Anti-Homomorphism in Fuzzy Ideals

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Abstract—The anti-homomorphic image of fuzzy ideals, fuzzy ideals of near-rings and anti ideals are discussed in this note. A necessary and sufficient condition has been established for near-ring anti ideal to be characteristic.

Keywords—Fuzzy Ideals, Anti fuzzy subgroup, Anti fuzzy ideals, Anti homomorphism, Lower α level cut.

I. INTRODUCTION

In 1971, Rosenfeld [11] constituted the elementary concepts of fuzzy subgroupoid, fuzzy ideals and fuzzy subgroups. Biswas [3] introduced the notion of anti fuzzy subgroups. Fuzzy subnear-rings are introduced by Abou-Zaid [1]. He studied fuzzy left (resp. right) ideals of a near-ring and gave some properties of fuzzy prime ideals of a near-ring. In [7], it has been established that homomorphic image of a fuzzy left (resp. right) ideal which has "sup property" is a fuzzy left (resp. right) ideal. In the year 1998, Sung M.H. et al. [12] proved the same result using the level fuzzy subsets and obtained some properties based on near-ring homomorphism. Properties of anti-homomorphic images of near-rings are discussed in [5]. Homomorphic images and pre images of anti fuzzy ideals are investigated by K.H. Kim et al. [9]. The notion of anti homomorphic image and pre image of fuzzy and anti fuzzy ideals are investigated in this paper. Also, near-ring anti homomorphic image and pre image of ideals are obtained.

A. Preliminaries

In this section, review of fuzzy set theoretic concepts are given briefly (for details one can refer [4], [11] and [10]). A fuzzy set μ of a set N is a function μ : N → [0, 1].

μ will be called a fuzzy left ideal [11], if μ(xy) ≥ μ(y); a fuzzy right ideal, if μ(xy) ≥ μ(x); anti fuzzy left ideal [3] if μ(xy) ≤ μ(y); anti fuzzy right ideal, if μ(xy) ≤ μ(x).

Let f : N → N' be a function and let μ and ν be fuzzy sets in N and N' respectively. Then f(μ) [11], the image of μ under f is a fuzzy set in N' defined by

\[ f(\mu)(y) = \begin{cases} \sup \{\mu(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \]

for all y ∈ N'. \( f^{-1}(\nu) \) [11], the preimage of ν under f is a fuzzy set in N given by

\[ f^{-1}(\nu)(x) = \nu(f(x)) \]

for all x ∈ N.

Similar to an α level cut [4], we have lower level cut [9] as follows:

Let μ be a fuzzy set in a set N. For α ∈ [0, 1], the lower α level cut of μ is denoted by \( \alpha N_\mu \) and is given by

\[ \alpha N_\mu = \{n \in N : \mu(n) \leq \alpha \} \]

Definition 1.1: [1], [7] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if

I-1. \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \),
I-2. \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \),
I-3. \( \mu(y + x - y) \geq \mu(x) \) and
I-4. \( \mu(xy) \geq \mu(y) \) where \( x, y \in N \)

are satisfied. If axioms (I-1), (I-2), (I-3) with (I-5) are satisfied then \( \mu \) is an anti fuzzy left N-ideal.

From the definition of right near-ring [6], ideals can be defined as follows:

Definition 1.2: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy right N-ideal if (I-1), (I-2), (I-3) and
I-6. \( \mu(xy) \geq \mu(y) \) where \( x, y \in N \)

are satisfied. If (I-7) is postulated instead of (I-5) of fuzzy N-ideal of left near-ring, μ is a fuzzy right N-ideal where
I-7. \( \mu((x + z) - y) \geq \mu(z) \)

holds, μ is a fuzzy right N-ideal.

Definition 1.3: [9] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal if

AI-1. \( \mu(x - y) \leq \max\{\mu(x), \mu(y)\} \),
AI-2. \( \mu(xy) \leq \max\{\mu(x), \mu(y)\} \),
AI-3. \( \mu(y + x - y) \leq \mu(x) \) and
AI-4. \( \mu(xy) \leq \mu(x) \) where \( x, y \in N \).

If axioms (AI-1), (AI-2), (AI-3) with the following (AI-5) are satisfied then μ is an anti fuzzy right N-ideal:
AI-5. \( \mu((x + z) - y) \leq \mu(z) \).

Definition 1.4: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal when AI-1 to AI-3 along with
AI-6. \( \mu(xy) \leq \mu(y) \) where \( x, y \in N \)

are postulated. If axioms (AI-1), (AI-2), (AI-3) with (AI-7) are satisfied then μ is an anti fuzzy right N-ideal:
AI-7. \( \mu((x + z) - y) \leq \mu(z) \).

Recall that, a function \( f : N \rightarrow N' \) of near-rings is called an anti-homomorphism [5] when

1. \( f(n + m) = f(m) + f(n) \)
2. \( f(nm) = f(m)f(n) \), for all \( n, m \in N \).

A surjective anti-homomorphism is called an anti-epimorphism (≥).
II. MAIN RESULTS

A. Fuzzy Ideals

It is to be noted that the anti homomorphic image pre-image of a fuzzy groupoid is again a fuzzy groupoid. Where as, 

\textbf{Result 2.1:} An anti homomorphic pre-image of a right (left) ideal is a left (right) ideal.

\textbf{Proof:} Let \( \nu \) be a fuzzy left ideal. Then

\[
\mu(xy) = \nu(f(xy)) \\
= \nu(f(y)f(x)) \\
\geq \nu(f(x)) \\
= \mu(x),
\]
a right ideal. When \( \nu \) is a fuzzy right ideal,

\[
\mu(xy) = \nu(f(xy)) \geq \nu(f(y)) = \mu(y),
\]
a left ideal. \( \square \)

\textbf{Result 2.2:} Anti-homomorphic image of a fuzzy left (right) ideal, with sup property, is a fuzzy right (left) ideal.

\textbf{Proof:} Let \( \mu \) be a fuzzy left ideal with sup property. Given \( f(x), f(y) \) in \( f(N) \), let \( x_0 \in f^{-1}[f(x)], y_0 \in f^{-1}[f(y)] \) be such that

\[
\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t)
\]
respectively. Then

\[
\nu(f(xy)) = \nu(f(y)x) = \sup_{t \in f^{-1}[f(xy)]} \mu(t) \geq \mu(y_0x_0) \geq \mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t) = \nu(f(x)),
\]
implies \( \nu \) is a fuzzy right ideal. Similarly, when \( \mu \) is a fuzzy right ideal with above property, we have \( \nu \) is a fuzzy left ideal. \( \square \)

In sequel to the above results, the following also can be established for the case of anti fuzzy left (right) ideals.

\textbf{Result 2.3:} An anti homomorphic pre-image of an anti fuzzy right (left) ideal is an anti fuzzy left (right) ideal.

\textbf{B. Fuzzy Ideals in Near-rings}

\textbf{Result 2.4:} An anti homomorphic image of an anti fuzzy right (left) ideal with sup property, is an anti fuzzy left (right) ideal.

\textbf{Result 2.5:} (\cite{5} Theorem 2.2) Anti homomorphic image of a right near-ring (left near-ring) is a left near-ring (right near-ring).

\textbf{Result 2.6:} Let \( f : N \rightarrow N' \) be an anti-epimorphism of near-rings. If \( \nu \) is a fuzzy (left/right) ideal in the right (left) near-ring \( N' \), then \( \mu \), which is \( f^{-1}(\nu) \) is a fuzzy (left/right) ideal in the left (right) near-ring \( N \).

\textbf{Proof:} Let \( \nu \) be a fuzzy left ideal of right near-ring \( N' \). The proof of conditions (I-1), (I-2) and (I-3) of definition 1.1 are similar to that of proof of [7] Theorem 2.12. For any \( x, y \in N \), we have

\[
\mu(xy) = \nu(f(xy)) = \nu(f(y)f(x)) \geq \nu(f(x)) = \mu(y).
\]
Thus \( \mu \) is a fuzzy left ideal of the left near-ring \( N \). When \( \nu \) is a right ideal of right near-ring \( N' \), for any \( x, y, z \in N \) we have,

\[
\mu((x + z)y - xy) = \nu(f((x + z)y - xy)) = \nu(f(y)f((x + z) - f(xy)) = \nu(f(y)(f(x) + f(z)) - f(y)f(x)) \geq \nu(f(z)) = \mu(z),
\]
\( \mu \) is a fuzzy right ideal of left near-ring \( N \).

\begin{align*}
\mu(y(x + z) - yx) & = \nu(f((x + z)y - xy)) = \nu(f(x + z)f(y) - f(y)f(x)) \\
& = \nu(f(x) + f(z)f(y) - f(x)f(y)) \geq \nu(f(z)) = \mu(z) .
\end{align*}

Thus \( \mu \) is a fuzzy left ideal of right near-ring \( N \). Let \( \nu \) is a fuzzy left ideal of left near-ring \( N' \). Then

\[
\mu(xy) = \nu(f(xy)) = \nu(f(y)f(x)) \geq \nu(f(x)) = \mu(x)
\]
for all \( x, y \in N \), implies \( \mu \) is a left ideal of right near ring \( N \). \( \square \)

\textbf{Result 2.7:} Let \( f : N \rightarrow N' \) be an anti-epimorphism of near-rings. If \( \mu \) is a fuzzy (left/right) ideal in the left (right) near-ring \( N \) with sup property, then \( \nu = f(\mu) \) is a fuzzy (left/right) ideal in the right (left) near-ring \( N' \).
Proof: Let \( \mu \) be a fuzzy left ideal of the left near-ring \( N \) with sup property and \( \nu \) be the image of \( \mu \) under \( f \). Let \( x_0 \in f^{-1}[f(x)] \), \( y_0 \in f^{-1}[f(y)] \) such that

\[
\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t).
\]

\[
\nu[f(x)f(y)] = \sup_{t \in f^{-1}[f(yz) \Rightarrow f(yz)]} \mu(t) \geq \mu(y_0x_0) \geq \mu(x_0) = \sup_{t \in f^{-1}[f(z)]} \mu(t) = \nu[f(z)].
\]

That is \( \nu \) is a fuzzy left ideal of right near-ring \( N' \).

If \( \mu \) is a fuzzy right ideal of left near-ring. For any \( f(z) \in f(N) \), let \( z_0 \in f^{-1}[f(z)] \) such that

\[
\nu(z_0) = \sup_{t \in f^{-1}[f(z)]} \mu(z).
\]

Now,

\[
\nu[f((x + z)y - (xy))] = \nu[f(y)(x + z) - f(xy)] = \nu[f(y)f(x) + f(z)] - f(y)f(x)] = \sup_{t \in f^{-1}[f(y)(x + z) - f(y)f(x)]} \mu(t) \geq \mu(y_0x_0 + z_0) - y_0x_0 \geq \mu(z_0) = \sup_{t \in f^{-1}[f(z)]} \mu(t) = \nu[f(z)].
\]

That is, \( \nu \) is a fuzzy right ideal of right near-ring.

The image and pre-image of the fuzzy ideal of a fuzzy right near-ring \( N \) can be proved to be the fuzzy ideal of a right near-ring \( N' \).

\( \square \)

C. Anti Fuzzy Ideals in Near-rings

Definition 2.8: A left \( N \)-ideal \( A \) of a near-ring is said to be characteristic [2], [8] if

\[
f(A) = A \quad \forall f \in \text{Aut}(N)
\]

where \( \text{Aut}(N) \) is set of all automorphism of \( N \).

Anti fuzzy left \( N \)-ideal of \( \mu \) of a near-ring \( N \) is said to be anti fuzzy characteristic if

\[
\mu f(x) = \mu(x) \quad \forall x \in N, f \in \text{Aut}(N).
\]

Lemma 2.9: Let \( \mu \) be an anti fuzzy left \( N \)-ideal of a near-ring \( N \) and let \( x \in N \). Then \( \mu(x) = s \) if and only if \( x \in sN_\mu \) and \( x \not\in tN_\mu \forall s > t \).

Proof is obvious.

The proof of the following theorem is analogous to the proof of theorem 3.9 [2], [8].

Theorem 2.10: Let \( \mu \) be an anti fuzzy \( N \)-ideal of a near-ring \( N \). Then each lower \( \alpha \) level left \( N \)-ideal of \( \mu \) is characteristic iff \( \mu \) is an anti fuzzy characteristic of \( N \).

K. H. Kim et al. [9] proved the following theorems:

Result 2.11 ([9], Theorem 3.19 (1)): Let \( f : N \to N' \) be an epimorphism of near-rings. Let \( \nu \) be an anti fuzzy left \( N' \)-ideal and \( \mu \) be the pre-image of \( \nu \) under \( f \). Then \( \mu \) is an anti-fuzzy left \( N \)-ideal.

Result 2.12 ([9], Theorem 3.19 (2)): Let \( f : N \to N' \) be a surjective homomorphism of near-rings. If \( \mu \) is an anti fuzzy left \( N' \)-ideal then \( f^{-1}(\mu) \) is an anti fuzzy left \( N \)-ideal.

Result 2.13: Let \( f : N \to N' \) be an anti-epimorphism of near-rings. Let \( \nu \) be an anti-fuzzy (left/right) ideal of right (left) near-ring \( N' \) then \( \mu \), the pre-image of \( \nu \) under \( f \), is an anti-fuzzy (left/right) ideal of left (right) near-ring \( N \).

The proof of AI-4 and AI-5 are similar to the proof of result 2.6.

Result 2.14: Let \( f : N \to N' \) be an anti epimorphism. If \( \mu \) is an anti fuzzy (left/right) ideal of left (right) near-ring \( N \) with sup property, \( f(\mu) \) is an anti fuzzy (left/right) ideal of right (left) near-ring \( N' \).

The proof of AI-6 and AI-7 are similar to that of result 2.7.

The references

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