Constructive Proof of the Existence of an Equilibrium in a Competitive Economy with Sequentially Locally Non-Constant Excess Demand Functions

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Abstract—In this paper we will constructively prove the existence of an equilibrium in a competitive economy with sequentially locally non-constant excess demand functions. And we will show that the existence of such an equilibrium in a competitive economy implies Sperner’s lemma. We follow the Bishop style constructive mathematics.

Keywords—Sequentially locally non-constant excess demand functions, Equilibrium in a competitive economy. Constructive mathematics

I. INTRODUCTION

It is well known that Brouwer’s fixed point theorem cannot be constructively proved. Thus, the existence of an equilibrium in a competitive economy cannot be constructively proved. Sperner’s lemma, which is used to prove Brouwer’s theorem, however, can be constructively proved. Some authors have constructively presented an approximate version of Brouwer’s theorem using Sperner’s lemma. See [8] and [9]. Thus, Brouwer’s fixed point theorem is constructively, in the sense of constructive mathematics à la Bishop, proved in its approximate version.

Also in constructive mathematics a nonempty set is called an inhabited set. A set is inhabited if there exists an element of it.

II. EXISTENCE OF AN EQUILIBRIUM IN A COMPETITIVE ECONOMY

In constructive mathematics a nonempty set is called an inhabited set. A set is inhabited if there exists an element of it. Note that in order to show that \( S \) is inhabited, we cannot just prove that it is impossible to show \( S \) is empty: we must actually construct an element of \( S \). (see page 12 of [4]).

Also in constructive mathematics compactness of a set means total boundedness with completeness. First we present finite enumerability and \( \varepsilon \)-approximation to a set. A set \( S \) is finitely enumerable if there exist a natural number \( N \) and a mapping of the set \( \{1, 2, \ldots, N\} \) onto \( S \). An \( \varepsilon \)-approximation to \( S \) is a subset of \( S \) such that for each \( p \in S \) there exists \( q \) in that \( \varepsilon \)-approximation with \( |p - q| < \varepsilon \) (see page 12 of [4]).

According to [4] we have the following result.

Lemma 1: If \( \Delta \) is an \( n \)-dimensional simplex, and consider a function \( \varphi \) from \( \Delta \) to itself. Denote the \( i \)-th components of \( p \) and \( \varphi(p) \) by \( p_i \) and \( \varphi_i \) or \( \varphi_i(p) \).

According to [4] we have the following result.

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The notion that \( f \) has at most one fixed point in [2] is defined as follows;

**Definition 1 (At most one fixed point):** For all \( p, q \in \Delta \), if \( p \neq q \), then \( \varphi(p) \neq \varphi(q) \).

By reference to the notion of sequentially at most one maximum in [1], we define the property of sequential local non-constancy as follow:

**Definition 2 (Sequential local non-constancy of functions):** There exists \( \varepsilon > 0 \) with the following property. For each \( \varepsilon > 0 \) less than or equal to \( \varepsilon \) there exist totally bounded sets \( H_1, H_2, \ldots, H_m \), each of diameter less than or equal to \( \varepsilon \), such that \( \Delta = \cup_{i=1}^{m} H_i \), and if for all sequences \((p_n)_{n \geq 1}, (q_n)_{n \geq 1}\) in each \( H_i \), \( |\varphi(p_n) - p_n| \to 0 \) and \( |\varphi(q_n) - q_n| \to 0 \), then \( |p_n - q_n| \to 0 \).

If \( \varphi \) is a uniformly continuous function from \( \Delta \) to itself, according to [8] and [9] it has an approximate fixed point. This means

- For each \( \varepsilon > 0 \) there exists \( x \in \Delta \) such that \( |p - \varphi(p)| < \varepsilon \).
- Since \( \varepsilon > 0 \) is arbitrary, \( \inf_{p \in \Delta} |p - \varphi(p)| = 0 \).
- Then, \( \inf_{p \in H_i} |p - \varphi(p)| = 0 \),

for some \( H_i \) such that \( \cup_{i=1}^{m} H_i = \Delta \). Since \( n \) is finite, we can find such an \( H_i \).

Now we show the following lemma.

**Lemma 2:** Let \( \varphi \) be a uniformly continuous function from \( \Delta \) to itself. Assume \( \inf_{p \in H_i} \varphi(p) = 0 \). If the following property holds:

- For each \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that if \( p, q \in H_i \), then \( |\varphi(p) - p| < \eta \) and \( |\varphi(q) - q| < \eta \), then \( |p - q| < \varepsilon \).

Then, there exists a point \( r \in H_i \) such that \( \varphi(r) = r \).

**Proof:** Choose a sequence \((p_n)_{n \geq 1}\) in \( H_i \) such that \( |\varphi(p_n) - p_n| \to 0 \). Compute \( N \) such that \( |\varphi(p_n) - p_n| < \eta \) for all \( n \geq N \). Then, for \( m, n \geq N \) we have \( |p_m - p_n| < \varepsilon \).

Since \( \varepsilon > 0 \) is arbitrary, \((p_n)_{n \geq 1}\) is a Cauchy sequence in \( H_i \), and converges to a limit \( r \in H_i \). The continuity of \( \varphi \) yields \( |\varphi(r) - r| = 0 \), that is, \( \varphi(r) = r \).

Consider a competitive exchange economy. There are \( n + 1 \) goods \( X_0, X_1, \ldots, X_n \), and a finite positive integer. The prices of the goods are denoted by \( p_i(\geq 0) \), \( i = 0, 1, \ldots, n \).

Let \( \bar{p} = (p_0, p_1, \ldots, p_n, p_{\bar{n}}) \) define

\[
\bar{p}_i = \frac{p_i}{\bar{p}}, \quad i = 0, 1, \ldots, n.
\]

Denote anew \( \bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{\bar{n}} \), respectively, by \( p_0, p_1, \ldots, p_n \). Then, \( p_0 + p_1 + \cdots + p_n = 1 \).

**Theorem 1:** In a competitive exchange economy, if the excess demand functions for the goods are uniformly continuous about their prices and satisfy the Walras law and the
condition of sequential local non-constancy, then there exists an equilibrium.

Proof: Assume \( \inf_{p \in H_i} |p - \varphi(p)| = 0 \). Choose a sequence \((r_m)_{m \geq 1} \in H_i \subset \Delta \) such that \( |\varphi(r_m) - r_m| \rightarrow 0 \).

We will prove the following condition.

For each \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that if \( p, q \in H_i \), \( |\varphi(p) - p| < \eta \) and \( |\varphi(q) - q| < \eta \), then \( p - q \leq \varepsilon \).

Assume that the set

\[
K = \{ (p, q) \in H_i \times H_i : |p - q| \geq \varepsilon \}
\]

is nonempty and compact\(^2\). Since the mapping \( (p, q) \rightarrow \max(|\varphi(p) - p|, |\varphi(q) - q|) \) is uniformly continuous, we can construct an increasing binary sequence \((\lambda_m)_{m \geq 1} \) such that

\[
\lambda_m = 0 \Rightarrow \inf_{(p, q) \in K} \max(|\varphi(p) - p|, |\varphi(q) - q|) < 2^{-m},
\]

\[
\lambda_m = 1 \Rightarrow \inf_{(p, q) \in K} \max(|\varphi(p) - p|, |\varphi(q) - q|) > 2^{-m-1}.
\]

It suffices to find \( m \) such that \( \lambda_m = 1 \). In that case, if \( |\varphi(p) - p| < 2^{-m-1}, |\varphi(q) - q| < 2^{-m-1} \), we have \( (p, q) \not\in K \) and \( |p - q| \leq \varepsilon \). Assume \( \lambda_1 = 0 \). If \( \lambda_0 = 0 \), choose \( (p_m, q_m) \in K \) such that \( \max(|\varphi(p_m) - p_m|, |\varphi(q_m) - q_m|) < 2^{-m-1} \), and if \( \lambda_m = 1 \), set \( p_m = q_m = r_m \). Then, \( |\varphi(p_m) - p_m| \rightarrow 0 \) and \( |\varphi(q_m) - q_m| \rightarrow 0 \), so \( |p_m - q_m| \rightarrow 0 \). Computing \( M \) such that \( |p_M - q_M| < \varepsilon \), we must have \( \lambda_M = 1 \).

Note that \( \varphi \) is sequentially locally non-constant uniformly continuous function from \( \Delta \) to itself. Thus, in view of Lemma 2.2 we have completed the proof of the existence of a point which satisfies

\[
\mathbf{p} = \varphi(p).
\]

III. FROM THE EXISTENCE OF A COMPETITIVE EQUILIBRIUM TO SPERNER’S LEMMA

In this section we will derive Sperner’s lemma from the existence of an equilibrium in a competitive economy\(^3\). Let partition an \( n \)-dimensional simplex \( \Delta \). Let \( K \) be the set of small \( n \)-dimensional simplices of \( \Delta \) constructed by partition. Vertices of these small simplices of \( K \) are labeled with the numbers \( 0, 1, 2, \ldots, n \) according to the following rules.

1) The vertices of \( \Delta \) are respectively labeled with \( 0 \) to \( n \). We label a point \((1,0,\ldots,0)\) with 0, a point \((0,1,\ldots,0)\) with 1, a point \((0,0,1,\ldots,0)\) with \( k \), a point \((0,0,0,1,\ldots,0)\) with \( n \), and all other coordinates are 0 and labeled with \( k \) for all \( k \in \{0,1,\ldots,n\} \).

2) If a vertex of \( K \) is contained in an \( n - 1 \)-dimensional face of \( \Delta \), then this vertex is labeled with some number which is the same as the number of a vertex of that face.

3) A vertex of \( K \) is contained in an \( n - 2 \)-dimensional face of \( \Delta \), then this vertex is labeled with some number which is the same as the number of a vertex of that face. And similarly for cases of lower dimension.

A vertex contained inside of \( \Delta \) is labeled with an arbitrary number among \( 0, 1, \ldots, n \).

Denote vertices of an \( n \)-dimensional simplex of \( K \) by \( x^0, x^1, \ldots, x^n \), the \( j \)-th component of \( x^i \) by \( x^i_j \), and the label of \( x^i_j \) by \( l(x^i) \). Let \( \tau \) be a positive number which is smaller than \( x^i_j(l(x^i)) \) for all \( i \), and define a function \( f(x^i) \) as follows:\(^4\)

\[
f(x^i) = (f_0(x^i), f_1(x^i), \ldots, f_n(x^i)),
\]

and

\[
f_j(x^i) = \begin{cases} x^i_j - \tau & \text{for } j = l(x^i), \\ x^i_j + \frac{\tau}{n} & \text{for } j \neq l(x^i). \end{cases}
\]

\( f_j \) denotes the \( j \)-th component of \( f \). From the labeling rules \( x^i_j(l(x^i)) > 0 \) for all \( x^i \), and so \( \tau > 0 \) is well defined. Since \( \sum_{j=0}^{n} f_j(x^i) = \sum_{j=0}^{n} x^i_j = 1 \), we have

\[
f(x^i) \in \Delta.
\]

We extend \( f \) to all points in the simplex by convex combinations of its values on the vertices of the simplex. Let \( y \) be a point in the \( n \)-dimensional simplex of \( K \) whose vertices are \( x^0, x^1, \ldots, x^n \). Then, \( y \) and \( f(y) \) are represented as follows;

\[
y = \sum_{i=0}^{n} \lambda_i x^i, \quad \text{and} \quad f(y) = \sum_{i=0}^{n} \lambda_i f(x^i), \quad \lambda_i \geq 0, \quad \sum_{i=0}^{n} \lambda_i = 1.
\]

It is clear that \( f \) is uniformly continuous. We verify that \( f \) is sequentially locally non-constant.

1) Assume that a point \( z \) is contained in an \( n - 1 \)-dimensional small simplex \( \delta^{n-1} \) constructed by partition of an \( n - 1 \)-dimensional face of \( \Delta \) such that its \( i \)-th coordinate is \( z_i = 0 \). Denote the vertices of \( \delta^{n-1} \) by

\[^3\text{Our result in this section is a variant of Uzawa equivalence theorem (7)? which (classically) states that the existence of a competitive equilibrium and Brouwer’s fixed point theorem are equivalent.}\]

\[^4\text{We refer to [10] about the definition of this function.}\]
\[ z^i, \ j = 0, 1, \ldots, n - 1 \ \text{and their } i\text{-th coordinate by } z^i_j. \]

Then, we have

\[ f_i(z) = \sum_{j=0}^{n-1} \lambda_j f_i(z^j). \]

Since all vertices of \( \delta^{n-1} \) are not labeled with \( i \), (5) means \( f_i(z^j) > z^i_j \) for all \( j = 0, 1, \ldots, n - 1 \).

Then, there exists no sequence \((z(m))_{m \geq 1}\) such that \( |f(z(m)) - z(m)| \rightarrow 0 \) in an \( n - 1\)-dimensional face of \( \Delta \).

2) Let \( z \) be a point in an \( n\)-dimensional simplex \( H_i \). Assume that no vertex of \( H_i \) is labeled with \( i \). Then

\[ f_i(z) = \sum_{j=0}^{n} \lambda_j f_i(x^j) = z_i + \left( 1 + \frac{1}{n} \right) \tau, \]

and so \( z \neq f(z) \). Then, there exists no sequence \((z(m))_{m \geq 1}\) such that \( |f(z(m)) - z(m)| \rightarrow 0 \) in \( H_i \).

3) Assume that \( z \) is contained in a fully labeled \( n\)-dimensional simplex \( H_i \), and rename vertices of \( H_i \) so that a vertex \( x^i \) is labeled with \( i \) for each \( i \). Then,

\[ f_i(z) = \sum_{j=0}^{n} \lambda_j f_i(x^j) = \lambda_i x^i + \sum_{j \neq i} \lambda_j x^j - \lambda_i \tau = z_i + \left( \frac{1}{n} \sum_{j \neq i} \lambda_j - \lambda_i \right) \tau \]

Consider sequences \((z(m))_{m \geq 1} = (z(1), z(2), \ldots), (z'(m))_{m \geq 1} = (z'(1), z'(2), \ldots)\) such that \( |f(z(m)) - z(m)| \rightarrow 0 \) and \( |f(z'(m)) - z'(m)| \rightarrow 0 \).

Let \( z(m) = \sum_{i=0}^{n} \lambda^i x^i \) and \( z'(m) = \sum_{i=0}^{n} \lambda'^i x^i \). Then, we have

\[ \frac{1}{n} \sum_{j \neq i} \lambda(j) - \lambda(i) \rightarrow 0, \]

and so \( \lambda'(m) - \lambda(m) \rightarrow 0 \) for all \( i \).

Therefore, we obtain

\[ \lambda^i(m) \rightarrow \frac{1}{n + 1}, \text{ and } \lambda'(m) \rightarrow \frac{1}{n + 1}. \]

These mean

\[ |z(m) - z'(m)| \rightarrow 0. \]

Thus, \( f \) is sequentially locally non-constant.

Now, using \( f \), we construct an excess demand function as follows;

\[ g_i(y) = f_i(y) - y_i \mu(y), \quad i = 0, 1, \ldots, n. \]

\( y \in \Delta \), and \( \mu(y) \) is defined by

\[ \mu(y) = \frac{\sum_{i=0}^{n} y_i f_i(y)}{\sum_{i=0}^{n} y_i}. \]

Each \( g_i(y) \) is uniformly continuous, and satisfies the Walras law as shown below. Multiplying \( y_i \) (the \( i\)-th component of \( y \)) to (7) for each \( i \), and adding them from 0 to \( n \) yields

\[ \sum_{i=0}^{n} y_i g_i = \sum_{i=0}^{n} y_i f_i(y) - \mu(y) \sum_{i=0}^{n} y_i^2 \]

\[ = \sum_{i=0}^{n} y_i f_i(y) - \frac{\sum_{i=0}^{n} y_i f_i(y) - \sum_{i=0}^{n} y_i^2}{\sum_{i=0}^{n} y_i} \]

\[ = \sum_{i=0}^{n} y_i f_i(y) - \sum_{i=0}^{n} y_i f_i(y) = 0. \]

Because of sequential local non-constancy of \( f \), \( g_i(y) \)'s are sequentially locally non-constant as excess demand functions described as follows:

1) In an \( n - 1\)-dimensional face of \( \Delta \) there exists no sequence \((z(m))_{m \geq 1}\) such that \( |f(z(m)) - z(m)| \rightarrow 0 \), and so there exists no sequence \( g(z(m))_{m \geq 1} \) such that \( \mu(g(z(m))) \rightarrow 1 \) and \( \max(g(z(m))), 0 \rightarrow 0 \).

2) In an \( n\)-dimensional simplex which is not fully labeled there exists no sequence \((z(m))_{m \geq 1}\) such that \( |f(z(m)) - z(m)| \rightarrow 0 \), and so there exists no sequence \( g(z(m))_{m \geq 1} \) such that \( \mu(g(z(m))) \rightarrow 1 \) and \( \max(g(z(m))), 0 \rightarrow 0 \).

3) In \( \Delta \) for all sequences \((x(m))_{m \geq 1}, (y(m))_{m \geq 1}\) such that \( |f(x(m)) - x(m)| \rightarrow 0 \) and \( |f(y(m)) - y(m)| \rightarrow 0 \), we have \( |x(m) - y(m)| \rightarrow 0 \). When \( |f(x(m)) - x(m)| \rightarrow 0 \), we have \( \mu(x(m)) \rightarrow 1 \) and \( \max(g(x(m))), 0 \rightarrow 0 \).

Similarly, when \( |f(y(m)) - y(m)| \rightarrow 0 \), we have \( \mu(y(m)) \rightarrow 1 \) and \( \max(g(y(m))), 0 \rightarrow 0 \).

Therefore, \( g \) is sequentially locally non-constant, and there exists an equilibrium. Let \( y^* = \{y_0^*, y_1^*, \ldots, y_n^*\} \) be the equilibrium price vector. Then,

\[ g_i(y^*) \leq 0 \text{ for all } i, \]

and

\[ g_i(y^*) = 0 \text{ for } i \text{ such that } y_i^* > 0, \]

and so \( f_i(y^*) = \mu(y^*) y_i^* \) for all such \( i \). \( \sum_{i=0}^{n} f_i(y^*) = \sum_{i=0}^{n} y_i^* = 1 \) implies \( \mu(y^*) \leq 1 \). On the other hand, \( g_i(y^*) \leq 0 \) (for all \( i \)) means \( \mu(y^*) \geq 1 \). Thus, \( \mu(y^*) = 1 \), and we obtain

\[ f_i(y^*) = y_i^* \text{ for all } i. \]

Let \( \Delta^* \) be a simplex of \( K \) which contains \( y^* \), and \( y_0^*, y_1^*, \ldots, y_n^* \) be the vertices of \( \Delta^* \). Then, \( y^* \) and \( f(y^*) \) are represented as

\[ y^* = \sum_{i=0}^{n} \lambda_i y_i \text{ and } f(y^*) = \sum_{i=0}^{n} \lambda_i f(y^*_i), \quad \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1. \]

(5) implies that if only one \( y_k^* \) among \( y_0^*, y_1^*, \ldots, y_n^* \) is labeled with \( i \), we have

\[ f_i(y^*) - y_i^* = \sum_{j=0}^{n} \lambda_j y_j^* + \sum_{j=0, j \neq k}^{n} \lambda_j \frac{y_j^*}{n} - \lambda_k \tau - y_i^* = \left( \frac{1}{n} \sum_{j=0, j \neq k}^{n} \lambda_j - \lambda_k \right) y_i^* \]
\(y^j_i\) is the \(i\)-th component of \(y^j\).

Since \(\tau > 0\), \(f_i(y^\ast) = y^\ast_i\) (for all \(i\)) is equivalent to
\[
\frac{1}{n} \sum_{j=0, j \neq k}^{n} \lambda_j - \lambda_k = 0.
\]

(9) is satisfied with \(\lambda_k = \frac{1}{n+1}\) for all \(k\). On the other hand, if no \(y^j\) is labeled with \(i\), we have
\[
f_i(y^\ast) = \sum_{j=0}^{n} \lambda_j y^j_i = y^\ast_i + \left(1 + \frac{1}{n}\right) \tau,
\]
and then (9) can not be satisfied. Thus, for each \(i\) one and only one \(y^j\) must be labeled with \(i\). Therefore, \(\Delta^\ast\) must be a fully labeled simplex. We have completed the proof of Sperner’s lemma.

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