On minimum cycle bases of the wreath product of wheels with stars

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Abstract—The length of a cycle basis of a graph is the sum of the lengths of its elements. A minimum cycle basis is a cycle basis with minimum length. In this work, a construction of a minimum cycle basis for the wreath product of wheels with stars is presented. Moreover, the length of minimum cycle basis and the length of its longest cycle are calculated.

Keywords—cycle space, minimum cycle basis, wreath product.

I. INTRODUCTION

For a given graph $G$, we denote the vertex set of $G$ by $V(G)$ and the edge set by $E(G)$. The set $E$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over $\mathbb{Z}_2$ with vector addition $X \oplus Y = (X \setminus Y) \cup (Y \setminus X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in E$. The cycle space, $\mathcal{C}(G)$, of a graph $G$ is the vector subspace of $(\mathbb{Z}_2, \oplus, \cdot)$ spanned by the cycles of $G$. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph $G$ the dimension of the cycle space is the cyclomatic number or the first Betti number, $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1$ (see [4]).

A basis $B$ for $\mathcal{C}(G)$ is called a cycle basis of $G$. The length, $|C|$, of the element $C$ of the cycle space $\mathcal{C}(G)$ is the number of its edges. The length, $l(B)$, of a cycles basis $B$ is the sum of the lengths of its elements: $l(B) = \sum_{C \in B} |C|$. A minimum cycle basis (MCB) is a cycle basis with minimum length.

The length of the longest element in an MCB is denoted by $\lambda(G)$. Since the cycle space is a matroid in which an element $C$ has weight $|C|$, the greedy algorithm can be used to extract an MCB (see [9]). The following results will be used frequently in the sequel (see [6]).

Lemma 2.2. (Jaradat, et al.) Let $A, B$ be sets of cycles of a graph $G$, and suppose that both $A$ and $B$ are linearly independent, and that $E(A) \cap E(B)$ induces a forest in $G$ (we allow the possibility that $E(A) \cap E(B) = \emptyset$). Then $A \cup B$ is linearly independent.

In the present article we continue what we initiate in [1], [4], and [7] by studying the problem of constructing an MCB for the wreath product of wheels and stars, where an MCB of graphs find applications in sciences and engineering: for examples, biochemistry, structural engineering, surface reconstruction and public transportations (See [2], [3] and [8]).

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The Cartesian product $G \Box H$ has the vertex set $V(G \Box H) = V(G) \times V(H)$ and the edge set $E(G \Box H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G) \text{ and } v_1 = v_2, \text{ or } v_1v_2 \in E(H) \text{ and } u_1 = u_2 \}$. (2) The Lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G_1[H]) = V(G_1) \times V(H)$ and the edge set $E(G_1[H]) = \{(u_1, v_1)(v_1, v_2) | u_1 = v_1 \text{ and } u_2v_2 \in E(H) \text{ or } u_1v_1 \in E(G_1) \}$. (3) The wreath product $G \rho H$ has the vertex set $V(G \rho H) = V(G) \times V(H)$ and the edge set $E(G \rho H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2 \text{ and } v_1v_2 \in H \text{, or } u_1u_2 \in G \text{ and there is } \alpha \in \text{Aut}(H) \text{ such that } \alpha(v_1) = v_2 \}$.

In the rest of this paper, we let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of the wheel $W_n$ (the star $S_n$), with $d_{W_n}(u_1) = n-1 \text{ and } \{v_1, v_2, \ldots, v_m\}$ be the vertex set $S_m$ with $d_{S_m}(v_1) = m-1$. Moreover, $N_{n-1}$ stands for the null graph with vertex set $\{v_2, v_3, \ldots, v_m\}$. Wherever they appear $a, b, c, d$ and $l$ stand for vertices and $E(B) = \cup_{C \in B} E(C)$ where $B \subseteq \mathcal{C}(G)$.

II. THE MINIMUM CYCLE BASIS OF $W_n \rho S_m$.

In this section, we present a minimum cycle basis of $W_n \rho S_m$. To proceed in our work we set the following sets of cycles:

$$R_{lab} = \{ (l, v_{j+1})(a, v_j)(b, v_j)(l, v_{j+1}) \}$$

$$M_{lab} = \{ (l, v_{j+1})(a, v_j)(b, v_{j+1})(l, v_{j+1}) \}$$

Also, for $j = 1, 2, \ldots, m$ we set the following cycle

$$U_{lab} = (l, v_j)(a, v_j)(b, v_j)(l, v_j).$$

Let

$$U_{lab} = \cup_{j=2}^m U_{lab}^{(j)}.$$

Lemma 2.1. $S_{lab} = U_{lab} \cup R_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{M}_{lab} \cup \mathcal{M}_{tba}$ is a linearly independent set.

Proof. Since each of $U_{lab}, R_{lab}, M_{lab}, M_{tba}, \mathcal{R}_{lab}, \mathcal{M}_{lab}$, and $\mathcal{R}_{lab}$ consists of edge disjoint cycles, as a result each of which is linearly independent. Now, any linear combination of cycles of $\mathcal{R}_{lab}$ contains an edge of the form $(l, v_{i+1})(a, v_i)$ for some $2 \leq i \leq m-1$ which occurs in no cycle of $U_{lab}$. Thus $U_{lab} \cup \mathcal{R}_{lab}$ is linearly independent. Any linear combination of cycles of $M_{lab}$ contains an edge of the form $(b, v_{i+1})(a, v_i)$ for some $2 \leq i \leq m-1$ which does not occur in any cycle of $U_{lab} \cup \mathcal{R}_{lab}$. Hence $U_{lab} \cup \mathcal{R}_{lab} \cup M_{lab}$ is linearly independent. Similarly, since any linear combination of cycles of $M_{tba}$ contains an edge of the form $(a, v_{i+1})(b, v_i)$ for some
2 ≤ i ≤ m − 1 which occurs in no cycle of \( U_{lab} \cup R_{lab} \cup M_{lab} \), as a result \( U_{lab} \cup R_{lab} \cup M_{lab} \) is linearly independent. Also, since any linear combination of cycles of \( R_{lab} \) contains an edge of the form \( (b, v_{i+1}) \) for some \( 2 ≤ i ≤ m − 1 \) which does not belong to any cycle of \( U_{lab} \cup R_{lab} \cup M_{lab} \), we have \( U_{lab} \cup R_{lab} \cup M_{lab} \) is linearly independent. Finally, any linear combination of cycles of \( R_{lab} \) contains an edge of the form \( (a, v_{i+1}) \) for some \( 2 ≤ i ≤ m − 1 \) which does not appear in any cycle of \( U_{lab} \cup R_{lab} \cup M_{lab} \). Therefore, \( S_{lab} \) is linearly independent.

Now, for each \( 2 ≤ j ≤ m − 2 \), we define the following set of cycles:

\[
E_{lab}^{(j)} = \{ E_{lab}^{(j)} = (l, v_{i+1−j})(a, v_{i})(b, v_{i+j}) \} \quad 2 ≤ i ≤ m − j \}
\]

Let

\[
E_{lab} = \bigcup_{j=2}^{m−2} E_{lab}^{(j)}
\]

**Lemma 2.2.** The set \( Z_{lab}^{(j)} = E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \) is linearly independent:

**Proof.** Since each of \( E_{lab}, E_{lab}, E_{lab}, E_{lab} \) and \( E_{lab} \) consists only of edge disjoint cycles and since

\[
E(E_{lab}^{(j)} \cap E_{lab}^{(j)}) = E(E_{lab}^{(j)} \cap E_{lab}^{(j)}) = E(E_{lab}^{(j)} \cap E_{lab}^{(j)}) = 0,
\]

\( E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \) is linearly independent by Lemma 1.2. Any linear combination of cycles of \( E_{lab}^{(j)} \) contains an edge of the form \( (l, v_{i+1})(b, v_{i+j}) \) for some \( 2 ≤ i ≤ m − j \) which does not occur in any cycle of \( E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \). Hence, \( E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \) is linearly independent. Since any linear combination of cycles of \( E_{lab}^{(j)} \) contains an edge of the form \( (l, v_{i+1})(b, v_{i+j}) \) for some \( 2 ≤ i ≤ m − j \) which belongs to no cycle of \( E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \cup E_{lab}^{(j)} \). Therefore, \( Z_{lab}^{(j)} \) is linearly independent.

**Lemma 2.3.** Any linear combination of cycles of \( (m−2)_{lab} \) contains at least one edge of the following forms:

\[
(a, v_{i})(b, v_{i+j}), (b, v_{i})(a, v_{i+j}), (b, v_{i})(l, v_{i+j}), (l, v_{i})(a, v_{i+j}) \quad 2 ≤ i ≤ m − 2 \text{ and } 2 ≤ i ≤ m − j.
\]

**Proof.** Assume that \( C \) is a linear combination of the cycles of \( \mathcal{O} = \{O_1, O_2, \ldots, O_r\} \subseteq \bigcup_{m−2} Z_{lab} \) and \( C \) does not contain any edge of the above stated forms. Then we split our work into the following cases:

**Case 1:** \( O \) contains at least one cycle of \( (m−2)_{lab} \) say \( E_{lab}^{(j, 0)} \) which does not occur in any other cycle of \( (m−2)_{lab} \). Thus, \( e \) does not belong to any other cycle of \( O \). Therefore \( e \in E(\bigcup_{i=1} O_i) = E(\mathcal{C}) \). This contradicts the fact that \( e \notin E(\mathcal{C}) \).

**Case 2:** \( O \) contains no cycle of \( (m−2)_{lab} \) but contains at least one cycle of \( (m−1)_{lab} \) say \( E_{lab}^{(j, 0)} \) which does not occur in any other cycle of \( (m−2)_{lab} \) \( (m−1)_{lab} \). Thus, \( e \) does not belong to any other cycle of \( O \). Therefore, \( e \in E(\bigcup_{i=1} O_i) = E(\mathcal{C}) \). This contradicts the fact that \( e \notin E(\mathcal{C}) \).

**Case 3:** \( O \) contains no cycle of \( (m−2)_{lab} \) or \( (m−2)_{lab} \) but contains at least one cycle of \( (m−2)_{lab} \) or \( (m−2)_{lab} \). Then we consider the following subcases:

**Subcase 3a:** \( O \) contains cycle of \( (m−2)_{lab} \). Let \( j \in (j ≤ m − 2) \) be the largest integer such that \( E_{lab}^{(j)} \cap O \neq \emptyset \). Let \( t_0 \) \( (2 ≤ t_0 ≤ m − j) \) be the largest integer greater such that \( E_{lab}^{(j)} \in O \), say \( Z_1 = E_{lab}^{(j)} \). Note that \( e \equiv (a, v_{i})(b, v_{i+j}) \in E(\mathcal{O}_1) \) and \( e \notin E(\mathcal{C}) \). Moreover, the only other cycle of \( (m−2)_{lab} \) \( (m−2)_{lab} \) \( (m−2)_{lab} \) that contains \( e \) is \( E_{lab}^{(j+1)} \). Hence, \( E_{lab}^{(j+1)} \in O \), say \( O_2 = E_{lab}^{(j+1)} \). Note that \( e \equiv (a, v_{i})(b, v_{i+j}) \) \( E(\mathcal{O}_2) \) which does not occur in \( O_1 \). Thus, \( e \in E(\bigcup_{i=1} O_i) \). Since \( e \notin E(\mathcal{C}) \) and \( E_{lab}^{(j+1)} \) is the only other cycle of \( (m−2)_{lab} \) \( (m−2)_{lab} \) \( (m−2)_{lab} \) which contains \( e \). By continuing in this process, we get the following: There is \( r \) such that

\[
O_r = E_{lab}^{(m−n, n)} \in O \quad O_r = E_{lab}^{(m−n, n)} \in O \quad O_r = E_{lab}^{(m−n, n)} \in O.
\]

Note that in both cases \( e_r = (a, v_{i})(b, v_{i+j}) \in O_r \) which occurs in no other cycles of \( (m−2)_{lab} \) \( (m−2)_{lab} \) \( (m−2)_{lab} \). Thus, \( e_r \in E(\bigcup_{i=1} O_i) \). This contradicts the fact that \( e_r \notin E(\mathcal{C}) \).

**Subcase 3b:** \( O \) contains cycle of \( (m−2)_{lab} \). We use the same arguments as in Case 3a to have a similar contradiction.

**Case 4:** \( O \) contains no cycle of \( (m−2)_{lab} \) \( (m−2)_{lab} \) \( (m−2)_{lab} \) but contains at least one cycle of \( (m−2)_{lab} \). As in Case 3a, let \( j \) \( (2 ≤ j ≤ m − 2) \) be the largest integer such that \( E_{lab}^{(j)} \cap O \neq \emptyset \). Let \( i_0 \) \( (2 ≤ i_0 ≤ m − j) \) be the largest integer greater such that \( E_{lab}^{(j)} \in O \), say \( O_1 = E_{lab}^{(j)} \). Note that \( e \equiv (a, v_{i})(b, v_{i+j}) \in E(\mathcal{O}_1) \) and \( e \notin E(\mathcal{C}) \). Moreover, the only other cycle of \( (m−2)_{lab} \) \( (m−2)_{lab} \) \( (m−2)_{lab} \) that contains \( e \) is \( E_{lab}^{(j+1)} \). Hence, \( E_{lab}^{(j+1)} \in O \), say \( O_2 = E_{lab}^{(j+1)} \). By continuing in the process as in Case 3, taking into account only Subcase 3a, we get a similar contradiction.

**Case 5:** \( O \) contains no cycle of \( (m−2)_{lab} \) \( (m−2)_{lab} \) \( (m−2)_{lab} \) but contains at least one cycle of \( (m−2)_{lab} \). Then \( O \) contains at least one cycle of \( (m−2)_{lab} \) say \( O_1 = E_{lab}^{(j)} \). Then, \( e \equiv (a, v_{i})(b, v_{i+j}) \in E(\mathcal{O}_1) \) and does not occur in any
other cycle of \((\bigcup_{j=2}^{m-2}z_{ij}^{(j)}) - (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)})\). Thus, \(e \in \bigcap_{i=1}^{n} O_i\).

This contradicts the fact that \(e \notin E(C)\). \(\square\)

**Remark 2.4.** By using same arguments as in Lemma 4.3, we can show, by taking into account Cases 1, 4, and 5 and Sub-case 3a, that any linear combination of \((\bigcup_{j=2}^{m-2}z_{ij}^{(j)}) - (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}c_{ij}^{(j)}) \cup (\bigcup_{j=2}^{m-2}e_{ij}^{(j)})\) contains at least one edge of the following forms: \((a, v_1)(b, v_{i+1})\), \((b, v_i)(a, v_{i+1})\), \((b, v_i)(l, v_{i+1})\), and \((l, v_i)(b, v_{i+1})\) where \(2 \leq j \leq m - 2 \) and \(2 \leq i \leq m - j\). \(\square\)

**Lemma 2.5.** The set \(S_{lb} \cup (\bigcup_{j=2}^{m-2}z_{ij}^{(j)})\) is linearly independent.

**Proof.** We use mathematical induction on \(m\) to show that \((\bigcup_{j=2}^{m-2}z_{ij}^{(j)})\) is linearly independent. If \(m = 4\), then \((\bigcup_{j=2}^{m-2}z_{ij}^{(j)})\) consists only of \(z_{ij}^{(2)}\). Thus, the result follows from Lemma 2.2. Assume that \(m\) is greater than 4 and the result is true for less than \(m\). Note that \((\bigcup_{j=2}^{m-3}z_{ij}^{(j)}) = \bigcap_{j=2}^{m-3}z_{ij}^{(j)} \cup (\bigcup_{j=2}^{m-2}z_{ij}^{(j)}).\) By a similar argument to that in proof of Lemma 2.3, any linear combination of cycles of \(z_{ij}^{(m-2)}\) contains at least one edge of the following: \((a, v_1)(b, v_{n_1})\), \((b, v_2)(a, v_{n_2})\), \((b, v_1)(l, v_{n_1})\), \((l, v_2)(a, v_{n_2})\), and \((a, v_2)(l, v_{n_2})\). Since non of them occurs in any cycle of \((\bigcup_{j=2}^{m-3}z_{ij}^{(j)})\), \((\bigcup_{j=2}^{m-2}z_{ij}^{(j)})\) is linearly independent. By Lemma 2.1 \(S_{lb}\) is linearly independent. Now, by Lemma 2.3 any linear combination of cycles of \((\bigcup_{j=2}^{m-2}z_{ij}^{(j)})\) contains at least one edge of the following:

\((a, v_1)(b, v_{i+1}), (l, v_i)(b, v_{i+1}), (l, v_i)(a, v_{i+1}), (a, v_1)(l, v_{i+1})\)

and \((a, v_2)(b, v_{i+1})\).

**Proof.** Let \(C\) be a linear combination of the cycles of \(O = \{O_1, O_2, \ldots, O_n\} \subseteq S_{lb} - R_{abld}.\) Then we consider the following cases:

**Case 1:** \(O\) contains at least one cycle of \(M_{lb}\), say \(O_1 = M_{lb}^{(1)}\). Note that \(e = (a, v_1)(b, v_i) \in O_1\) and does not occur in other cycles of \(M_{lb}\). And so \(e\) does not occur in any other cycle of \(S_{lb} - M_{lb}\). Therefore, \((l, v_{i+1})(b, v_i) \in E(O_i)\).

**Case 2:** \(O\) contains no cycle of \(M_{lb}\), but contains at least one cycle of \(R_{abld}\), say \(O_1 = R_{abld}^{(1)}\). Note that \(e = (l, v_{i+1})(b, v_i) \in E(O_i)\) and does not occur in other cycles of \(M_{lb}^{*} - M_{lb}\). And so \(e\) does not occur in any other cycle of \(S_{lb} - M_{lb}\). Thus, \((l, v_{i+1})(b, v_i) \in E(O_i)\).

**Case 3:** \(O\) contains no cycle of \(M_{lb} \cup R_{abld}\), but contains at least one cycle of \(R_{abld}\), say \(O_1 = R_{abld}^{(1)}\). Note that \(e = (l, v_{i+1})(b, v_i) \in E(O_i)\) and does not occur in other cycles of \(S_{lb} - M_{lb} \cup R_{abld}\). Thus, \((l, v_{i+1})(b, v_i) \in E(O_i)\).

**Case 4:** \(O\) contains no cycle of \(M_{lb} \cup R_{abld} \cup R_{abld}\), but contains at least one cycle of \(M_{lb}\), say \(O_1 = M_{lb}^{(1)}\). Note that \(e = (l, v_{i+1})(b, v_i) \in O_1\) and does not occur in other cycles of \(S_{lb} - M_{lb} \cup R_{abld} \cup R_{abld}\). And so \(e\) does not occur in any other cycle of \(S_{lb} - M_{lb} \cup R_{abld} \cup R_{abld}\). Thus, \((l, v_{i+1})(b, v_i) \in E(O_i)\).

**Case 5:** \(O\) contains no cycle of \(M_{lb} \cup R_{abld} \cup R_{abld} \cup M_{lb}\). Then \(O\) must contain at least one cycle of \(U_{abld}\), say \(O_1 = U_{abld}^{(1)}\). Note that \(e = (l, v_{i+1})(b, v_i) \in O_1\) and does not occur in other cycles of \(S_{lb} - M_{lb} \cup R_{abld} \cup R_{abld} \cup M_{lb}\). And so \(e\) does not occur in any other cycle of \(S_{lb} - M_{lb} \cup R_{abld} \cup R_{abld} \cup M_{lb}\). Thus, \((l, v_{i+1})(b, v_i) \in E(O_i)\).

**Lemma 2.7.** \((\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*})\) is linearly independent set.

**Proof.** We prove that \((\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*})\) is linearly independent using mathematical induction on \(n\). If \(n = 4\), then \((\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*}) = S_{u_{2}u_1}^{*}\) which is linearly independent by Lemma 2.5. Assume that \(n\) is greater than 4 and the result is true for less than \(n\). Note that \((\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*}) = (\bigcup_{i=2}^{n-2} S_{u_{i+1}u_i}^{*}) \cup (\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*}) \cup (\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*})\). By combining Lemmas 2.3 and 2.6, any linear combination of cycles \(S_{u_{i+1}u_i}^{*}\) contains an edge of the form \((u_{i+1}, v_1)(u_{j}, v_2)\) or \((u_1, v_j)(u_{i+1}, v_k)\) for \(2 \leq i, k \leq m\) which does not occur in any cycle of \((\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*})\). Thus, \((\bigcup_{i=2}^{n-1} S_{u_{i+1}u_i}^{*})\) is linearly independent. \(\square\)

Let \(E = \bigcup_{i=1}^{n} V(W_{u_i} \cap S_{m}) \) and \(C = W_{u_1} u_2\). Then, \(W_{u_1} u_2 \cap S_{m} = F \cup \mathcal{U} \) where \(K = W_{u_1} u_2 E \cup (F \cup C)\). Note that \(|E(K)| = |E(W_{u_1} u_2)\cap\mathcal{U}| = (2m - 1)(n - 1)\). Thus, \(\text{dim}(C) = (2m - 1)(n - 1) = n(m - 1) + 1\)
\[(u_2, v_j), (u_{2, j+1}), (u_{m, 1}), (u_{m, m})\] and 
\[(u_{2, j}), (u_{2, j+1})\] for \(2 \leq i \leq m, 2 \leq j \leq m - 1\) and \(2 \leq k \leq m - j\) which does not occur in any cycle of \((S_{u_1u_2u_3}, \cup (m - 2) Z_{u_1u_2u_3}) \cup (m - 1) S_{u_1u_2u_3}\). Thus, \(B(K)\) is linearly independent. Now,
\[|S_{u_1u_2u_3}| = |S_{lab}| + |S_{lab}| + 3|R_{lab}| + 2|M_{lab}| = (m - 1) + 3(m - 2) + 2(m - 2) = 6m - 11,
\]
and
\[|E_{u_1u_2u_3}| = |E_{lab}| + \sum_{j=2}^{m-2} |E_{lab}| = \sum_{j=2}^{m-2} (m - j - 1) = \frac{(m - 3)(m - 2)}{2} = \frac{m^2 - 5m + 6}{2}.
\]

Thus,
\[|Z_{u_1u_2u_3}| = 6|E_{lab}| = 6\left(\frac{m^2 - 5m + 6}{2}\right) = 3m^2 - 15m + 18,
\]
and
\[\text{Also set,}
\[T = \{(u_i, v_j, (u_i, v_j) (u_i, v_j)| 3 \leq j \leq m\}.
\]

Let
\[U(1) = \bigcup_{i=2}^{m} U(1)_{u_1u_2u_3+1} \text{ and } N = \bigcup_{i=2}^{m} N_{u_1}.
\]

\[\text{Lemma 2.9. } B(K) \cup U(1) \text{ is a cycle basis of } K \cup L \text{ and so of } K \cup L \cup \{(u_i, v_j)\}.
\]

\[\text{Proof. Note that } U(1) \text{ is the set of all triangle bounded faces of } W_{u_i} \Box u_1. \text{ Thus, } U(1) \text{ is a cycles basis of } L. \text{ Since } B(K) \text{ is a linearly independent set and } E(B(K)) \cap E(U(1)) = \emptyset, \text{ } B(K) \cup U(1) \text{ is linearly independent. Now,}
\]
\[|B(K) \cup U(1)| = |B(K)| + |U(1)| = (2(m - 1)^2(n - 1) - n(m - 1) + 1 + n - 1) = (2(m - 1)^2(n - 1) - n(m - 1) + 2(n - 1) - n + 2)\]
\[= \dim (K \cup U(1)).
\]

Thus, \(B(K) \cup U(1)\) is a cycle basis of \(K \cup L\). The second part follows from noting that the addition of \((u_i, v_j)(u_i, v_2)\) to the subgraph \(K \cup N\) does not create any cycles. \(\square\)

\[\text{Remark 2.10. Note that the addition of any edge of } F - \{(u_i, v_j)\} \text{ to the subgraph } K \cup L \cup \{(u_i, v_j)\}, \text{ creates at least one new cycle and the shortest cycle of } W_{n_1} \text{ contains the added edge is of length } 4. \text{ Moreover, any cycle contains } (u_i, v_j)(u_i, v_2) \text{ must contain at least another edge of } F. \\square
\]

\[\text{Lemma 2.11. } N \cup T \text{ is linearly independent.}
\]

\[\text{Proof. We use mathematical induction on } m \text{ to show that } N_{u_i} = \bigcup_{j=2}^{m} N_{u_i}^{(j)} \text{ is linearly independent. If } m = 2, \text{ then } N_{u_i} \text{ consists only of one cycle } C_{u_i}. \text{ Thus } N_{u_i} \text{ is linearly independent. Assume that } m \text{ is greater than } 2, \text{ and the result is true for less than } m. \text{ Note that } N_{u_i} = \bigcup_{j=2}^{m} N_{u_i}^{(j)} \bigcup N_{u_i}^{(m)}. \text{ Since the cycle } N_{u_i}^{(m)} \text{ contains the edge } (u_i, v_1)(u_i, v_m) \text{ which occurs in no cycle of } \bigcup_{j=2}^{m-1} C_{u_i}^{(j)}. \text{ Thus, } N_{u_i} \text{ is linearly independent for each } i. \text{ Now, } E(N_{u_i}) \cap E(N_{u_j}) = \{(u_i, v_1)(u_1, v_2)\} \text{ whenever } i \neq j. \text{ Therefore, } N \text{ is linearly independent. Similarly we can show that } T \text{ is linearly independent since } E(N) \cap E(T) = \{(u_i, v_1)(u_1, v_2), (u_2, v_1)(u_2, v_2)\} \text{ which is an edge set of a path, } \mathcal{N} \cup T \text{ is linearly independent by Lemma 2.2. \square
\]

\[\text{Theorem 4.12. } B(W_{n_1} \rho S_m) = B(K) \cup U(1) \cup N \cup T \text{ is a minimal cycle basis of } W_{n_1} \rho S_m.
\]

\[\text{Proof. By Lemmas 2.9 and 2.11 each of } N \cup T \text{ and } B(K) \cup U(1) \text{ is linearly independent. Note that any linear combination of cycles of } N \cup T \text{ must contains at least one edge of } \bigcup_{i=2}^{m} (u_i \Box S_m) \text{ which does not belong to any cycle of } B(K) \cup U(1). \text{ Thus, } B(W_{n_1} \rho S_m) \text{ is linearly independent. Note that}
\]
\[|N_{u_i}| = (m - 1) \text{ and } |T| = (m - 2),
\]
and,
\[|B(W_{n_1} \rho S_m)| = |B(K) \cup U(1)| + |N| + |T|.
\]
Thus by (2), (14) and (15), we get

\[ |B(W_n \rho S_m)| = 2(m - 1)^2(n - 1) - n(m - 1) + 2(n - 1) - n + 2 + (n - 1)(m - 1) + (m - 2) = (n - 1)(2m^2 - 4m + 3) - n + 2 + (n - 1)(m - 1) + (m - 2) = \dim C(W_n \rho S_m). \]

Hence, \( B(W_n \rho S_m) \) is a cycle basis of \( W_n \rho S_m \). Now, we show that \( B(W_n \rho S_m) \) is a minimum basis. Since any cycle of \( B(K) \cup U^{(1)} \) is of length 3 and \( B(K) \cup U^{(1)} \) is a basis for \( W_n \rho S_m - E(F) \), as a result \( |B(K) \cup U^{(1)}| \) is the size of a maximum linearly independent set of \( W_n \rho S_m - E(F) \) consisting of 3-cycles. But any cycle of \( W_n \rho S_m \) that contains an edge of \( F \) must be of length at least 4. Therefore, \( |B(K) \cup U^{(1)}| \) is the size of a maximum linearly independent set of \( W_n \rho S_m \) consisting of 3-cycles. Now, since each cycle of \( N \cup T \) is of length 4 and since the cycle space is a matroid, as a result \( B(W_n \rho S_m) \) is a minimum cycle basis for \( W_n \rho S_m \).

\[ \square \]

Corollary 3.13 \( l(W_n \rho S_m) = 8nm^2 - 17mn - 4m^2 + 8m + 5n - 13, \) and \( \lambda(W_n \rho S_m) = 4. \) \[ \square \]

III. Conclusion

The works in this paper present some important results of the minimum cycle basis and cycle spaces of the wreath product of graphs. Some applications in sciences and engineering are indicated. In particular we have drawn attention to the use of MCB in biochemistry, structural engineering, surface reconstruction and public transportations (See [2], [3] and [8]).

REFERENCES


