Generalized inverse eigenvalue problems for symmetric arrow-head matrices

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Abstract—In this paper, we first give the representation of the general solution of the following inverse eigenvalue problem (IEP): Given $X \in \mathbb{R}^{n \times p}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{p \times p}$, find nontrivial real-valued symmetric arrow-head matrices $A$ and $B$ such that $AXA = BX$. We then consider an optimal approximation problem: Given real-valued symmetric arrow-head matrices $\hat{A}, \hat{B} \in \mathbb{R}^{n \times n}$, find $(\hat{A}, \hat{B}) \in S_E$ such that $\|A - A\|^2 + \|B - B\|^2 = \min_{A, B \in S_E} (\|A - \hat{A}\|^2 + \|B - \hat{B}\|^2)$, where $S_E$ is the solution set of IEP. We show that the optimal approximation solution $(\hat{A}, \hat{B})$ is unique and derive an explicit formula for it.

Keywords—partially prescribed spectral information, symmetric arrow-head matrix, inverse problem, optimal approximation.

I. INTRODUCTION

Throughout this paper, we denote the real $m \times n$ matrix space by $\mathbb{R}^{m \times n}$, the set of all symmetric matrices in $\mathbb{R}^{n \times n}$ by $\mathbf{SR}^{n \times n}$, the transpose and the Moore-Penrose generalized inverse of a real matrix $A$ by $A^T$ and $A^+$, respectively. $I_n$ represents the identity matrix of size $n$. For $A, B \in \mathbb{R}^{m \times n}$, an inner product in $\mathbb{R}^{m \times n}$ is defined by $(A, B) = \text{trace}(B^TA)$, then $\mathbb{R}^{m \times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{R}^{n \times p}$, the Kronecker product of $A$ and $B$ is defined by $A \otimes B = [a_{ij}b_{kl}] \in \mathbb{R}^{mp \times nq}$. Also, for an $m \times n$ matrix $A = [a_{ij}]$, where $a_{ij}, i = 1, \ldots, m$, is the $i$-th column vector of $A$, the stretching function $\text{Vec}(A)$ is defined by $\text{Vec}(A) = [a_1^T, a_2^T, \ldots, a_n^T]^T$.

Definition 1 An $n \times n$ matrix $A$ is called an arrow-head matrix if

$$A = \begin{bmatrix}
    a_1 & b_1 & b_2 & \cdots & b_{n-1} \\
    c_1 & a_2 & 0 & \cdots & 0 \\
    c_2 & 0 & a_3 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_{n-1} & 0 & 0 & \cdots & a_n
\end{bmatrix}.$$

If $b_i = c_i$, $i = 1, \ldots, n-1$, then $A$ is a symmetric arrow-head matrix.

The application background and the computations of the eigenvalues and eigenvectors of this kind of matrices can see [1, 2, 3, 4]. The inverse problem of constructing the symmetric arrow-head matrix from spectral data has been investigated by Peng et al. [5], and Borges et al. [6]. In this paper we will further consider generalized inverse eigenvalue problems for symmetric arrow-head matrices, which can be described as follows:

Problem IEP. Given $X \in \mathbb{R}^{n \times p}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{p \times p}$, find nontrivial real-valued symmetric arrow-head matrices $A$ and $B$ such that

$$AXA = BX.$$

(1)

Problem II. Given real-valued symmetric arrow-head matrices $\hat{A}, \hat{B} \in \mathbb{R}^{n \times n}$, find $(\hat{A}, \hat{B}) \in S_E$ such that

$$\|\hat{A} - \hat{A}\|^2 + \|\hat{B} - \hat{B}\|^2 = \min_{(A, B) \in S_E} (\|A - \hat{A}\|^2 + \|B - \hat{B}\|^2),$$

where $S_E$ is the solution set of IEP.

The paper is organized as follows. In Section 2, using the Kronecker product and stretching function $\text{Vec}(\cdot)$ of matrices, we give an explicit representation of the solution set $S_E$ of Problem IEP. In Section 3, we show that there exists a unique solution in Problem II and present the expression of the unique solution $(\hat{A}, \hat{B})$ of Problem II. Finally, in Section 4, a numerical algorithm to acquire the optimal approximation solution under the Frobenius norm sense is described and a numerical example is provided.

II. THE SOLUTION OF PROBLEM IEP

To begin with, we introduce two lemmas.

Lemma 1: [7] If $L \in \mathbb{R}^{m \times q}$, $b \in \mathbb{R}^m$, then $Ly$ has a solution $y \in \mathbb{R}^q$ if and only if $LL^Tb = b$. In this case, the general solution of the equation can be described as $y = L^Tb + (I_q - L^TL)z$, where $z \in \mathbb{R}^q$ is an arbitrary vector.

Lemma 2: [8] Let $D \in \mathbb{R}^{m \times n}, H \in \mathbb{R}^{m \times l}, J \in \mathbb{R}^{l \times s}$. Then

$$\text{Vec}(DHJ) = (J^T \otimes D)\text{Vec}(H).$$

Let $S_0$ be the set of all real-valued symmetric arrow-head matrices, then $S_0$ is a linear subspace of $\mathbf{SR}^{n \times n}$, and the dimension of $S_0$ is $d = 2n - 1$.

Define $Y_{ij}$ as

$$Y_{ij} = \begin{cases}
    \frac{\sqrt{2}}{2}(e_i e_j^T + e_j e_i^T), & i = 1, \ j = 2, \ldots, n; \\
    e_i e_j^T, & i = j = 1, \ldots, n,
\end{cases}$$

(3)
where \( e_i, i = 1, \cdots, n \), is the \( i \)-th column vector of the identity matrix \( I_n \). It is easy to verify that \( \{ Y_{ij} \} \) forms an orthonormal basis of the subspace \( S_\theta \), that is,

\[
(Y_{ij}, Y_{kl}) = \begin{cases} 
0, & i \neq k \text{ or } j \neq l, \\
1, & i = k \text{ and } j = l.
\end{cases}
\]  

(4)

Now, if \( A, B \in \mathbb{R}^{n \times n} \) are symmetric arrow-head matrices, then \( A, B \) can be expressed as

\[
A = \sum_{i,j} \alpha_{ij} Y_{ij}, \quad B = \sum_{i,j} \beta_{ij} Y_{ij},
\]

(5)

where the real numbers \( \alpha_{ij}, \beta_{ij}, \) \( i = 1, j = 2, \cdots, n; \) \( i = j = 1, \cdots, n \), are yet to be determined. Substituting (5) into (1), we have

\[
\sum_{i,j} \alpha_{ij} Y_{ij} X \Lambda - \sum_{i,j} \beta_{ij} Y_{ij} X = 0.
\]

(6)

Let

\[
\alpha = [\alpha_{11}, \cdots, \alpha_{nn}, \alpha_{n1}, \cdots, \alpha_{1n}]^T, \\
\beta = [\beta_{11}, \cdots, \beta_{nn}, \beta_{n1}, \cdots, \beta_{1n}]^T, \\
G = [\text{Vec}(Y_{11}), \cdots, \text{Vec}(Y_{nn}), \\
\text{Vec}(Y_{12}), \cdots, \text{Vec}(Y_{1n})] \in \mathbb{R}^{n^2 \times d}
\]

and

\[
M = (A^T X^T \otimes I_n) G, \quad N = (X^T \otimes I_n) G.
\]

(7)

Using Lemma 2, we see that the equation of (6) is equivalent to

\[
M \alpha - N \beta = 0.
\]

(9)

It follows from Lemma 1 that the equation of (9) with unknown vector \( \alpha \) has a solution if and only if

\[
E_M N \beta = 0,
\]

(10)

where \( E_M = I_{np} - MM^+ \). Using Lemma 1 again, we know that the equation of (10) with respect to \( \beta \) is always solvable and the general solution to the equation is

\[
\beta = (I_d - (E_M N)^+ E_M N) u,
\]

(11)

where \( u \in \mathbb{R}^d \) is an arbitrary vector.

Substituting (11) into (9) and applying Lemma 1, we obtain

\[
\alpha = M^+ N (I_d - (E_M N)^+ E_M N) u + F_M v,
\]

(12)

where \( F_M = I_d - M^+ M \), and \( v \in \mathbb{R}^d \) is an arbitrary vector.

In summary of above discussion, we have proved the following result.

Theorem 1: Suppose that \( X \in \mathbb{R}^{n \times p}, \Lambda \in \mathbb{R}^{p \times p} \), and \( \Lambda \) is a diagonal matrix. Let \( \{ Y_{ij} \}, \ G, M, N \) be given as in (3), (7) and (8). Write \( d = 2n - 1, \ E_M = I_{np} - MM^+, \ F_M = I_d - M^+ M \). Then the solution set \( S_E \) of Problem IEP can be expressed as

\[
S_E = \{ (A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} : \ A = K(\alpha \otimes I_n), \ B = K(\beta \otimes I_n) \},
\]

(13)

where

\[
K = [Y_{11}, \cdots, Y_{nn}, Y_{12}, \cdots, Y_{1n}] \in \mathbb{R}^{n \times nd},
\]

(14)

\( \alpha, \beta \) are, respectively, given by (12) and (11) with \( u, v \in \mathbb{R}^d \) being arbitrary vectors.

III. THE SOLUTION OF PROBLEM II

It follows from Theorem 1 that the set \( S_E \) is always nonempty. It is easy to verify that \( S_E \) is a closed convex subset of \( \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \). From the best approximation theorem [9], we know there exists a unique solution \( (\hat{A}, \hat{B}) \) in \( S_E \) such that (2) holds.

We now focus our attention on seeking the unique solution \( (\hat{A}, \hat{B}) \) in \( S_E \). For the real-valued symmetric arrow-head matrices \( \hat{A} \) and \( \hat{B} \), it is easily seen that \( \hat{A}, \hat{B} \) can be expressed as the linear combinations of the orthonormal basis \( \{ Y_{ij} \} \), that is,

\[
\hat{A} = \sum_{i,j} \gamma_{ij} Y_{ij}, \quad \hat{B} = \sum_{i,j} \delta_{ij} Y_{ij},
\]

(15)

where \( \gamma_{ij}, \delta_{ij}, \) \( i = 1, j = 2, \cdots, n; \) \( i = j = 1, \cdots, n \), are uniquely determined by the elements of \( \hat{A} \) and \( \hat{B} \).

Let

\[
\gamma = [\gamma_{11}, \cdots, \gamma_{nn}, \gamma_{12}, \cdots, \gamma_{1n}]^T, \\
\delta = [\delta_{11}, \cdots, \delta_{1n}, \delta_{12}, \cdots, \delta_{1n}]^T.
\]

(16, 17)

Then, for any pair of matrices \( (A, B) \in S_E \) in (13), by the relations of (4) and (15) we see that

\[
f = \| A - \hat{A} \|^2 + \| B - \hat{B} \|^2
\]

\[
= \| \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij} \|^2 + \| \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij} \|^2
\]

\[
= \left( \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij} \right)^2 \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Y_{ij} \right) + \left( \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij} \right)^2 \sum_{i,j} (\beta_{ij} - \delta_{ij}) Y_{ij} \right)
\]

\[
= \sum_{i,j} (\alpha_{ij} - \gamma_{ij})^2 + \sum_{i,j} (\beta_{ij} - \delta_{ij})^2
\]

\[
= \| \alpha - \gamma \|^2 + \| \beta - \delta \|^2.
\]
Substituting (11) and (12) into the relation of $f$, we have

$$ f = ||M^+NWu + F_Mv - \gamma||^2 + ||Wu - \delta||^2 $$

$$ = u^TWN(MM^T)^+NWu - 2\gamma^TMM^+NWu $$

$$ - 2\gamma^TF_Mv + u^TF_Mv + \gamma^T\gamma $$

$$ + u^TWu - 2u^TW\delta + \delta^T\delta, $$

where $W = I_d - (E_MN)^+E_MN$. Therefore,

$$ \frac{\partial f}{\partial u} = 2WN^T(MM^T)^+NWu $$

$$ - 2WN^T(MM^T)^+\gamma + 2W\delta, $$

$$ \frac{\partial f}{\partial v} = 2F_Mv - 2F_M\gamma, $$

which yields

$$ Wu = (I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(MM^T)^+\gamma), \tag{18} $$

$$ F_Mv = F_M\gamma. \tag{19} $$

Upon substituting (18) and (19) into (11) and (12), we obtain

$$ \hat{\alpha} = M^+NW(I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(MM^T)^+\gamma) $$

$$ \hat{\beta} = W(I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(MM^T)^+\gamma). \tag{20} $$

By now, we have proved the following result.

**Theorem 2:** Let the real-valued symmetric arrow-head matrices $\hat{A}$ and $\hat{B}$ be given. Then Problem II has a unique solution and the unique solution of Problem II can be expressed as

$$ \hat{A} = K(\hat{\alpha} \otimes I_n), \tag{22} $$

$$ \hat{B} = K(\hat{\beta} \otimes I_n), \tag{23} $$

where $\hat{\alpha}$, $\hat{\beta}$ are given by (20) and (21), respectively.

**IV. A NUMERICAL EXAMPLE**

Based on Theorem 1 and Theorem 2 we can describe an algorithm for solving Problem IEP and Problem II as follows.

**Algorithm 1.**

1. Input $\hat{A}$, $\hat{B}$, $\Lambda$, $X$.
2. Form the orthonormal basis $\{Y_i\}$ by (3).
3. Compute $G$, $M$, $N$ according to (7) and (8), respectively.
5. Form vectors $\gamma, \delta$ by (15), (16) and (17).
6. Compute $K$, $\hat{\alpha}$, $\hat{\beta}$ by (14), (20) and (21), respectively.
7. Compute the unique solution $(\hat{A}, \hat{B})$ of Problem II by (22) and (23).

**Example 1.** Given

$$ \tilde{A} = \begin{pmatrix} -4 & 2 & 5 & 1 & 2 & 11 \\ 2 & -3 & 0 & 0 & 0 & 0 \\ 5 & 0 & -6 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & -4 & 0 \\ 11 & 0 & 0 & 0 & 0 & -44 \end{pmatrix}, $$

$$ \tilde{B} = \begin{pmatrix} -7 & 2 & 19 & 9 & 3 & 15 \\ 2 & -13 & 0 & 0 & 0 & 0 \\ 19 & 0 & -8 & 0 & 0 & 0 \\ 9 & 0 & 0 & -6 & 0 & 0 \\ 3 & 0 & 0 & 0 & -3 & 0 \\ 15 & 0 & 0 & 0 & 0 & -28 \end{pmatrix} \tag{19} $$

and

$$ A = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\} = \text{diag}\{2.1709, 0.84882, 0.73245\}, $$

$$ X = [x_1, x_2, x_3] = \begin{bmatrix} -0.27362 & 0.019321 & -0.090308 \\ 0.071468 & 0.00087224 & -0.0056621 \\ 0.70165 & 0.085116 & -0.3485 \\ -0.91 & 0.035512 & -0.16064 \\ -0.050465 & -0.91 & -0.39781 \\ -0.030195 & -0.024079 & 0.91 \end{bmatrix}. $$

Using Algorithm 1, we obtain the unique solution of Problem II as follows.

$$ \hat{A} = \begin{bmatrix} -3.8904 & 1.8001 & 4.7078 \\ 1.8001 & -2.7001 & 0 \\ 4.7078 & 0 & -5.6494 \\ 0.90628 & 0 & 0 \\ 1.8874 & 0 & 0 \\ 10.383 & 0 & 0 \end{bmatrix}, $$

$$ \hat{B} = \begin{bmatrix} -6.6982 & 2.0161 & 20.037 \\ 2.0161 & -13.105 & 0 \\ 20.037 & 0 & -8.4365 \\ 9.1354 & 0 & 0 \\ 3.1709 & 0 & 0 \\ 15.857 & 0 & 0 \end{bmatrix}. \tag{20} $$
We define the residual as
\[ \text{res}(\lambda_i, x_i) = \| (\lambda_i \hat{\mathbf{A}} - \hat{\mathbf{B}}) x_i \|, \]
and the numerical results shown as follows.

<table>
<thead>
<tr>
<th>((\lambda_i, x_i))</th>
<th>((\lambda_1, x_1))</th>
<th>((\lambda_2, x_2))</th>
<th>((\lambda_3, x_3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Res((\lambda_i, x_i))</td>
<td>1.7468e-014</td>
<td>1.0116e-014</td>
<td>3.6636e-014</td>
</tr>
</tbody>
</table>

Furthermore, we can figure out
\[ \| \hat{\mathbf{A}} - \tilde{\mathbf{A}} \| = 2.7319, \quad \| \hat{\mathbf{B}} - \tilde{\mathbf{B}} \| = 2.57. \]

REFERENCES


Yongxin Yuan received his Ph.D from Nanjing University of Aeronautics and Astronautics in China. His research interests are mainly on numerical algebra and matrix theory, especially in the theory and computation of the inverse problems in Structural Dynamics.