Some New Subclasses of Nonsingular H-matrices

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Abstract—In this paper, we obtain some new subclasses of nonsingular H-matrices by using \( \alpha \) diagonally dominant matrix.

Keywords—H-matrix, diagonal dominance, \( \alpha \) diagonally dominant matrix.

I. INTRODUCTION

Let \( A \) be a \( n \times n \) matrix, \( M(A) \) the diagonal dominance matrix of \( A \), where

\[
m_{ij} = \begin{cases} 
|a_{ii}| & i = j, \\
-|a_{ij}| & i \neq j,
\end{cases} \quad i, j = 1, 2, \ldots, n.
\]

Then we call \( M(A) \) is the comparison matrix of \( A \). Suppose \( A \) is a nonnegative matrix and \( \sigma > \rho(B) \) the spectral radius of \( B \), then \( A \) is called a nonsingular M-matrix. This class of matrices has been much studied \cite{[3]}.

If \( M(A) \) is nonsingular M-matrix, then \( A \) is called a nonsingular H-matrix. If

\[
|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n,
\]

then we say \( A \) is strictly diagonally dominant. If there exist positive number \( x_1, x_2, \ldots, x_n \) such that

\[
x_i a_{ii} > \sum_{j \neq i} x_j |a_{ij}|, \quad i = 1, 2, \ldots, n,
\]

then we say \( A \) is generalized strictly diagonally dominant \cite{[3]}.

A matrix \( A \) be a nonsingular H-matrix is equivalent to that \( A \) be a generalized strictly diagonally dominant matrix \cite{[3]}. H-matrices have important applications, for instance, in iterative methods of numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. But how to determine whether an \( n \times n \) complex matrix is a nonsingular H-matrix is not easy in practice. In this paper, we will give some new subclasses of nonsingular H-matrices.

II. MAIN RESULTS

We will use the following notations:

\[
R_i = \left| a_{ii} \right|, \quad S_i = \left| a_{ij} \right|, \quad i = 1, 2, \ldots, n,
\]

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when \( \sum_{j \in \mathbb{N}_2^1} |a_{ij}| 0, \sum_{j \in \mathbb{N}_2^1} |a_{ji}| 0, \) we denote \( c_i \sim \alpha, f_i \sim \gamma, \) according to the hypothesis of this paper, we have \( c_i > 0, f_i > 0. \)

We denote
\[
\{ \begin{aligned}
& c_i, i \in \frac{\alpha}{2}, \\
& f_i, i \in \frac{\gamma}{2}.
\end{aligned}
\]

There must exits a small enough positive number \( \varepsilon, \) such that
\[
0 < \varepsilon < \frac{\alpha}{\gamma}, \quad \left\{ \begin{aligned}
& c_i, i \in \frac{\alpha}{2}, \\
& f_i, i \in \frac{\gamma}{2}.
\end{aligned} \right.
\]

We choose positive diagonal matrix
\[
diag \left( d_1, d_2, \cdots, d_n \right)
\]
and
\[
diag \left( e_1, e_2, \cdots, e_n \right),
\]
where
\[
d_i \left\{ \begin{aligned}
& x_i, i \in \frac{\alpha}{2}, \\
& x_i + \varepsilon, i \in \frac{\gamma}{2}.
\end{aligned} \right.
\]

In the follows, we just need to prove that is a strictly \( \alpha \) diagonally dominant matrix.

For \( \forall i \in \frac{\alpha}{2}, \) according to (3) we have
\[
|a_{ij}| x_i y_i \left( 1 + b_i \right) \left( \alpha \sum_{j \neq i} |a_{ij}| x_j y_i + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i \right)
\]
and
\[
\sum_{j \neq i} |a_{ji}| y_j x_i + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i \left( 1 + \varepsilon \right)
\]
\[
\sum_{j \neq i} |a_{ji}| y_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j + \sum_{j \neq i} |a_{ji}| \left( y_j + \varepsilon \right)
\]
\[
\alpha R_i \left( \right) \left( 1 + \alpha \right) S_i \left( \right)
\]
\[
\alpha R_i \left( \right) \left( 1 + \alpha \right) S_i \left( \right)
\]

Case three: \( \sum_{j \in \mathbb{N}_2^1} |a_{ij}| 0, \sum_{j \in \mathbb{N}_2^1} |a_{ji}| 0. \) As the same proof of case two, we can obtain
\[
|b_{ii}| > R_i \left( \right) ^\alpha C_i \left( \right) ^{1 - \alpha}
\]

Case four: \( \sum_{j \in \mathbb{N}_2^1} |a_{ij}| 0, \sum_{j \in \mathbb{N}_2^1} |a_{ji}| 0 \) according to (4) we have:
\[
\varepsilon < c_i \Leftrightarrow \sum_{j \in \mathbb{N}_2^1} |a_{ij}| < b_i \sum_{j \in \mathbb{N}_2^1} |a_{ij}| x_j
\]
\[
\Leftrightarrow \left( 1 + b_i \right) \sum_{j \neq i} |a_{ij}| x_j < \sum_{j \neq i} |a_{ij}| x_j + \varepsilon \sum_{j \neq i} |a_{ij}|
\]
\[
\Leftrightarrow \left( 1 + b_i \right) \sum_{j \neq i} |a_{ij}| y_j > \sum_{j \neq i} |a_{ij}| y_j + \varepsilon \sum_{j \neq i} |a_{ij}|
\]
\[
(6)
\]
With (5), (6) and the hypothesis of the paper, we have:
\[ x_i (1 - \alpha) \left( \sum_{j \in N^1_i \setminus N^2_i} |a_{ji}| y_j + \sum_{j \in N^2_i \setminus N^2_i} (y_j + \varepsilon) |a_{ji}| \right) \geq (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

\[ \alpha R_i (\cdot) + (1 - \alpha) S_i (\cdot). \]

For any \( i \notin \mathcal{N}_i, i \in \mathcal{N}_i \setminus \mathcal{N}_i \), from the choice of \( \varepsilon \) and the positive diagonal matrices \( D \) and \( E \), we know that \( 0 < d_i, \varepsilon_i \leq 1, \) for any \( i \notin \mathcal{N}_i \).

Case one: \( i \in \mathcal{N}_i \cap \mathcal{N}_i \)

\[ |b_{ii}| |a_{ii}| \geq \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

\[ + (1 - \alpha) \left( \sum_{j \in N^1_i} |a_{ji}| y_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ji}| \right) \]

\[ > \alpha \left( \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

\[ + (1 - \alpha) \left( \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ji}| y_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ji}| \right) \]

\[ \alpha R_i (\cdot) + (1 - \alpha) S_i (\cdot). \]

Case three: \( i \notin \mathcal{N}_i \), \( i \in \mathcal{N}_i \), as the same proof of case two, we can obtain

\[ |b_{ii}| > \alpha R_i (\cdot) + (1 - \alpha) S_i (\cdot). \]

Case four: \( i \notin \mathcal{N}_i \), \( i \notin \mathcal{N}_i \), from (2) we have

\[ (y_i + \varepsilon) |a_{ii}| (x_i + \varepsilon) \]

\[ \alpha R_i (\cdot) + (1 - \alpha) S_i (\cdot). \]

Case two: \( i \in \mathcal{N}_i \), \( i \notin \mathcal{N}_i \), if \( \alpha / 1 \), from (2) we have

\[ (y_i + \varepsilon) |a_{ii}| \]

\[ \geq (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

\[ + (1 - \alpha) \left( \sum_{j \in N^1_i} |a_{ji}| y_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ji}| \right) \]

Hence

\[ |b_{ii}| (y_i + \varepsilon) |a_{ii}| \]

\[ \geq (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

\[ + (1 - \alpha) \left( \sum_{j \in N^1_i} |a_{ji}| y_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ji}| \right) \]

\[ \alpha R_i (\cdot) + (1 - \alpha) S_i (\cdot). \]

Since

\[ (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) > \alpha R_i (\cdot), \]

\[ (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) > \alpha R_i (\cdot), \]

\[ (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

\[ (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

\[ (y_i + \varepsilon) \alpha \left( \sum_{j \in N^1_i} |a_{ij}| x_j + \sum_{j \in \mathcal{N}_i \setminus \mathcal{N}_i} |a_{ij}| \right) \]

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we have
\[ |b_{ii}| (y_i + \varepsilon) |a_{ii}| (x_i + \varepsilon) \geq (y_i + \varepsilon) \alpha \left( \sum_{j \in N_i} |a_{ij}| x_j + \sum_{j \notin \tilde{I}} |a_{ij}| \right) (x_i + \varepsilon) \]
\[ + (1 - \alpha) \left( \sum_{j \in N_i} |a_{ji}| y_j + \sum_{j \notin \tilde{I}} |a_{ji}| \right) \]
\[ > \alpha R_i (\cdot) + (1 - \alpha) S_i (\cdot) \, . \]

We see that for any \( i \in \langle n \rangle \), we have \( |b_{ii}| > \alpha R_i (\cdot) + (1 - \alpha) S_i (\cdot) \). According to Lemma 1, we know that matrix \( B \) is a nonsingular H-matrix, so matrix \( A \) is a nonsingular H-matrix.

Let \( \{ a_{ij} \}_{n \times n} \in C^{n \times n}, 0 < x_i, y_i < 1, i \in \langle n \rangle \) satisfy the equation (2), we denote
\[ K_\alpha \left\{ i \in \langle n \rangle | |a_{ii}| > \frac{\alpha}{x_i} \sum_{j \notin \tilde{I}} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \notin \tilde{I}} |a_{ji}| y_j \right\} . \]

**Theorem 2:** Let \( \{ a_{ij} \}_{n \times n} \in C^{n \times n}, \) for \( \alpha \in (0, 1) \), if \( 0 < x_i < 1, \) \( 0 < y_i < 1, \) \( i \in \langle n \rangle \) satisfy the inequations (2) and
\[ |a_{ii}| \geq \frac{\alpha}{x_i} \sum_{j \notin \tilde{I}} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \notin \tilde{I}} |a_{ji}| y_j, \quad i \in \langle n \rangle \]
and \( K_\alpha \not= \emptyset \), for any \( i_0 \in \langle n \rangle \setminus K_\alpha \), there exists a nonzero elements chain \( a_{i_0, i_1}, a_{i_1, i_2}, \ldots, a_{i_k, i_0} \not= 0 \) such that \( i_k \in K_\alpha \), then \( A \) is a nonsingular H-matrix.

**Proof:** We structure two positive diagonal matrices: \( D = \text{diag} (x_1, x_2, \ldots, x_n) \) and \( \text{diag} (y_1, y_2, \ldots, y_n) \), and notes \( B = \{ b_{ij} \} \in \text{EAD} \). So for any \( i \in \langle n \rangle \), we have
\[ |b_{ii}| \geq \alpha R_i (B) + (1 - \alpha) S_i (B) \, . \]

Obviously, \( K_\alpha \) can be note
\[ K_\alpha \left\{ i \in \langle n \rangle | |b_{ii}| > \alpha R_i (B) + (1 - \alpha) S_i (B) \right\} , \]
for any \( i_0 \in K_\alpha \), we have \( b_{i_0, i_1}, b_{i_1, i_2}, \ldots, b_{i_k, i_0} \not= 0 \) such that \( i_k \in K_\alpha \). So according to Lemma 2, we know that matrix \( B \) is a nonsingular H-matrix, so matrix \( A \) is a nonsingular H-matrix.

From Theorem 2, we can get the following corollary.

**Corollary** Let \( \{ a_{ij} \}_{n \times n} \in C^{n \times n} \) be irreducible, for \( \alpha \in (0, 1) \), if \( 0 < x_i < 1, \) \( 0 < y_i < 1, \) \( i \in \langle n \rangle \) satisfy the inequations (2) and (8), \( \tilde{I} \not= \emptyset \), where

\[ \tilde{I} \left\{ i \in S (A) \mid y_i \frac{|a_{ii}|}{x_i} / \tilde{R}_i (A) \right\} , \]
or
\[ \tilde{I} \left\{ i \in S (A) \mid y_i \frac{|a_{ii}|}{x_i} / \tilde{C}_i (A) \right\} , \]

then \( A \) is a nonsingular H-matrix.