Banach lattices with weak Dunford-Pettis property

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Abstract—We introduce and study the class of weak almost Dunford-Pettis operators. As an application, we characterize Banach lattices with the weak Dunford-Pettis property. Also, we establish some sufficient conditions for which each weak almost Dunford-Pettis operator is weak Dunford-Pettis. Finally, we derive some interesting results.

Keywords—weak almost Dunford-Pettis operator, almost Dunford-Pettis set, weak Dunford-Pettis operator, weak Dunford-Pettis set. A norm bounded subset $A$ of Banach space $F$ is said to be weak Dunford-Pettis if every weakly compact operator defined on $E$ (and taking their values in a Banach space $F$) is weak Dunford-Pettis (respectively, almost Dunford-Pettis, that is, the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence $(x_n)$ consisting of pairwise disjoint elements in $E$).

I. INTRODUCTION AND NOTATION

As many Banach spaces do not have the Dunford-Pettis property, a weak notion is introduced, called the weak Dunford-Pettis property. A Banach space (respectively, Banach lattice) $E$ has the Dunford-Pettis (respectively, weak Dunford-Pettis) property if every weakly compact operator defined on $E$ (and taking their values in a Banach space $F$) is Dunford-Pettis (respectively, almost Dunford-Pettis, that is, the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence $(x_n)$ consisting of pairwise disjoint elements in $E$).

On the other hand, whenever Aliprantis-Burkinshaw [1] and Kalton-Saab [4] studied the domination property of Dunford-Pettis operators, they used the class of weak Dunford-Pettis operators which satisfies the domination property [4]. Let us recall from [2] that an operator $T$ from a Banach space $X$ into another $Y$ is said to be weak Dunford-Pettis if the sequence $(f_n(T(x_n)))$ converges to 0 whenever $(x_n)$ converges weakly to 0 in $X$ and $(f_n)$ converges weakly to 0 in $Y$. Alternatively, $T$ is weak Dunford-Pettis if $T$ maps relatively weakly compact sets of $X$ into Dunford-Pettis sets of $Y$ (see Theorem 5.99 of [2]). A norm bounded subset $A$ of a Banach lattice $E$ is said to be weak Dunford-Pettis if every weakly null sequence $(f_n)$ of $E$ converges uniformly to zero on the set $A$, that is, $\sup_{x \in A} |f_n(x)| \to 0$ (see Theorem 5.98 of [2]).

In [3], we introduced a new class of sets we call almost Dunford-Pettis set. A norm bounded subset $A$ of a Banach lattice $E$ is said to be almost Dunford-Pettis set if every disjoint weakly null sequence $(f_n)$ of $E'$ converges uniformly to zero on the set $A$, that is, $\sup_{x \in A} |f_n(x)| \to 0$.

As weak Dunford-Pettis operators, we introduce a new class of operators that we call weak almost Dunford-Pettis operator. An operator $T$ from a Banach space $X$ into a Banach lattice $F$ is said to be weak almost Dunford-Pettis if $T$ maps relatively weakly compact sets of $X$ into weak Dunford-Pettis sets of $F$.

II. MAIN RESULTS

Recall from [5] that an operator from a Banach lattice $E$ into a Banach space $X$ is said to be almost Dunford-Pettis if the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence $(x_n)$ consisting of pairwise disjoint elements in $E$.
The following result gives a characteristics of weak almost Dunford-Pettis operators from a Banach space into a Banach lattice in term of weakly compact operators and the adjoint of almost Dunford-Pettis operators.

**Theorem 2.1:** For an operator $T$ from a Banach space $X$ into a Banach lattice $F$, the following statements are equivalent:

1) $T$ is weak almost Dunford-Pettis operator.

2) If $S$ is a weakly compact operator from an arbitrary Banach space $Z$ into $X$, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.

3) If $S$ is a weakly compact operator from $\ell^1$ into $X$, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.

4) For all weakly null sequence $(x_n)_{n \in \mathbb{N}} \subset X$, and for all disjoint weakly null sequence $(f_n)_{n \in \mathbb{N}} \subset F^*$ it follows that $f_n(T(x_n)) \rightarrow 0$.

**Proof:** (1) $\Rightarrow$ (2) Let $(f_n)$ be a disjoint weakly null sequence in $F^*$, we have to prove that $|f_n(T(x_n))| \rightarrow 0$, which is impossible with $|f_n(T(x_n))| > \varepsilon$ for all $n$. Thus, the sequence $|(T \circ S)'(f_n)|$ converges to 0 for the norm of $Z'$.

5) Then $|f_n(T(x_n))| \rightarrow 0$.

6) The sequence $(S((\lambda_i)_{i=1}^\infty))_{i=1}^\infty$ is weakly compact (Theorem 5.26 of [2]), so by our hypothesis $(T \circ S)'(f_n)$ is an almost Dunford-Pettis operator. Thus $|(T \circ S)'(f_n)| \rightarrow 0$ and the desired conclusion follows from the inequality

$$|f_n(T(x_n))| \leq \sup_{\lambda \in B_1} |f_n(T(S((\lambda_i)_{i=1}^\infty)))|$$

for each $n$, where $(\lambda_i)_{i=1}^\infty$ is the canonical basis of $l^1$.

(4) $\Rightarrow$ (1) Let $W$ be a relatively weakly compact subset of $X$, and let $(f_n)$ be a disjoint weakly null sequence in $F^*$. If $(f_n)$ does not converge uniformly to zero on $T(W)$, then there exist a sequence $(x_n)_{n \in W}$, a subsequence of $(f_n)$ (which we shall denote by $(f_n)$ again), and some $\varepsilon > 0$ satisfying $|f_n(T(x_n))| > \varepsilon$ for all $n$.

Since $W$ is weakly compact, we can assume that $x_n \rightarrow x$ weakly in $X$. Then $T(x_n) \rightarrow T(x)$ weakly in $F$ and so, by our hypothesis, we have $0 < \varepsilon < \{f_n(T(x_n))\} \leq |f_n(T(x_n) - x)| + |f_n(T(x))| \rightarrow 0$, which is impossible. Thus, $(f_n)$ converges uniformly to zero on $T(W)$, and this shows that $T(W)$ is an almost Dunford-Pettis set, which ends the proof of the Theorem.

Let us recall that, an operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is said to be order bounded if for each $z \in E^+$, the set $\{T([-z,z])\}$ is order bounded set in $F$. An operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is said to be regular if it can be written as a difference of two positive operators. Note that, every regular operator is order bounded but an order bounded operator is not necessary regular (see [2], Example 1.16, p. 13).

**Remark 2.2:** Each order interval $[-z,z]$ of a Banach lattice $E$ is an almost Dunford-Pettis set for each $z \in E^+$. In fact, if $(f_n)$ be a disjoint weakly null sequence in $E'$, then by Remark 1 of Wnuk [5], $(f_n)$ is a disjoint weakly null sequence in $E'$. Hence $\sup_{x \in [-z,z]} |f_n(x)| = |f_n(z)| \rightarrow 0$ for each $z \in E^+$.

As a consequence, if $T : E \rightarrow F$ is an order bounded operator from a Banach lattice $E$ into another $F$, then $T([-z,z])$ is an almost Dunford-Pettis set in $F$, and then $|f_n \circ T|(z) = \sup_{x \in [-z,z]} |f_n(x)| \rightarrow 0$ for each $z \in E^+$.

We will need the following characterizations, which are just Theorem 2.4 of [3].

**Theorem 2.3:** [3] Let $T : E \rightarrow F$ be an order bounded operator from a Banach lattice $E$ into another Banach lattice $F$, and let $A$ be a norm bounded solid subset of $E$. The following statements are equivalent:

1) $T(A)$ is an almost Dunford-Pettis set.

2) $\{T(x_n), n \in \mathbb{N}\}$ is an almost Dunford-Pettis set for each disjoint sequence $(x_n)_{n \in \mathbb{N}}$ in $A^+ = A \cap E^+$.

3) $f_n(T(x_n)) \rightarrow 0$ for each disjoint sequence $(x_n)_{n \in \mathbb{N}}$ in $A^+$ and for every disjoint weakly null sequence $(f_n)_{n \in \mathbb{N}}$ of $E'$.

**Proof:** (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) To prove that $T(A)$ is an almost Dunford-Pettis set, it suffice to show that $\sup_{x \in A} |f_n(T(x))| \rightarrow 0$ for every disjoint weakly null sequence $(f_n)$ of $F'$. Otherwise, there exists a sequence $(f_n) \in E'$ satisfying $\sup_{x \in A} |f_n(T(x))| > \varepsilon$ for some $\varepsilon > 0$ and all $n$. For every $n$ there exists $x_n \in A^+$ such that $|T(f_n)(x_n)| > \varepsilon$. Since $|T(f_n)(x_n)| \rightarrow 0$ for every $z \in E^+$ (see Remark 2.2), then by an easy inductive argument shows that there exist a subsequence $(y_n)$ of $(x_n)$ and a subsequence $(g_n)$ of $(f_n)$ such that

$$|T'(g_{n+1})(y_{n+1})| > \varepsilon$$

and $|T'(g_{n+1})(y_{n+1})| |n| < \frac{1}{n}$ for all $n \geq 1$. Put $x = \sum_{i=1}^\infty 2^{-i} y_i$ and $x_n = (y_{n+1} - 4^n \sum_{i=1}^n y_i - 2^{-n} x)$. By Lemma 4.35 of [2] the sequence $(x_n)_{n \in \mathbb{N}}$ is disjoint. Since $0 \leq x_n \leq y_{n+1}$ for every $n$, and $(y_{n+1})_{n \in \mathbb{N}}$ in $A^+$ then $(x_n)_{n \in \mathbb{N}}$ in $A^+$.

From the inequalities

$$|T'(g_{n+1})(y_{n+1})| \geq |T'(g_{n+1})(y_{n+1} - 4^n \sum_{i=1}^n y_i - 2^{-n} x)|$$

$$\geq \varepsilon - \frac{1}{n} - 2^{-n} |T'(g_{n+1})(y_{n+1})|$$

for all $n \geq 1$. This contradicts the fact that $(y_n)$ is disjoint, and so the sequence $(y_n)$ is a subsequence of $(x_n)$ which is weakly null in $E^+$.

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we see that $|T'(g_{n+1})| (x) > \frac{2}{\varepsilon}$ must hold for all $n$ sufficiently large (because $2^{-n} |T'(g_{n+1})| (x) \rightarrow 0$).

In view of $|T'(g_{n+1})| (x) = \sup_{z \in E} |g_{n+1} (T(z))| : |z| \leq x$, for each $n$ sufficiently large there exists some $z_n \in E$ with $|g_{n+1} (T(z_n))| > \frac{2}{\varepsilon}$. Since $(z_n')$ and $(z_n')$ are both norm bounded disjoint sequence in $A^+$, it follows from our hypothesis that

$$\varepsilon < |g_{n+1} (T(z_n))| \leq |g_{n+1} (T(z'_n))| + |g_{n+1} (T(z_n'))| \rightarrow 0$$

which is impossible. This proves that $T(A)$ is an almost Dunford-Pettis set.

For ordered bounded operators between two Banach lattices, we give a characterization of weak almost Dunford-Pettis operators.

**Theorem 2.4:** Let $T$ be an ordered bounded operator from a Banach lattice $E$ into another $F$. Then the following assertions are equivalent:

1) $T$ is weak Dunford-Pettis operator.

2) $f_n (T(x)) \rightarrow 0$ for all weakly null sequence $(x_n)$ in $E$ consisting of pairwise disjoint terms, and for all weakly null sequence $(f_n)$ in $F'$ consisting of pairwise disjoint terms.

**Proof:** (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let $(x_n)$ be a weakly null sequence in $E$, and let $(f_n)$ be a disjoint null sequence in $F'$. We have to prove that $f_n (T(x_n)) \rightarrow 0$.

Let $A$ be the solid hull of the weakly relatively compact subset $\{x_n, n \in N\}$ of $E$, by Theorem 4.34 of [2], $(x_n) \rightarrow 0$ weakly for each disjoint sequence $(z_n)$ in $A^+$ and so, by our hypothesis, we have $g_n (T(z_n)) \rightarrow 0$ for each disjoint weakly null sequence $(g_n)$ in $F'$ and for each disjoint sequence $(z_n)$ in $A^+$, then Theorem 2.3, implies that $T(A)$ is an almost Dunford-Pettis set, and hence $\sup_{y \in T(A)} \|f_n (y)\| \rightarrow 0$.

Therefore,

$$|f_n (T(x_n))| \leq \sup_{x \in A} |f_n (T(x))| \leq \sup_{y \in T(A)} \|f_n (y)\| \rightarrow 0$$

holds and the proof is finished.

Now for positive operators between two Banach lattices, we give other characterizations of weak almost Dunford-Pettis operators.

**Theorem 2.5:** Let $E$ and $F$ be two Banach lattices. For every positive operator $T$ from $E$ into $F$, the following assertions are equivalent:

1) $T$ is weak Dunford-Pettis.

2) If $S$ is a weakly compact operator from an arbitrary Banach space $Z$ into $E$, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.

3) If $S$ is a weakly compact operator from $\ell^1$ into $E$, then the adjoint of the operator product $T \circ S$ is almost Dunford-Pettis.

4) For all weakly null sequence $(x_n)$ in $E$, and for all disjoint weakly null sequence $(f_n)$ in $F'$ it follows that $f_n (T(x_n)) \rightarrow 0$.

5) $f_n (T(x_n)) \rightarrow 0$ for every weakly null sequence $(x_n)$ in $E'$ and for all disjoint weakly null sequence $(f_n)$ in $F'$.

6) $f_n (T(x_n)) \rightarrow 0$ for all weakly null sequence $(x_n)$ in $E$ consisting of pairwise disjoint terms, and for all weakly null sequence $(f_n)$ in $F'$ consisting of pairwise disjoint terms.

7) For all disjoint weakly null sequences $(x_n) \subset E^+$, $(f_n) \subset (F')^+$ it follows that $f_n (T(x_n)) \rightarrow 0$.

8) $f_n (T(x_n)) \rightarrow 0$ for every disjoint weakly null sequence $(x_n)$ in $E^+$ and for all weakly null sequence $(f_n)$ in $F'$.

9) $f_n (T(x_n)) \rightarrow 0$ for every disjoint weakly null sequence $(x_n)$ in $E^+$ and for all weakly null sequence $(f_n)$ in $(F')^+$.

10) $f_n (T(x_n)) \rightarrow 0$ for every weakly null sequence $(x_n)$ in $E$ and for all weakly null sequence $(f_n)$ in $(F')^+$.

11) $f_n (T(x_n)) \rightarrow 0$ for every weakly null sequence $(x_n)$ in $E^+$ and for all weakly null sequence $(f_n)$ in $(F')^+$.

12) $f_n (T(x_n)) \rightarrow 0$ for every weakly null sequence $(x_n)$ in $E^+$ and for all weakly null sequence $(f_n)$ in $(F')^+$.

**Proof:** (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) Follows from Theorem 2.1.

(6) $\Leftrightarrow$ (4) Follows from Theorem 2.4.

(4) $\Leftrightarrow$ (5) Obvious.

(5) $\Rightarrow$ (6) Let $(x_n)$ be a weakly null sequence in $E$ consisting of pairwise disjoint elements, and let $(f_n)$ be a weakly null sequence in $F'$, consisting of pairwise disjoint elements, it follows from Remark 1 of Wnuk [5] that $x_n^+ \rightarrow 0$ and $x_n^- \rightarrow 0$ weakly in $E^+$. Hence by (5), $f_n (T(x_n)) = f_n (T(x_n^+)) - f_n (T(x_n^-)) \rightarrow 0$.

(6) $\Rightarrow$ (7) Obvious.

(7) $\Rightarrow$ (8) Assume by way of contradiction that there exists a disjoint weakly null sequence $(x_n) \subset E^+$ and a weakly null sequence $(f_n) \subset F'$ such that $f_n (T(x_n)) \not\rightarrow 0$. The inequality $|f_n (T(x_n))| \leq |f_n (T(x_n))| \not\rightarrow 0$ implies $|f_n (T(x_n))| \not\rightarrow 0$. Then there exists some $\varepsilon > 0$ and a subsequence of $(f_n) (T(x_n))$ (which we shall denote by $(f_n) (T(x_n))$ again) satisfying $|f_n (T(x_n))| > \varepsilon \forall n$.

On the other hand, since $(x_n) \rightarrow 0$ weakly in $E$, then $T(x_n) \rightarrow 0$ weakly in $F$. Now an easy inductive argument shows that there exist a subsequence $(z_n)$ of $(x_n)$ and a subsequence $(g_n)$ of $(f_n)$ such that $\forall n \geq 1$

$$|g_n (T(z_n))| > \varepsilon \text{ and } (4^n \sum_{i=1}^{n} |g_i|)(T(z_{n+1})) < \frac{1}{n}$$

Put $h = \sum_{n=0}^{\infty} 2^{-n} |g_n|$ and $h_n = (|g_n| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n} h)^+$. By Lemma 4.35 of [2] the sequence $(h_n)$ is disjoint. Since $0 \leq h_n \leq |g_n|$ for all $n \geq 1$ and $(g_n) \not\rightarrow 0$ weakly in $F'$ then it follows from Theorem 4.34 of [2] that $(h_n) \not\rightarrow 0$ weakly in $F'$.

From the inequalities

$$h_n (T(z_{n+1})) \geq (|g_n| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n} h)(T(z_{n+1}))$$

we see that $h_n (T(z_{n+1})) > \frac{2}{n}$ must hold for all $n$ sufficiently large (because $2^{-n} h(T(z_{n+1})) \rightarrow 0$), which contradicts with our hypothesis (7).

(8) $\Rightarrow$ (9) Obvious.
consisting of pairwise disjoint elements, and let shows that there exist a subsequence for all inequality \( f_n(T(|x_n|)) \leq f_n(T(|x_n|)) \) implies \( f_n(T(|x_n|)) \to 0 \). Then there exists some \( \varepsilon > 0 \) and a subsequence of \( f_n(T(|x_n|)) \) (which we shall denote by \( f_n(T(|x_n|)) \)) again satisfying \( f_n(T(|x_n|)) \to \varepsilon \) for all \( n \).

On the other hand, since \( f_n \to 0 \) weakly in \( F' \), then \( T'(f_n) \to 0 \) weakly in \( E' \). Now an easy inductive argument shows that there exist a subsequence \( (x_n) \) of \( (|x_n|) \) and a subsequence \( (g_n) \) of \( (f_n) \) such that \( \forall n \geq 1 \)

\[
T'(g_n)(z_n) > \varepsilon \quad \text{and} \quad T'(g_n+1)(\varepsilon - \sum_{i=1}^{\infty} z_i) < \frac{1}{n}
\]

Put \( z = \sum_{n=1}^{\infty} 2^{-n} z_n \) and \( y_n = (z_{n+1} - \sum_{i=1}^{\infty} z_i - 2^{-n} z) \). By Lemma 4.35 of [2] the sequence \( (y_n) \) is disjoint. Since \( 0 \leq y_n \leq \varepsilon \) for all \( n \geq 1 \) and \( (z_n) \to 0 \) weakly in \( E \), then it follows from Theorem 4.34 of [2] that \( (y_n) \to 0 \) weakly in \( E \).

From the inequalities

\[
T'(g_n+1)(y_n) \geq T'(g_n+1)(\varepsilon - \sum_{i=1}^{\infty} z_i - \sum_{i=1}^{\infty} \frac{z}{2^n}) \\
\geq \varepsilon - \frac{1}{n} - 2^{-n} T'(g_n+1)(z)
\]

we see that \( g_{n+1}(T(y_n)) = T'(g_n+1)(y_n) > \varepsilon \) must hold for all \( n \) sufficiently large (because \( 2^{-n} T'(g_n+1)(z) \to 0 \)), which contradicts with our hypothesis (9).

(10) \(\implies\) (11) Obvious.

(11) \(\implies\) (6) Let \( (x_n) \) be a weakly null sequence in \( E \) consisting of pairwise disjoint elements, and let \( (f_n) \) be a weakly null sequence in \( F' \), consisting of pairwise disjoint elements, it follows from Remark 1 of Wnuk [5] that \( |x_n| \to 0 \) in \( \sigma(E, E') \), and \( |f_n| \to 0 \) in \( \sigma(F', F'') \). Hence by (11), \( |f_n(T(|x_n|))| \to 0 \). Now, from \( |f_n(T(|x_n|))| \leq |f_n(T(|x_n|))| \) for each \( n \), we derive that \( |f_n(T(|x_n|))| \to 0 \). Thus, (12) \(\implies\) (8) Obvious.

(5) \(\implies\) (12) The proof is similar of the proof (7) \(\implies\) (8).

An application of Theorem 2.5, gives other characterizations of Banach lattices with the weak Dunford-Pettis property.

**Corollary 2.6:** For a Banach lattice \( E \) the following statements are equivalent:

1) \( E \) has the weak Dunford-Pettis property.
2) The identity operator \( I_E : E \to E \) is weak almost Dunford-Pettis, that is, every relatively weakly compact set of \( E \) is almost Dunford-Pettis set.
3) Every weakly compact operator \( T \) from an arbitrary Banach space \( X \) to \( E \) has an adjoint \( T^* : E' \to X' \) which is almost Dunford-Pettis.
4) Every weakly compact operator \( T : E \to E' \) has an adjoint \( T^* \) which is almost Dunford-Pettis.
5) For all weakly null sequence \( (x_n)_n \subset E \), and for all disjoint weakly null sequence \( (f_n)_n \subset E' \) it follows that \( f_n(x_n) \to 0 \).
6) \( f_n(x_n) \to 0 \) for every weakly null sequence \( (x_n)_n \subset E' \) and for all disjoint weakly null sequence \( (f_n)_n \subset E' \).
7) For all disjoint weakly null sequences \( (f_n)_n \subset E' \), \( (x_n)_n \subset E \) it follows that \( f_n(x_n) \to 0 \).
8) For all disjoint weakly null sequences \( (f_n)_n \subset (E')^+ \), \( (x_n)_n \subset E^+ \) it follows that \( f_n(x_n) \to 0 \).
9) \( f_n(x_n) \to 0 \) for every disjoint weakly null sequence \( (x_n)_n \subset E' \) and for all weakly null sequence \( (f_n)_n \subset E' \).
10) \( f_n(x_n) \to 0 \) for every disjoint weakly null sequence \( (x_n)_n \subset E' \) and for all weakly null sequence \( (f_n)_n \subset (E')^+ \).
11) \( f_n(x_n) \to 0 \) for every weakly null sequence \( (x_n)_n \subset E \) and for all weakly null sequence \( (f_n)_n \subset (E')^+ \).
12) \( f_n(x_n) \to 0 \) for every weakly null sequence \( (x_n)_n \subset E' \) and for all weakly null sequence \( (f_n)_n \subset (E')^+ \).
13) \( f_n(x_n) \to 0 \) for every weakly null sequence \( (x_n)_n \subset E' \) and for all weakly null sequence \( (f_n)_n \subset E' \).

**Proof:** (1) \(\equiv\) (8) follows from Proposition 1 of Wnuk [5].
(2) \(\equiv\) (3) \(\equiv\) ... \(\equiv\) (13) follows from Theorem 2.5.

The following consequence of Theorem 2.5 gives a sufficient conditions under which the class of positive weak almost Dunford-Pettis operators coincide with that of positive weak Dunford-Pettis operators.

**Corollary 2.7:** Let \( E \) and \( F \) be two Banach lattices. Then each positive weak almost Dunford-Pettis operator from \( E \) into \( F \) is weak Dunford-Pettis if one of the following assertions is valid:

1) The lattice operation of \( E \) are weak sequentially continuous;
2) The lattice operation of \( F' \) are weak sequentially continuous.

**Proof:** (1) Assume that \( T : E \to F \) is a positive weak almost Dunford-Pettis operator. Let \( (x_n) \) be a weakly null sequence in \( E \), and let \( (f_n) \) be a weakly null sequence in \( F' \). We have to prove that \( f_n(T(x_n)) \to 0 \).

Since the lattice operation of \( E \) are weak sequentially continuous, then the positive sequences \( (x_n^+) \) and \( (x_n^-) \) converge weakly to zero. Thus, Theorem 2.5 (12) imply that

\[
f_n(T(x_n^+)) \to 0 \quad \text{and} \quad f_n(T(x_n^-)) \to 0.
\]

Finally, from \( f_n(T(x_n)) = f_n(T(x_n^+)) = f_n(T(x_n^-)) \) for each \( n \), we conclude that \( f_n(T(x_n)) \to 0 \). This shows that \( T \) is weak Dunford-Pettis.

(2) Assume that \( T : E \to F \) is a positive weak almost Dunford-Pettis operator. Let \( (x_n) \) be a weakly null sequence in \( E \), and let \( (f_n) \) be a weakly null sequence in \( F' \). We have to prove that \( f_n(T(x_n)) \to 0 \).

Since the lattice operation of \( F' \) are weak sequentially continuous, then the positive sequences \( (f_n^+) \) and \( (f_n^-) \) converge weakly to zero. Thus, Theorem 2.5 (10) imply that

\[
f_n^+(T(x_n)) \to 0 \quad \text{and} \quad f_n^-(T(x_n)) \to 0.
\]

Finally, from \( f_n(T(x_n)) = f_n(T(x_n^+)) = f_n(T(x_n^-)) \) for each \( n \), we conclude that \( f_n(T(x_n)) \to 0 \). This shows that \( T \) is weak Dunford-Pettis.

The preceding Corollary, gives a sufficient conditions under which the weak Dunford-Pettis property and the Dunford-Pettis property coincide.

**Corollary 2.8:** Let \( E \) be a Banach lattice. Then \( E \) has the Dunford-Pettis property if and only if it has the weak Dunford-Pettis property, if one of the following assertions is valid:
1) The lattice operation of $E$ are weak sequentially continuous; 
2) The lattice operation of $E'$ are weak sequentially continuous.

Our consequence of Theorem 2.5 we obtain the domination property for weak almost Dunford-Pettis operators.

**Corollary 2.9:** Let $E$ and $F$ be two Banach lattices. If $S$ and $T$ are two positive operators from $E$ into $F$ such that $0 \leq S \leq T$ and $T$ is weak almost Dunford-Pettis operator, then $S$ is also weak almost Dunford-Pettis operator.

**Proof:** Let $(x_n)_n$ be a weakly null sequence in $E^+$ and $(f_n)_n$ be a weakly null sequence in $(F')^+$. According to (11) of Theorem 2.5, it suffices to show that $f_n(S(x_n)) \to 0$. Since $T$ is weak almost Dunford-Pettis, then Theorem 2.5 implies that $f_n(T(x_n)) \to 0$. Now, by using the inequalities $0 \leq f_n(S(x_n)) \leq f_n(T(x_n))$ for each $n$, we see that $f_n(S(x_n)) \to 0$.

Now, we look at the duality property of the class of positive weak almost Dunford-Pettis operators.

**Theorem 2.10:** Let $E$ and $F$ be two Banach lattices and let $T$ be a positive operator from $E$ into $F$. If the adjoint $T'$ is weak almost Dunford-Pettis from $F'$ into $E'$, then $T$ itself is weak almost Dunford-Pettis.

**Proof:** Let $(x_n)_n$ be a weakly null sequence in $E^+$, and let $(f_n)_n$ be a weakly null sequence in $(F')^+$. We have to prove that $f_n(T(x_n)) \to 0$.

Let $\tau : E \to E''$ be the canonical injection of $E$ into its topological bidual $E''$. Since $\tau$ is a lattice homomorphism, the sequence $(\tau(x_n))$ is weakly null in $(E'')^+$. And as the adjoint $T'$ is weak almost Dunford-Pettis from $F'$ into $E'$, we deduce by Theorem 2.1 that $\tau(x_n)(T'(f_n)) \to 0$. But $\tau(x_n)(T'(f_n)) = T'(f_n)(x_n) = f_n(T(x_n))$ for each $n$. Hence $f_n(T(x_n)) \to 0$ and this ends the proof.

We end this paper by a consequence of Theorem 2.10, we obtain Proposition 2 of Wnuk [5].

**Corollary 2.11:** Let $E$ be a Banach lattice. If $E'$ has the weak Dunford-Pettis property, then $E$ itself has the weak Dunford-Pettis.

**REFERENCES**