Lower Bounds of Some Small Ramsey Numbers

Decha Samana* and Vites Longani

Abstract—For positive integer \( s \) and \( t \), the Ramsey number \( R(s, t) \) is the least positive integer \( n \) such that for every graph \( G \) of order \( n \), either \( G \) contains \( K_s \) as a subgraph or \( \overline{G} \) contains \( K_t \) as a subgraph. We construct the circulant graphs and use them to obtain lower bounds of some small Ramsey numbers.

Keywords—Lower bound, Ramsey numbers, Graphs, Distance line.

I. INTRODUCTION

For positive integer \( s \) and \( t \), the Ramsey number \( R(s, t) \) is the least positive integer \( n \) such that for every graph \( G \) of order \( n \), either \( G \) contains \( K_s \) as a subgraph or \( \overline{G} \) contains \( K_t \) as a subgraph.

The problem of determining Ramsey numbers is known to be very difficult. The few known exact values and several bounds for different graphs are scattered among many technical paper [1].

Table 1. Known nontrivial values and some lower bounds for Ramsey numbers \( R(s, t) \).

<table>
<thead>
<tr>
<th>( s )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
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<td>3</td>
<td>6*</td>
<td>9*</td>
<td>14*</td>
<td>18*</td>
<td>23*</td>
<td>28*</td>
<td>36*</td>
<td>40</td>
<td>46</td>
</tr>
<tr>
<td>4</td>
<td>18*</td>
<td>25*</td>
<td>35</td>
<td>49</td>
<td>56</td>
<td>73</td>
<td>92</td>
<td>98</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>43</td>
<td>58</td>
<td>80</td>
<td>101</td>
<td>126</td>
<td>144</td>
<td>171</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>102</td>
<td>113</td>
<td>132</td>
<td>169</td>
<td>179</td>
<td>253</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Exact Ramsey numbers

For small Ramsey numbers \( R(s, t) \), the general method in establishing a lower bound is to construct a graph \( G \) which does not contain \( K_s \) and the \( \overline{G} \) of \( G \) does not contain \( K_t \). In this paper, we construct the circulant graphs and use them to obtain lower bounds for some small Ramsey numbers.

Definition 1. Let \( G \) be a circulant graph with \( n \) vertices and \( i, j \) be vertices in \( G \). The line distance of line \( \{i, j\} \), denoted by \( d_{ij} \), is defined as

\[
d_{ij} = \min\{|i - j|, n - |i - j|\}
\]

and a line distance set is a set of the line distances.

In section II, we construct line distance sets in order to find lower bounds of some Ramsey numbers.

II. THE MAIN RESULTS

In this section, we find lower bounds of \( R(3, 10), R(3, 11) \), and \( R(3, 12) \) by constructing line distance sets of \( G \) and \( \overline{G} \).

Since \( G \) and \( \overline{G} \) have symmetric patterns, in verifying that \( G \) does not contain \( K_s \) and \( \overline{G} \) does not contain \( K_t \) we can have one vertex fixed and only need to consider other \( s - 1 \) vertices for the case of \( K_s \) and other \( t - 1 \) vertices for the case of \( K_t \).

Theorem 1. \( R(3, 10) \geq 39 \).

Proof: The graph \( G \) of order 38 in Figure 3a has line distance set as \( \{1, 4, 11, 13, 19\} \), and the graph \( \overline{G} \) in Figure 3b has line distance set as \( \{2, 3, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18\} \).

It can be verified that \( G \) contains no \( K_3 \) and \( \overline{G} \) contains no \( K_{10} \). According to the definition of Ramsey numbers, we have that \( R(3, 10) \geq 39 \).

Next, we have a lower bound of \( R(3, 11) \).
Hence \( \text{Theorem 2.} \quad R(3, 11) \geq 46. \)

\textit{Proof:} We have 6 line distance sets of \( G \) and \( \overline{G} \) of order 45, see Figure 4 and Figure 5.

\[
\begin{align*}
\{1, 3, 5, 12, 19\}, \\
\{2, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 20, 21, 22\}; \\
\{2, 6, 7, 10, 21\}, \\
\{1, 3, 4, 5, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18\}; 19, 20, 22\}; \\
\{3, 4, 12, 14, 20\}, \\
\{1, 2, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19, 21, 22\}; \\
\{3, 10, 11, 12, 16\}, \\
\{1, 2, 4, 5, 6, 7, 8, 9, 13, 14, 15, 17, 18, 19, 20, 21, 22\}; \\
\{5, 6, 8, 17, 21\}, \\
\{1, 2, 3, 4, 7, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 22\}; \\
\{6, 13, 20, 21, 22\}, \\
\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19\}.
\end{align*}
\]

It can be verified from each \( G \) and \( \overline{G} \) that \( G \) does not contain \( K_3 \) and \( \overline{G} \) does not contain \( K_{11} \). Hence \( R(3, 11) \geq 46. \)

\[\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{image}
\caption{lower bound of Ramsey number \( R(3, 10) > 38 \).}
\end{figure}\]

\[\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{image}
\caption{lower bound of Ramsey number \( R(3, 11) > 45 \).}
\end{figure}\]

Next, we have a lower bound for \( R(3, 12) \).

\textit{Theorem 3.} \( R(3, 12) \geq 49 \)

\textit{Proof:} We have 12 line distance sets of \( G \) and \( \overline{G} \) of order 48.

\[
\begin{align*}
\{1, 3, 8, 14, 18, 24\}, \\
\{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 19, 20, 21, 22, 23\}; \\
\{2, 3, 8, 14, 15, 24\}, \\
\{1, 4, 5, 6, 7, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23\}; \\
\{2, 3, 8, 17, 24\}, \\
\{1, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23\}; \\
\{2, 7, 8, 21, 24\}, \\
\{1, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 22, 23\}; \\
\{2, 8, 9, 14, 21, 24\}, \\
\{1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23\}; \\
\{3, 8, 9, 10, 22, 24\}, \\
\{1, 2, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23\}; \\
\{5, 6, 8, 15, 22, 24\}, \\
\{1, 2, 3, 4, 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 23\}; \\
\{6, 8, 9, 10, 13, 24\}, \\
\{1, 2, 3, 4, 5, 7, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}; \\
\{6, 8, 9, 19, 22, 24\}.
\end{align*}
\]
\{1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 23\};
\{6, 8, 10, 11, 15, 24\},
\{1, 2, 3, 4, 5, 7, 9, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23\};
\{8, 10, 15, 21, 22, 24\},
\{1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 13, 14, 16, 17, 18, 19, 20, 23\};
\{8, 14, 18, 21, 23, 24\},
\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15, 16, 17, 19, 20, 22\}.

It can be verified from each $G$ and $\overline{G}$ that $G$ does not contain $K_3$ and $\overline{G}$ does not contain $K_{12}$.

Hence $R(3, 12) \geq 49$.

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\[ \text{REFERENCES} \]


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