An Alternative Proof for the NP-completeness of Top Right Access point-Minimum Length Corridor Problem

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Abstract—In the Top Right Access point Minimum Length Corridor (TRA-MLC) problem [1], a rectangular boundary partitioned into rectilinear polygons is given and the problem is to find a corridor of least total length and it must include the top right corner of the outer rectangular boundary. A corridor is a tree containing a set of line segments lying along the outer rectangular boundary and/or on the boundary of the rectilinear polygons. The corridor must contain at least one point from the boundaries of the outer rectangle and also the rectilinear polygons. Gutierrez and Gonzalez [1] proved that the MLC problem, along with some of its restricted versions and variants, are NP-complete. In this paper, we give a shorter proof of NP-Completeness of TRA-MLC by finding the reduction in the following way.

Connected vertex cover in 2-connected planar graph with maximum degree 4

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Top-Right Access Point Minimum Length Corridor (TRA-MLC)

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I. INTRODUCTION

In the Minimum-Length Corridor (MLC) problem [1], a rectangular boundary partitioned into rectilinear polygons is given and the problem is to find a corridor of least total length. A corridor is a tree containing a set of line segments lying along the outer rectangular boundary and/or on the boundary of the rectilinear polygons. The corridor must contain at least one point from the boundaries of the outer rectangle and also the rectilinear polygons. An access point of a corridor is any point on the rectangular boundary. If this access point is constrained to be at the top right corner of the outer rectangular boundary, then this problem is referred to as TRA-MLC. In the MLC problem, and in its variants, it is assumed that the rectangular boundary and the partitions are orthogonal. In fig. 1, we can see an instance of TRA-MLC and the thick line refers to an optimal corridor with top right access point included.

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Fig 1: An Optimal corridor for an instance of TRA-MLC

The decision version of TRA-MLC can be defined as follows

Instance: A Pair \((F, R)\) and a real number \(k\), where \(F\) is a rectangular boundary and \(R\) is a set of rectilinear partitions \(R_1, R_2, ..., R_p\).

Question: Does there exist a set \(S\) of line segments which form a tree such that \(L(S) \leq k\), where \(L(S)\), called the edge length, is the sum of the lengths of the line segments in \(S\).

This problem has many applications in laying optical fibre cables for data communication and electrical wiring in floor plans. We can consider \((F, R)\) as floor plan with the rectilinear partitions representing \(p\) rooms. The corridor refers to placement of cables. There are many other applications which include signal communication in circuit layout design [1].

The Minimum Length Corridor (MLC) problem was first posed by Naoki Katoh [2] as an architectural design problem and its restricted version MLC-R was introduced by Eppstein [3]. An extensive study of these problems and their variants is made by Gutierrez and Gonzalez [1]. They also proved that the decision version of MLC problem, along with some of its restricted versions and variants, are NP-complete. To do this, they reduced the planar 3-SAT problem to TRA-MLC and TRA-MLC-R problems. From these two problems they found polynomial reductions to other variants of MLC. In the next section of this paper, we attempt to give an alternative proof of NP-completeness of the TRA-MLC problem. The proof which we are going to present is shorter and it uses popularly known graph theoretic concepts. Considering the reductions given in [1], we can say that the variants of MLC and TRA-MLC are NP-Complete. We find the polynomial reduction in the following way.
Connected vertex cover in 2-connected planar graph with maximum degree 4

Top-Right Access Point Minimum Length Corridor (TRA-MLC)

Garey and Johnson proved that the problem of finding connected vertex cover in planar graphs with maximum degree 4 (CVC) is NP-complete [4]. As a first step, we attempted to prove in [5] that a restricted version: connected vertex cover in 2-connected planar graphs with maximum degree 4 (hereafter referred to as CVC-2) is also NP-complete by finding polynomial reduction from CVC to CVC-2. Now, in this paper, we find a polynomial reduction from CVC-2 to TRA-MLC thereby proving TRA-MLC is NP-complete.

To prove that any problem P to be NP-complete we need to show that
1. $P \in NP : x$ is a yes instance of $P$ if and only if there exists a concise certificate $c(x)$, and it is verifiable by a polynomial time algorithm.
2. Some known NP-complete problem $P'$ is polynomially reducible to $P$: For any given instance $x$ of $P'$, we should be able to construct an instance $y$ of $P$ within polynomial in $|x|$ time, such that $x$ is a yes instance of $P'$ if and only if $y$ is a yes instance of $P$.

For more explanation on NP-completeness, reader is referred to [6, 7].

II. THE PROOF

Theorem: TRA-MLC is NP-complete.

Proof: It can be understood, from [1], that TRA-MLC $\in$ NP. Now, we construct an instance of TRA-MLC, from the given instance of the problem of connected vertex cover in 2-connected planar graph with maximum degree 4. Let an instance of CVC-2 be given by a 2-connected planar graph $G_1$ and an integer $K$. Assume that $G_1$ has $n$ vertices with maximum degree 4 and $m$ edges and $K$ is the upper bound on the size of the vertex cover.

Our construction begins with finding a planar representation $G'$ of $G_1$ [8]. Let us consider a vertex $x$ on the external face of $G'$ and let the degree of $x$ be $d$. We will now replace $x$ with an edge $(u, v)$ in the following way. We know that $d$ will be equal to 2, 3, or 4. If $d$ is 2, we add one edge to each of $u, v$ and the degrees of $u, v$ will be 2. If $d$ is 3, then $u, v$ will be of degree 2 and 3 respectively as we add one incident edge to $u$, and two remaining consecutive edges in clockwise order around $x$ to $v$. In the case of $d$ being 4, we add frist two edges in clockwise order around $x$ to $u$, and the remaining 2 edges to $v$ making the degrees of $u, v$ to be equal to 3. Let us call the new graph as $G$ having $n+1$ vertices and $m+1$ edges and clearly this graph is also a 2-connected planar graph with maximum degree 4. Fig. 2(a), 2(b) show the example of $G'$, $G$.

Now, we find the weak visibility representation [9] of $G$ by selecting the edge $(u, v)$ for st-numbering by taking $s = u$ and $t = v$. (All the vertices of $G$ will be distinctly numbered from 1 to $n + 1$ making $u = v_1$ and $v = v_{n+1}$ (we refer to $u$ as $v_1$ and $v$ as $v_{n+1}$ from now onwards). Then find the orthogonal representation for $G$ followed by a grid embedding, as described in [10], on a discrete grid of squares with all points of the form $(6i, 6j)$ where $i, j$ are integers. Let $p_i$ denote the point in the grid corresponding to a vertex $v_i$ in $G$ and all these points will have coordinates of the form $(6i, 6j)$. It is easy to find out the coordinates of the corners of the smallest rectangle which encloses the grid embedding and let us call the four corners, in clockwise order starting from bottom-left, as $(x_1, y_1), (x_1, y_2), (x_2, y_2)$ and $(x_2, y_1)$. Fig. 3 shows the grid embedding of the graph $G$.

After obtaining the grid embedding, we add some more line segments to it to get an instance of TRA-MLC as follows. Refer to fig. 4, which is an instance $R$ of TRA-MLC for the construction. Let $h = y_2 - y_1$ (height of the rectangle) and let $d = 6n^2 - h - 6$ (as the area of the rectangle is $O(n^2)$ [12], $d$ will be non-negative). Now, we draw a rectangle with $A = (x_1 - 6, y_1 - d), B = (x_1 - 6, y_2 + 6), C = (x_2 + 6, y_2 + 6)$.
and $B = (x_2 + 6, y_1 - d)$ as corners in clockwise order starting from bottom-left corner. This rectangle $ABCD$ completely encloses the grid embedding. The degrees of $v_1, v_{n+1}$ in $G$ are not more than 3, and hence by the way the transformations are done in [10], the points $p_1$ and $p_{n+1}$ would not have edges at the bottom and on top respectively. Now, we shall draw vertical lines joining $p_{n+1}$ to the horizontal line $BC$ and $p_1$ to the horizontal line $AD$ and let us call the intersection points as $E$ and $F$ respectively.

![Figure 4: Instance $R$ of TRA-MLC](constructed from $G$)

For all the line segments $(p_i, p_j)$ where $1 \leq i, j \leq (n + 1)$ corresponding to the edges in $G$, we draw unit squares on both the sides (top and bottom for the horizontal component, left and right for the vertical component), as shown in fig. 4, leaving a line segment of length 2 units at both the end points $p_i$ and $p_j$. For the line segment $(p_{n+1}, E)$ draw unit squares on either sides leaving two units at $p_{n+1}$ as $H$. Let us call a point on the line $(p_{n+1}, E)$, which is at a distance of 3 units from $p_{n+1}$ as $H$. For the line segment $EH$, we draw unit squares at the bottom of the line. The resultant rectangle, $(ABCD)$ is a rectangual grid, say $R$, divided into rectilinear partitions and this forms the instance of TRA-MLC.

For any point $p_i$ corresponding to a vertex $v_i$ in $G$, there will be $d_i$ (degree of $v_i$) line segments having one end at $p_i$, and let us call parts of these line segments, each of 3 units of length from $p_i$, together with $p_i$ as $p_i$’s region. For any line segment $(p_i, p_j)$, corresponding to the edge $(v_i, v_j)$ in $G$, remove the line segments in $p_i$ & $p_j$’s regions and let us call the remaining line segment as edge component of $(v_i, v_j)$ (This line segment is sufficient to cover all the squares as it touches the corner points of the two squares in $p_i$ and $p_j$ regions). Fig. 5 shows a line segment corresponding to an edge in $G$.

![Figure 5: A line segment in $R$ representing an edge in $G$](constructed from $G$)

Now find the lengths of all the $(m + 1)$ edge components and let this total length be $l$. Add length of $CE$ and length of $EH$ to $l$. Now let us consider the integer $L = l + 3(m + K + 2)$. We prove that the given 2-connected planar graph $G_1$ with maximum degree 4 will have a connected vertex cover of size $C_1 \leq K$ if and only if the instance $R$ of TRA-MLC will have a corridor $RLT$, in which top right access point is included, and with a length $C' \leq L$.

First assume that $V_1$ is a connected vertex cover of $G_1$ and with size $C_1 \leq K$. If the vertex $x$ in $G_1$ belongs to $V_1$ then we take $V = \{v_1, v_{n+1}\} \cup (V_1 - \{x\})$ which is a subset of the vertices of graph $G$. If $x \notin V_1$ then we take $V = \{v_{n+1}\} \cup V_1$. Now $V$ forms a connected vertex cover of size $C_1 + 1$ which is less than or equal to $K + 1$ for the new graph $G$. The vertex $v_{n+1}$ will always be present in the vertex cover $V$. Find a tree $T$ induced by $V$ in $G$ with $C_1$ edges. In
the rectangular grid $R$ corresponding to $G$, any corridor should include at least all the \textit{edge components} in order to cover the unit squares drawn on both the sides of these \textit{edge components}. Also, any corridor must include the line segments $CE$ and $EH$ to cover the squares incident on them. Let us construct a rectilinear tree (a corridor) $RLT$ starting with all these \textit{edge components} along with the line segments $CE$ and $EH$. For the line segment corresponding to any edge $(v_i,v_j)$, belonging to the tree $T$, add length 3 line segments, in both $p_i$, $p_j$ regions, on the line $(p_i,p_j)$ along with $p_i$, $p_j$ to $RLT$. For any edge $(v_k,v_b)$, which is not in $T$, either $v_k$ or $v_b$ or both must be present in $V$ and without loss of generality let us assume that $v_k \in V$. Now, add length 3 line segment on the line $(p_k,p_b)$, in $p_b$'s region along with $p_k$, to $RLT$. Finally, add length 3 line segment $(p_{n+1},H)$ to $RLT$. This line segment will be along the borders of the two rectilinear partitions formed by the sides of the outer rectangle $AB$ and $CD$. For any rectilinear region corresponding to the face of the graph $G$, there will be at least one point $p_i \in RLT$ which corresponds to a vertex covering the edge incident on it and which is on the border of the face. Now $RLT$ will be a rectilinear tree along the sides of the rectilinear partitions and the outer rectangle. It includes the top right access point $C$ and has a length $C' = l + 3(m + 1) + 3C_1$ which is less than or equal to $L$. So $RLT$ becomes the required corridor.

Conversely, suppose the instance $R$ of TRA-MLC has a corridor $RLT$ including the top right access point $C$, and it is of length $C' \leq L$. As mentioned above, $RLT$ should include all the \textit{edge components} to cover all the unit squares drawn along the line segments representing $G$. Also it should include the lines $CE$ and $EH$ to cover the squares drawn along these lines. So the length of these line segments together is $l$ and this should be a part of $C'$. The length of the remaining line segments in the corridor $RLT$ will be at the most $3(m + k + 2)$ as $C' \leq L$. The line segments $H_{p_{n+1}}, E_{p_1}$ and $(C,p_1)$ connect the outer rectangle to the inner grid embedding of $G$. Among these, $RLT$ cannot include $E_{p_1}$ and $(C,p_1)$ because, by way of construction of $R$, each of their lengths will be greater than $6n^2$ and hence it is greater than $3(m + k + 2)$. So $RLT$ must include the line segment $(H,p_{n+1})$ to connect the rectangular boundary to the inner rectilinear partitions and hence the length of the remaining part of $RLT$ will be at the most $3(m + 1) + 3K$. There are $m + 1$ \textit{edge components} corresponding to the edges in $G$ and in order to connect them together into a tree, at least one length 3 line segment connecting the \textit{edge component} to one of its incident points should be present in $RLT$. These line segments together will have a length of $3(m + 1)$ and the remaining line segments in $RLT$ will have length at most $3K$. This extra length comes from the length 3 line segments on the other side of some of the \textit{edge components} which are included in $RLT$ i.e. For a maximum of $K$ \textit{edge components} the length 3 line segments joining to both the incident points are present in $RLT$. Now let us consider a subset $V$ of the vertices of $G$, containing all the vertices corresponding to the points for which at least one length 3 line segment in their region is included in $RLT$. The set $V$ will obviously cover all the $(m+1)$ edges in $G$. If we consider the edges corresponding to the \textit{edge components} for which the length 3 line segments on both the sides along with both the incident points are in $RLT$, they will be at the most $K$. As these line segments are part of a tree $RLT$ in $R$, we can say that the corresponding $K$ edges in $G$ form a tree connecting vertices of $V$ and hence $|V|$ is at the most $K + 1$. To find a corresponding vertex cover in the original graph $G_1$, let us take a subset $V_1$ of the vertices of $G_1$, with the vertices in the set $\{V - \{v_1, v_{n+1}\}\}$. If $P_1 \in RLT$ then corresponding $v_1$ will also be in $V$, and hence we add $x$ to $V_1$. The size of $V_1$ will be at the most $K$ and it forms a connected vertex cover for $G_1$. Hence the proof.

III. CONCLUSIONS

The proof given in this paper is shorter and it uses the most commonly known concepts of graph theory. The restricted version TRA-MLC-I, imposes a constraint on TRA-MLC that all the rectilinear partitions should be rectangles. We are hopeful that, in future, a shorter proof of the complexity of this problem can also be given.

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