_2(n) \\ 0 & 0 & \varphi_2(v) \end{pmatrix}$$

$$= \begin{pmatrix} t' & m' & d' \\ 0 & u' & n' \\ 0 & 0 & v' \end{pmatrix}.$$

Therefore, ψ is surjective and hence a ring isomorphism.

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