Some properties of b-weakly compact operators on Banach lattice

Na Cheng  and Zi-li Chen

Abstract—We investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

Keywords—b-weakly compact, Dunford-Pettis operator, M-weakly compact operator, semi-compact operator, weakly sequentially continuous lattice operations, order continuous norm, positive Schur property.

I. INTRODUCTION

RECALL that a subset A of a Riesz space E is called b-order bounded in E if it is order bounded in \((E^\sim)^\sim\). A Riesz space is said to have property (b) if A \subseteq E is order bounded whenever A is order bounded in \((E^\sim)^\sim\). Note that every perfect Riesz space and therefore every norm dual has property (b). Every reflexive Banach lattice has property (b). Every KB space has property (b) and if \((E^\sim)^\sim\) is retractable on E then E has property (b). On the other hand, by considering A = \{e_n\} \subseteq c_0, we see that c_0 does not have property (b). An operator \(T : E \to X\), mapping each b-order bounded subset of Banach lattice E into a relatively weakly compact subset of Banach space X is called a b-weakly compact operator. The collection of b-weakly compact operators will be denoted by \(W_b(E,F)\). Then \(W_b(E,F)\) is a closed subspace of \(L(E,F)\), the vector space of all continuous operators from E into F. Operators mapping order intervals into relatively weakly compact sets are called weakly operators and denoted by \(W_0(E,F)\). The collection of weakly compact operators will be denoted by \(W(E,F)\). Then \(W(E,F) \subseteq W_0(E,F) \subseteq W_b(E,F)\), [9] gave examples to show that these inclusions may be proper. An operator is said to be a Dunford-Pettis operator if it carries relatively weakly compact subsets onto norm totally bounded subsets. An operator \(T\) from a Banach lattice E into a Banach lattice F is said to be M-weakly compact if each disjoint bounded sequence \((x_n)\) of E, we have \(\lim_n \|T(x_n)\| = 0\).

An operator \(T\) from a Banach lattice E into a Banach lattice F is called L-weakly compact if for each disjoint bounded sequence \((y_n)\) in the solid hull of \(T(B_E)\), we have \(\lim_n \|y_n\| = 0\) where \(B_E\) is the closed unit ball of E.

In 2003, S.Alpay and B.Altin [9] studied the property (b). They proved that Banach lattice F is a KB-space if and only if it has order continuous norm and property (b) [9, Theorem 2.1]. They also gave the definition of b-weakly compact operator. They characterized that \(T : E \to X\) is b-weakly compact operator if and only if for each b-order bounded \(A \subseteq E\) and disjoint sequence \((x_n)\) in A satisfies \(\lim_n \|T(x_n)\| = 0\) [9, Theorem 2.8]. In 2006, S.Alpay and B.Altin [10] investigate Riesz spaces and Banach lattices enjoying property (b). They proved that if Banach F is Dedekind complete, then the space of order bounded operators from Banach E into F has property (b) if and only if F has property (b) [10, Theorem 2]. Every order closed Riesz subspace of a Dedekind complete Riesz space E with property (b) has property (b) [10, Theorem 2]. In 2007, S.Alpay and B.Altin [11] characterized the b-weak compactness of T in terms of its mapping properties [11, Theorem 1, Theorem 2, Theorem 4]. In 2007, B.Altin [13] investigated the order structure of b-weakly compact operator. In 2009, S.Alpay and B.Altin [12] gave characterized of KB-spaces in terms of b-weakly compact operators. A Banach lattice F is KB-space if and only if for each Banach lattice E and positive disjointness preserving operator \(T : E \to F\) is b-weakly compact. In 2009, B. Aqzzouz and A. Elbour, and J. Hmichane [14] establish necessary and sufficient conditions under which b-weakly compact operators between Banach lattices have b-weakly compact adjoint or operators with b-weakly compact adjoint are themselves b-weakly compact. If \(T : E \to F\) between Banach lattices is a b-weakly compact operator, then its adjoint \(T' : F' \to E'\) is b-weakly compact if and only if \(F'\) or \(E'\) is a KB-space. Each operator \(T : E \to F\) is b-weakly compact whenever its adjoint \(T' : F' \to E'\) is b-weakly compact if and only if \(E\) or \(F\) is a KB-space.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is an AM-space if and only if the norm is additive on the positive cone of the dual. An element \(e > 0\) in a Riesz space is said to be an order unit whenever for each \(x\) there exists some \(\lambda > 0\) with \(|x| \leq \lambda e\). Now if a Banach lattice E has an order unit \(e > 0\), then \(A_e = E\) holds, and the norm \(\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda e\}\) is equivalent to the original norm of E. In other words, if a Banach lattice E has an order unit e, then E can be renormed in such a way that it becomes an AM-space having \([-e, e]\) as its closed unit ball. A Banach lattice has order continuous norm if and only if every order bounded disjoint sequence id norm convergent to zero. A Banach lattice E is said to be a KB-space, whenever every increasing norm bounded sequence of \(E^+\) is norm convergent. For example, each reflexive Banach lattice is KB-space. Also, each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a KB-
space. In fact, the Banach lattice \( c_0 \) has an order continuous norm but it is not a KB-space. However, if \( E \) is a Banach lattice, the topological dual \( E' \) is a KB-space if and only if its norm is order continuous. The Banach lattice \( E \) has the positive Schur property if each weakly null sequence with positive sequence in \( E \) converges to zero in norm. A Banach lattice \( E \) is said to have weakly sequentially continuous lattice operations whenever \( x_n \xrightarrow{w} 0 \) in \( E \) implies \( |x_n| \xrightarrow{w} 0 \) in \( E \). In an AM-space the lattice operations are weakly sequentially continuous. Also, every Banach lattice with the Schur property (i.e., \( x_n \xrightarrow{w} 0 \) implies \( |x_n| \xrightarrow{0} \)) has weakly sequentially continuous lattice operations. Thus, for example, the Banach lattice \( C[0,1], l_1, l_1 \oplus C[0,1] \) all have weakly sequentially continuous lattice operations.

The goal of this paper is to investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly compact. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact. All notions concerning Banach lattices and not explained here can be found in [1] and [2].

II. PROPERTIES OF B-WEAKLY COMPACT OPERATORS

**Theorem 1:** For Banach lattice \( F \), each positive b-weakly compact operator from AM-space into \( F \) is Dunford-Pettis.

**Proof:** Let \( \rho(x) = \|Tx\| \) for every \( x \in E \), then \( \rho \) is a continuous lattice seminorm on \( E \). Suppose \( T : E \to F \) is not a Dunford-Pettis operator, since AM-space has weakly sequentially continuous lattice operators, there exists a sequence \( \{x_n\} \subset E_+ \) with \( x_n \xrightarrow{w} 0 \) and \( \|Tx_n\| \geq 1 \).

Corollary 2.3.5 of [2] shows that for every \( 0 < c < 1 \), there exists a subsequence \( (k(n))_{n=1}^\infty \subset N \) and a disjoint sequence \( \{y_n\} \subset E_+ \) such that

\[
y_n \leq x_{k(n)} \cdot \|Ty_n\| \geq c
\]

for all \( n \in N \). Since \( y_n \leq x_{k(n)} \) and \( x_n \xrightarrow{w} 0 \), the uniform boundedness theorem implies that the sequence \( y_n \) is bounded. Observing that \( (y_1 + \cdots + y_n)\) is a monotone norm bounded sequence, there exists \( x^0 \in E_+ \) such that

\[
0 \leq y_1 + \cdots + y_n \leq x^0
\]

together with the fact that \( T \) is b-weakly compact, it follows that

\[
\|Ty_n\| \to 0 (n \to \infty)
\]

This gives a contradiction. \( \Box \)

**Theorem 2:** Let \( E \) and \( F \) be two Banach lattices, if every positive b-weakly compact operator \( T : E \to F \) is Dunford-Pettis, then the norm of \( F \) is order continuous or the lattice operations of \( E \) are weakly sequentially continuous.

**Proof:** If the norm of \( F \) is not order continuous and the lattice operations of \( E \) are not weakly sequentially continuous, A.W. Wickstead constructed in the proof of Theorem 2 of [4] two positive operators \( S, T : E \to F \) such that \( 0 \leq S \leq T \) and \( T \) is compact and hence it is b-weakly compact, Proposition 2.2 of [6] implies \( S \) is b-weakly compact, but it is not Dunford-Pettis. \( \Box \)

**Theorem 3:** Let \( E \) and \( F \) be two Banach lattices, if every positive b-weakly compact operator \( T : E \to F \) is weakly compact, then one of the following statements is valid:

(a) The norm of the topological dual \( E' \) is order continuous.

(b) \( F \) is reflexive.

**Proof:** Suppose that neither the norm of \( E' \) is order continuous nor \( F \) is reflexive, then there exist a sublattice \( H \) of \( E \) which is isomorphic to \( l_1 \) and a positive projection \( P : F \to l_1 \).

On the other hand, since the closed unit ball \( B_F \) of \( F \) is not weakly compact, there exists a sequence \( (e_n) \) in \( B_F \) which does not have any weakly convergent subsequence.

Consider the operator \( S : l_1 \to F \) defined by

\[
S(x_n) = \sum_{n=1}^\infty x_n e_n
\]

It is easy to see that \( S \cdot P \) is o-weakly compact, since \( l_1 \) is a KB-space, it is b-weakly compact, but it is not weakly compact. \( \Box \)

**Theorem 4:** Let \( E \) and \( F \) be two Banach lattices, if each positive o-weakly compact operator \( T : E \to F \) is L-weakly compact, then one of the following conditions holds.

(a) \( F \) are KB-spaces.

(b) \( E' \) has the positive Schur property.

**Proof:** Suppose \( F \) is not a KB-space, Theorem 2.4.12 of [4] implies that \( F \) contains a sublattice isomorphism to \( c_0 \).

Applying Theorem 3.1 of [3] it suffices to show each disjoint weak null sequence \( (x_n^\infty)_{n=1} \) is norm convergent to 0. For each \( x \in E \) define \( T : E \to c_0 \) by

\[
Tx = (x_n^\infty)_1
\]

Theorem 17.5 of [1] implies \( T \) is a weakly compact operator, hence it is o-weakly compact, it is L-weakly compact. Theorem 18.13 of [1] implies

\[
T' : l_1 \to E'
\]

is M-weakly compact. As

\[
T'(e_n) = x_n^\infty
\]

for all \( n \in N \), where \( e_n \) is the sequence with \( n \)th entry equals to 1 and all others are zero, we conclude that

\[
\|x_n^\infty\| \to 0 (n \to \infty)
\]

Recall that A continuous operator \( T : E \to F \) is said to be semi-compact if for each \( \epsilon > 0 \), there exists some \( u \in F^+ \) such that \( T(U) \subset [-u, u] + \epsilon V \) where \( U, V \) denote the closed unit balls of \( E \) and \( F \), respectively. Each compact operator, M-weakly compact (L-weakly compact) operator between Banach lattice is semi-compact. However, a semi-compact operator need not be compact, weakly compact, M-weakly compact (L-weakly compact). For instance, the identity operator \( I : l_\infty \to l_\infty \) is semi-compact, but it does not have any one of the above mentioned compactness properties.
Theorem 5: Let $E$ and $F$ be nonzero Banach lattices such that $F$ is $\sigma$-Dedekind complete. Then the following statements are equivalent.

1) Each positive semi-compact operator $T : E \to F$ is $b$-weakly compact.

2) At least one of the following conditions holds:
   a) The norm of $E$ is order continuous.
   b) The norm of $F$ is order continuous.

Proof: 2) $\Rightarrow$ 1) Suppose that $E$ has order continuous norm and $T : E \to F$ is a positive semi-compact operator.
Theorem 12.9 of [1] implies that each order interval of Banach lattice $E$ is weakly compact, together with the fact that $T$ is a positive semi-compact operator, it follows that $T$ is weakly compact. Hence, $T$ is $b$-weakly compact.

2) $\Rightarrow$ 1) Suppose that $F$ has order continuous norm and $T : E \to F$ is a positive semi-compact operator. For each $\epsilon > 0$ there exists some $u \in F^+$ such that

$$U \subseteq [-u, u] + \epsilon V$$

$U$ and $V$ denote the closed unit balls of $E$ and $F$, respectively.

Theorem 12.9 of [1] implies that the order interval $[-u, u]$ in $F$ is weakly compact, combined with Theorem 10.17 of [1] show that $T(U)$ is relatively weakly compact, it follows that $T$ is weakly compact. Hence, $T$ is $b$-weakly compact.

1) $\Rightarrow$ 2) Assume by way of contradiction that neither $E$ nor $F$ has an order continuous norm. To finish the proof, we have to construct a positive semi-compact operator $T : E \to F$ which is not $b$-weakly compact.

Since the norm on $E$ is not order continuous, applying Theorem 12.13 of [1] that there exists some $x \in E^+$ and a sequence $(x_n) \subset [0, y]$ which does not converge to zero in norm. We may assume that $\|x_n\| = 1$ for all $n$.

Hence, by lemma 2.1 of [15] there exists a positive disjoint sequence $(g_n)$ of $E'$ with $\|g_n\| \leq 1$ such that $g_n(x_n) = 1$ for all $n$ and $g_n(x_m) = 0$ for $n \neq m$.

For all $x \in E$, define the positive operator $R : E \to \ell_\infty$ by

$$R(x) = (g_1(x), g_2(x), \cdots)$$

Note that $R(B_E) \subset B_{\ell_\infty}.$

On the other hand, as the norm on $F$ is not order continuous, applying Theorem 12.13 of [1] that there exists some $y \in F^+$ and a sequence $(y_n) \subset [0, y]$ which does not converge to zero in norm. We may assume that $\|y_n\| = 1$ for all $n$.

Since $\sum_{i=1}^n y_i \leq y$ holds for all $n$, and $F$ is $\sigma$-Dedekind complete, for all $(\alpha_1, \alpha_2, \cdots) \in \ell_\infty$, define the positive operator $S : \ell_\infty \to F$ by

$$S(\alpha_1, \alpha_2, \cdots) = \sum_{i=1}^n \alpha_i y_i$$

Defines a lattice isomorphism from $\ell_\infty$ into $F$ where $\lim_{n \to \infty} \sum_{i=1}^n \alpha_i y_i$ denotes the order limit of the partial sum $\sum_{i=1}^n \alpha_i y_i$.

Since the sequence $(y_n)$ is order bounded and disjoint, for each $(\alpha_1, \alpha_2, \cdots) \in B_{\ell_\infty}$, we see that

$$|S(\alpha_1, \alpha_2, \cdots)| = \lim_{n \to \infty} \sum_{i=1}^n |\alpha_i| y_i \leq (\sup_{i} |\alpha_i|) \cdot y \leq y$$

Hence $S(\alpha_1, \alpha_2, \cdots) \in [-y, y]$, and we have $S(B_{\ell_\infty}) \subset [-y, y]$.

Now consider the operator $T = S \circ R : E \to F$ by

$$T(x) = \lim_{n \to \infty} \sum_{i=1}^n g_i(x) y_i$$

it is positive, and we have

$$T(B_E) = S(R(B_E)) \subset S(B_{\ell_\infty}) \subset [-y, y]$$

It is clear that $T$ is semi-compact.

On the other hand, for all $n$, we have

$$T(x_n) = \lim_{n \to \infty} \sum_{i=1}^n g_i(x_n) y_i = y_n$$

It follows that $\|T(x_n)\| = \|u_n\| = 1$. As the sequence $(x_n)$ is order bounded and disjoint in $E$, it is clear that $T$ is not order weakly compact. Hence, $T$ is not $b$-weakly compact.

Theorem 6: Let $E$ and $F$ be nonzero Banach lattices. Then the following statements are equivalent.

1) Each positive semi-compact operator $T' : F' \to E'$ is $b$-weakly compact.

2) At least one of the following conditions holds:
   a) The norm of $E'$ is order continuous.
   b) The norm of $F'$ is order continuous.

Proof: 1) $\Rightarrow$ 2) Assume by way of contradiction that neither $E'$ nor $F'$ has an order continuous norm. To finish the proof, we have to construct a positive semi-compact operator $T' : F' \to E'$ which is not $b$-weakly compact.

Since the norm on $E'$ is not order continuous, applying Theorem 2.6 of [15] that there exists a disjoint sequence $(x_n) \subset E'$ with $\|x_n\| \leq 1$ for all $n$ and there exists some $0 \leq x' \in E'$ with $x'(x_n) = 1$ for all $n$. Moreover, the components $x'_n$ of $x'$, in the carrier $C_{x_n}$, from an order bounded disjoint sequence in $(E')^+$ such that $x'_n(x_m) = x'(x_m) = 1$ for all $n$ and $x'_n(x_m) = 0$ for $n \neq m$.

Note that $0 \leq x'_n \leq x'$ holds for all $n$.

For all $x \in E$, define the positive operator $R : E \to \ell_1$ by

$$R(x) = (x'(x_n))_{n=1}^\infty$$

Since $\sum_{i=1}^\infty |x_n'(x)| \leq \sum_{i=1}^\infty x'_n(|x|) \leq x'(|x|)$ holds for each $x \in E$, the operator $R$ is well defined.

On the other hand, as the norm on $F'$ is not order continuous, applying Theorem 12.13 of [1] that there exists some $f' \in F'_+$ and a disjoint sequence $(f'_n) \subset [0, f']$ which does not converge to zero in norm. We may assume that $\|f'_n\| = 1$ for all $n$. Hence, for each $n$, we can choose $f_n \in F'_+$ with $\|f_n\| = 1$ and $f'_n(f_n) \geq \frac{1}{2} \|f'_n\| = \frac{1}{2}.$

For all $(\lambda_n) \in \ell_\infty$ consider the positive operator $S : \ell_\infty \to F'$ defined by

$$S(\lambda_n) = \sum_{n=1}^\infty \lambda_n f_n$$

Since $(\lambda_n) \in \ell_\infty$ and $\sum_{n=1}^\infty \|\lambda_n f_n\| = \sum_{n=1}^\infty |\lambda_n|$, it follows that $S$ is well defined.
Now, for all $x \in E$, consider the operator $T = S \circ R : E \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} x_n'(x)f_n$$

Its adjoint $T' : F' \to E'$ defined by

$$T'(g') = \sum_{n=1}^{\infty} g'_n(f_n)x'_n$$

for all $g' \in F'$. Since $\ell_\infty$ is an AM-space with unit, it follows that $R'$ is semi-compact, hence $T'$ is semi-compact.

On the other hand, note that the sequence $f'_n$ is order bounded and disjoint, and

$$\|T'(f'_n)\| = \|\sum_{i=1}^{\infty} f'_n(f_n)x'_n\|$$

$$\geq \|f'_n(f_n)x'_n\| \geq \frac{1}{2} \|x'_n\|$$

$$\geq \frac{1}{2} x'_n(x_n) \geq \frac{1}{2}$$

Hence, $T'$ is not o-weakly compact, it is not b-weakly compact. □

### III. CONCLUSIONS

In this paper, we investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

### ACKNOWLEDGMENT

The authors wish to thank the referee for many valuable remarks.

### REFERENCES


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