Anti-Homomorphism in Fuzzy Ideals

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Abstract—The anti-homomorphic image of fuzzy ideals, fuzzy ideals of near-rings and anti ideals are discussed in this note. A necessary and sufficient condition has been established for near-ring anti ideal to be characteristic.

Keywords—Fuzzy Ideals, Anti fuzzy subgroup, Anti fuzzy ideals, Anti homomorphism, Lower α level cut.

I. INTRODUCTION

In 1971, Rosenfeld [11] constituted the elementary concepts of fuzzy subgroupoid, fuzzy ideals and fuzzy subgroupoids. Biswas [3] introduced the notion of anti fuzzy subgroups. Fuzzy subnear-rings are introduced by Abou-Zaid [1]. He studied fuzzy left (resp. right) ideals of a near-ring and gave some properties of fuzzy prime ideals of a near-ring. In [7], it has been established that homomorphic image of a fuzzy left (resp. right) ideal which has “sup property” is a fuzzy left (resp. right) ideal. In the year 1998, Sung M.H. et al. [12] proved the same result using the level fuzzy subsets and obtained some properties based on near-ring homomorphism. Properties of anti-homomorphic images of near-rings are discussed in [5]. Homomorphic images and pre images of anti fuzzy ideals are investigated by K.H. Kim et al. [9]. The notion of anti-homomorphic image and pre image of fuzzy and anti fuzzy ideals are investigated in this paper. Also, near-ring anti homomorphic image and pre image of ideals are obtained.

A. Preliminaries

In this section, review of fuzzy set theoretic concepts are given briefly (for details one can refer [4], [11] and [10]). A fuzzy set μ of a set N is a function μ : N → [0, 1].

μ will be called a fuzzy left ideal [11], if μ(xy) ≥ μ(y); a fuzzy right ideal, if μ(xy) ≥ μ(x); anti fuzzy left ideal [3] if μ(xy) ≤ μ(y); anti fuzzy right ideal, if μ(xy) ≤ μ(x);

Let f : N → N′ be a function and let μ and ν be fuzzy sets in N and N′ respectively. Then f(μ) [11], the image of μ under f is a fuzzy set in N′ defined by

\[ f(μ)(y) = \begin{cases} \sup \{μ(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{Otherwise} \end{cases} \]

for all y ∈ N′. \( f^{-1}(ν) \) [11], the preimage of ν under f is a fuzzy set in N given by

\[ f^{-1}(ν)(x) = ν(f(x)) \]

for all x ∈ N.

Similar to an α level cut [4], we have lower level cut [9] as follows:

Let μ be a fuzzy set in a set N. For α ∈ [0, 1], the lower α level cut of μ is denoted by \( αN_μ \) and is given by

\[ αN_μ = \{n ∈ N : μ(n) ≤ α\}. \]

Definition 1.1: [1], [7] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if

I-1. \( μ(x - y) ≥ \min\{μ(x), μ(y)\} \),
I-2. \( μ(xy) ≥ \min\{μ(x), μ(y)\} \), for all \( x, y ∈ N \),
I-3. \( μ(y + x - y) ≥ μ(x) \) and
I-4. \( μ(xy) ≥ μ(y) \) where \( x, y ∈ N \)

are satisfied. If axioms (I-1), (I-2), (I-3) with (I-5) are postulated instead of (I-5) of fuzzy N-ideal of left near-ring, μ is a fuzzy right N-ideal.

From the definition of right near-ring [6], ideals can be defined as follows:

Definition 1.2: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if (I-1), (I-2), (I-3) and
I-6. \( μ(xy) ≥ μ(y) \) where \( x, y ∈ N \)

are satisfied. If (I-7) is postulated instead of (I-5) of fuzzy N-ideal of left near-ring, μ is a fuzzy right N-ideal where
I-7. \( μ(y(x + z) - yx) ≥ μ(z) \).

Definition 1.3: [9] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal if
A1-1. \( μ(x - y) ≤ \max\{μ(x), μ(y)\} \),
A1-2. \( μ(xy) ≤ \max\{μ(x), μ(y)\} \),
A1-3. \( μ(y + x - y) ≤ μ(x) \) and
A1-4. \( μ(xy) ≤ μ(x) \) where \( x, y ∈ N \).

If axioms (A1-1), (A1-2), (A1-3) with the following (A1-5) are satisfied then μ is an anti fuzzy right N-ideal;
A1-5. \( μ((x + z)y - xy) ≤ μ(z) \).

Definition 1.4: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal when A1-1 to A1-3 along with
A1-6. \( μ(xy) ≤ μ(y) \) where \( x, y ∈ N \)

are postulated. If axioms (A1-1), (A1-2), (A1-3) with (A1-7) are satisfied then μ is an anti fuzzy right N-ideal;
A1-7. \( μ(y(x + z) - yx) ≤ μ(z) \).

Recall that, a function f : N → N′ of near-rings is called an anti-homomorphism [5] when
1. \( f(n + m) = f(m) + f(n) \)
2. \( f(nm) = f(m)f(n) \), for all \( n, m ∈ N \).

A surjective anti-homomorphism is called an anti-epimorphism (\( \cong \)).
II. MAIN RESULTS

A. Fuzzy Ideals

It is to be noted that the anti homomorphic image pre-image of a fuzzy groupoid is again a fuzzy groupoid. Where as,

Result 2.1: An anti homomorphic pre-image of a right (left) ideal is a left (right) ideal.

Proof: Let \( \nu \) be a fuzzy left ideal. Then

\[
\mu(xy) = \nu(f(xy)) = \nu(f(y)f(x)) \geq \nu(f(x)) = \mu(x),
\]
a right ideal. When \( \nu \) is a fuzzy right ideal,

\[
\mu(xy) = \nu(f(xy)) = \nu(f(y)f(x)) \geq \nu(f(y)) = \mu(y),
\]
a left ideal. □

Result 2.2: Anti-homomorphic image of a fuzzy left (right) ideal, with supremum property, is a fuzzy right (left) ideal.

Proof: Let \( \mu \) be a fuzzy left ideal with sup property. Given \( f(x), f(y) \) in \( f(N) \), let \( x_0 \in f^{-1}[f(x)], y_0 \in f^{-1}[f(y)] \) be such that

\[
\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t)
\]
respectively. Then

\[
\nu[f(x)f(y)] = \nu[f(y)x] = \sup_{t \in f^{-1}[f(y)x]} \mu(t) \geq \mu(y_0 x_0) \geq \mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t) = \nu[f(x)],
\]
implies \( \nu \) is a fuzzy right ideal. Similarly, when \( \mu \) is a fuzzy right ideal with above property, we have \( \nu \) is a fuzzy left ideal. □

In sequel to the above results, the following also can be established for the case of anti fuzzy left (right) ideals.

Result 2.3: An anti homomorphic pre-image of an anti fuzzy right (left) ideal is an anti fuzzy left (right) ideal.

Result 2.4: An anti homomorphic image of an anti fuzzy right (left) ideal with sup property, is an anti fuzzy left (right) ideal.

B. Fuzzy Ideals in Near-rings

Result 2.5: ([5] Theorem 2.2) Anti homomorphic image of a right near-ring (left near-ring) is a left near-ring (right near-ring).

Result 2.6: Let \( f : N \rightarrow N' \) be an anti-epimorphism of near-rings. If \( \nu \) is a fuzzy (left/right) ideal in the right (left) near-ring \( N' \), then \( \mu \), which is \( f^{-1}(\nu) \) is a fuzzy (left/right) ideal in the left (right) near-ring \( N \).

Proof:

Let \( \nu \) be a fuzzy left ideal of right near-ring \( N' \). The proof of conditions (I-1), (I-2) and (I-3) of definition 1.1 are similar to that of proof of [7] Theorem 2.12. For any \( x, y \in N \), we have

\[
\mu(xy) = \nu(f(xy)) = \nu(f(y)f(x)) \geq \nu(f(x)) = \mu(x),
\]

Thus \( \mu \) is a fuzzy left ideal of the left near-ring \( N \). When \( \nu \) is a right ideal of right near-ring \( N' \), for any \( x, y, z \in N \), we have,

\[
\mu((x + z)y - xy) = \nu(f((x + z)y - xy)) = \nu(f(x + z) - f(y)) = \nu(f(x + f(z) - f(y))) \geq \nu(f(z)) = \mu(z),
\]

\( \mu \) is a fuzzy right ideal of left near-ring \( N \).

Let \( \nu \) be a right ideal of left near-ring \( N' \). For any \( x, y, z \in N \), we have

\[
\mu(y(x + z) - yx) = \nu(f((x + z)y - xy)) = \nu(f(x) + f(z)y - f(x)y) \geq \nu(f(z)) = \mu(z).
\]

Thus \( \mu \) is a fuzzy left ideal of right near-ring \( N \). Let \( \nu \) is a fuzzy left ideal of left near-ring \( N' \). Then

\[
\mu(xy) = \nu(f(xy)) = \nu(f(y)f(x)) \geq \nu(f(x)) = \mu(x)
\]

for all \( x, y \in N \), implies \( \mu \) is a left ideal of right near ring \( N \). □

Result 2.7: Let \( f : N \rightarrow N' \) be an anti-epimorphism of near-rings. If \( \mu \) is a fuzzy (left/right) ideal in the left (right) near-ring \( N \) with sup property, then \( \nu = f(\mu) \) is a fuzzy (left/right) ideal in the right (left) near-ring \( N' \).
Proof: Let $\mu$ be a fuzzy left ideal of the left near-ring $N$ with sup property and $\nu$ be the image of $\mu$ under $f$. Let $x_0 \in f^{-1}[f(x)]$, $y_0 \in f^{-1}[f(y)]$ such that

$$\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t).$$

$$\nu[f(x)f(y)] = \sup_{t \in f^{-1}[f(y)]} \mu(t) \geq \mu(y_0) \geq \mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t) = \nu[f(x)].$$

That is $\nu$ is a fuzzy left ideal of right near-ring $N'$.

If $\mu$ is a fuzzy right ideal of left near-ring. For any $f(z) \in f(N)$, let $z_0 \in f^{-1}[f(z)]$ such that

$$\mu(z_0) = \sup_{t \in f^{-1}[f(z)]} \mu(t).$$

Now,

$$\nu[f((x+z)y) - (xy)] = \nu[f(y)[(x+z) - f(xy)] = \nu[f(x)(f(x) + f(z)) - f(y)f(x)] = \sup_{t \in f^{-1}[f(z)]} \mu(t) \geq \mu(y_0[x_0 + z_0] - y_0x_0) \geq \mu[z_0] = \sup_{t \in f^{-1}[f(z)]} \mu(t) = \nu[f(x)].$$

That is, $\nu$ is a fuzzy right ideal of right near-ring.

The proof of the following theorem is analogous to the proof of theorem 3.9 [2], [8].

**Theorem 2.10:** Let $\mu$ be an anti fuzzy $N$-ideal of a near-ring $N$. Then each lower $\alpha$ level left $N$-ideal of $\mu$ is characteristic iff $\mu$ is an anti fuzzy characteristic of $N$.

K. H. Kim et al. [9] proved the following theorems:

**Result 2.11 ([9], Theorem 3.19 (1)):** Let $f : N \rightarrow N'$ be an epimorphism of near-rings. Let $\nu$ be an anti-fuzzy left $N'$-ideal and $\mu$ be the pre-image of $\nu$ under $f$. Then $\mu$ is an anti-fuzzy left $N$-ideal.

**Result 2.12 ([9], Theorem 3.19 (2)):** Let $f : N \rightarrow N'$ be a surjective homomorphism of near-rings. If $\mu$ is an anti fuzzy left $N'$-ideal then $f^{-1}(\mu)$ is an anti fuzzy left $N$-ideal.

**Result 2.13:** Let $f : N \rightarrow N'$ be an anti-epimorphism of near-rings. Let $\nu$ be an anti-fuzzy (left/right) ideal of right (left) near-ring $N'$ then $\mu$, the pre-image of $\nu$ under $f$, is an anti-fuzzy (left/right) ideal of left (right) near-ring $N$.

The proof of AI-4 and AI-5 are similar to the proof of result 2.6.

**Result 2.14:** Let $f : N \rightarrow N'$ be an anti epimorphism. If $\mu$ is an anti fuzzy (left/right) ideal of left (right) near-ring $N$ with sup property, $f(\mu)$ is an anti fuzzy (left/right) ideal of right (left) near-ring $N'$.

The proof of AI-6 and AI-7 are similar to that of result 2.7.

### References


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