A new preconditioned AOR method for Z-matrices

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Abstract—In this paper, we present a preconditioned AOR-type iterative method for solving the linear systems $Ax = b$, where $A$ is a Z-matrix. And give some comparison theorems to show that the rate of convergence of the preconditioned AOR-type iterative method is faster than the rate of convergence of the AOR-type iterative method.

Keywords—Z-matrix, AOR-type iterative method, precondition, comparison.

I. INTRODUCTION

For solving linear system

$$Ax = b,$$  

(1)

where $A$ is an $n \times n$ square matrix, and $x$ and $b$ are $n$-dimensional vectors, the basic iterative method is

$$Mx^{k+1} = Nx^k + b, k = 0, 1, \ldots ,$$  

(2)

where $A = M - N$ and $M$ is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c, k = 0, 1, \ldots ,$$  

where $T = M^{-1}N$, $c = M^{-1}b$.

Assuming $A$ has unit diagonal entries and let $A = I - L - U$ where $I$ is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of $A$, respectively. Then, (I) the iteration matrix of the classical Gauss-Seidel-type method is given by

$$T = (I - L)^{-1}U$$  

(3)

(II) the iteration matrix of the classical SOR-type method is given by

$$L_r = (I - rL)^{-1}[(1 - r)I + rU]$$  

(4)

where $r \neq 0$ is a parameter called the relaxation parameter.

(III) the iteration matrix of the classical AOR-type method is given by

$$L_{r,w} = (I - L)^{-1}[(1 - w)L + (w - r)U]$$  

(5)

where $w$ and $r$ are real parameters and $w \neq 0$.

Transform the original system (1) into the preconditioned form

$$PAx = Pb.$$  

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, k = 0, 1, \ldots ,$$

where $PA = M_p - N_p$ and $M_p$ is nonsingular. Thus the equation above can also be written as

$$x^{k+1} = T x^k + c, k = 0, 1, \ldots ,$$

where $T = M_p^{-1}N_p$, $c = M_p^{-1}Pb$.

In paper [1], Meijun Wu et al. presented the preconditioned AOR-type iterative method with

$$P_a = I + S_a$$  

(6)

$$\begin{pmatrix}
1 & -\alpha_1 a_{12} & -\alpha_2 a_{23} & \cdots & -\alpha_{n-1} a_{n-1,n} \\
1 & 1 & -\alpha_2 a_{23} & \cdots & -\alpha_{n-1} a_{n-1,n} \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
\end{pmatrix}$$

and $\alpha_i (i = 1, 2, \ldots , n - 1)$ are nonnegative real numbers, and obtained some comparison results.

In this paper, we will present the preconditioned AOR-type iterative method with

$$P_a = I + K_a$$  

(7)

$$\begin{pmatrix}
1 & -\beta_1 a_{12} & -\beta_2 a_{23} & \cdots & -\beta_{n-1} a_{n-1,n} \\
1 & 1 & -\beta_2 a_{23} & \cdots & -\beta_{n-1} a_{n-1,n} \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
\end{pmatrix}$$

and $\beta_i (i = 1, 2, \ldots , n - 1)$ are nonnegative real numbers.

In the following, we consider three splittings for $A$:

$$A = \begin{pmatrix}
(\tilde{D} - \tilde{L})^{-1}U & \tilde{D} - \tilde{L} - U \\
\tilde{D} - \tilde{L} - U & \tilde{D} - \tilde{L} - U \\
\tilde{D} - \tilde{L} - U & \tilde{D} - \tilde{L} - U \\
\tilde{D} - \tilde{L} - U & \tilde{D} - \tilde{L} - U \\
\tilde{D} - \tilde{L} - U & \tilde{D} - \tilde{L} - U \\
\end{pmatrix}$$  

(8)

where $\tilde{D}$, $\tilde{L}$ and $\tilde{U}$ are diagonal, strictly lower and strictly upper triangular parts of $A$, respectively.

In view of (8), the iteration matrices associated with $\tilde{A}$ are:

$$\tilde{T} = (\tilde{D} - \tilde{L})^{-1}\tilde{U}$$  

(9)

$$\tilde{L}_r = (\tilde{D} - \tilde{L})^{-1}[(1 - r)\tilde{D} + r\tilde{U}]$$  

(10)

$$\tilde{L}_{r,w} = (\tilde{D} - \tilde{L})^{-1}[(1 - w)\tilde{D} + (w - r)\tilde{L} + w\tilde{U}]$$  

(11)

In this paper, we will discuss the preconditioned iterative methods with the preconditioner $P_a$ for solving Z-matrices linear systems and present comparison theorems of these methods.

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II. COMPARISON RESULTS OF PRECONDITIONED AOR-TYPE METHODS WITH PRECONDITIONER $P_\beta$

We need the following definitions and results.

**Definition 2.1** (Young [3]). $A$ is a Z-matrix if $a_{ij} \leq 0$, for all $i, j = 1, 2, \ldots, n$, $i \neq j$.

**Lemma 2.2** (Young [3]). Let $A \geq 0$ be an irreducible matrix. Then

1. $A$ has a positive real eigenvalue equals to its spectral radius;
2. To $\rho(A)$ there corresponds an eigenvector $x > 0$;
3. $\rho(A)$ is a simple eigenvalue of $A$.

**Lemma 2.3** (Varga [4]). Let $A$ be a nonnegative matrix. Then

1. If $\alpha x \leq Ax$ for some nonnegative vector $x$, $x \neq 0$, then $\alpha \leq \rho(A)$;
2. If $\alpha x \leq \beta x$ for some positive vector $x$, then $\alpha \leq \beta$. Moreover, if $A$ is irreducible and if $0 \neq \alpha x \leq Ax$ for some nonnegative vector $x$, then $\alpha \leq \rho(A) \leq \beta$ and $x$ is a positive vector.

**Lemma 2.4** ([5]). Let $A = M - N$ be an M-splitting of $A$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is a nonsingular M-matrix.

**Lemma 2.5** ([6]). Let $A$ be a Z-matrix. Then $A$ is a nonsingular M-matrix if and only if there is a positive vector $x$ such that $Ax \geq 0$.

For the linear system (1), we consider its preconditioned form

$$P_\beta Ax = P_\beta b$$

with the preconditioner $P_\beta = I + K_\beta$ in this section.

We apply the AOR method to it and have the corresponding preconditioned AOR iteration matrix

$$\hat{L}_{r,w} = (D_\beta - rL_\beta)^{-1}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta],$$

where $D_\beta$, $-L_\beta$ and $-U_\beta$ are diagonal, strictly lower and strictly upper triangular parts of $A_\beta = P_\beta A$, respectively.

Now we give the main results as follows.

**Theorem 2.1** Let $A = I - L - U \in R^{n \times n}$ be a nonsingular Z-matrix, $L_{r,w}$ and $U_{r,w}$ be the iteration matrices given by (5) and (12). Assume that $0 < r < w < 1$, and $0 < \beta < 1$, $i = 1, 2, \ldots, n$.

(i) If $\rho(L_{r,w}) < 1$, then

$$\rho(\hat{L}_{r,w}) \leq \rho(L_{r,w}) < 1$$

(ii) Let $A$ be irreducible. Assume that $a_{i,i-1}a_{i-1,i} < 1, i = 2, \ldots, n$, then

(i) $\rho(\hat{L}_{r,w}) > \rho(L_{r,w})$ if $\rho(L_{r,w}) > 1$;
(ii) $\rho(\hat{L}_{r,w}) = \rho(L_{r,w})$ if $\rho(L_{r,w}) = 1$;
(iii) $\rho(\hat{L}_{r,w}) < \rho(L_{r,w})$ if $\rho(L_{r,w}) < 1$.

**Proof.** Let

$$M = \frac{1}{w}(I - rL),$$

$$N = \frac{1}{w}[(1-w)I + (w-r)L + wU],$$

$$E_\beta = \frac{1}{w}(D_\beta - rL_\beta),$$

$$F_\beta = \frac{1}{w}[(1-w)D_\beta + (w-r)\beta L + wU_\beta],$$

$$M_\beta = \frac{1}{w}(I + K_\beta)(I - rL),$$

$$N_\beta = \frac{1}{w}(I + K_\beta)[(1-w)I + (w-r)L + wU].$$

Then, we have

$$A = M - N, \quad A_\beta = E_\beta - F_\beta = M_\beta - N_\beta$$

(i) Since $A$ is a nonsingular Z-matrix and $0 < r < w < 1$, $w \neq 0$, it is clear that $M = \frac{1}{w}(I - rL)$ is a nonsingular M-matrix and the splitting

$$A = M - N = \frac{1}{w}(I - rL) - \frac{1}{w}[(1-w)I + (w-r)L + wU]$$

is an M-splitting. Since $\rho(L_{r,w}) < 1$, it follows from Lemma 2.4 that $A$ is a nonsingular M-matrix. Then by Lemma 2.5, there is a positive vector $x$ such that $Ax \geq 0$, so $A_\beta x = (I + K_\beta)Ax \geq 0$.

By Lemma 2.5, $A_\beta$ is also a nonsingular M-matrix. Obviously, we can get $D_\beta$ is an positive diagonal matrix, and $L_\beta$ is nonnegative. From $r > 0$ we know that $E_\beta$ is a Z-matrix. Since $rD_\beta^{-1}L_\beta \geq 0$ is a strictly lower triangular matrix so that $\rho(rD_\beta^{-1}L_\beta) = 0 < 1$, we have $(I - rD_\beta^{-1}L_\beta)^{-1} \geq 0$. Then

$$E_\beta = (I - rD_\beta^{-1}L_\beta)^{-1}D_\beta^{-1} \geq 0$$

Hence $E_\beta$ is an nonsingular M-matrix. By $F_\beta \geq 0$ we can prove that $A_\beta = E_\beta - F_\beta$ is an M-splitting. It follows from Lemma 2.4 that

$$\rho(L_{r,w}) = \rho(E_\beta^{-1}F_\beta) < 1.$$
From the expression of \( L_{r,w} \), we obtain the following equality

\[
[(1 - w)I + (w - r)L + wU]x = \lambda(I - rL)x
\]

which is equivalent to

\[
[(1 - w - r)I + (w - r + \lambda r)L + wU]x = 0,
\]

or

\[
(\lambda - 1)(I - rL)xw(L + U - I) = 0
\]

(13)

Let \( K_2U = K_1 + K_2 \), where \( K_1 \) and \( K_2 \) are the diagonal and lower triangular parts of \( K_2U \), respectively. So

\[
A_{\beta} = D_{\beta} - L_{\beta} - U_{\beta}
= (I - K_1) - (L - K_1 + K_2L) - (U + K_2)
\]

where \( D_{\beta} = I - K_1, L_{\beta} = L - K_1 + K_2L, U_{\beta} = U + K_2 \)

By (13) and (14), we have

\[
\hat{L}_{r,w}x = xx
\]

\[
= (D_{\beta} - rL_{\beta})^{-1}[(1 - w)D_{\beta} + (w - r)L_{\beta} + wU_{\beta}
- \lambda(D_{\beta} - rL_{\beta})]x
\]

\[
= (D_{\beta} - rL_{\beta})^{-1}[(1 - w - r)D_{\beta}
+ (w - r + \lambda r)L_{\beta} + wU_{\beta}]x
\]

\[
= (D_{\beta} - rL_{\beta})^{-1}[(1 - w - \lambda)(I - K_1)
+ (w - r + \lambda r)(L - K_2 + K_2L) + w(U + K_2)]x
\]

\[
= (D_{\beta} - rL_{\beta})^{-1}[(1 - w - \lambda)(I - K_1)
+ (w - r + \lambda r)(K_3L) + wK_2]x
\]

\[
= (D_{\beta} - rL_{\beta})^{-1}[-(1 - w - \lambda)K_1
+ (w - r + \lambda r)(K_3L) + wK_2]x
\]

\[
= (D_{\beta} - rL_{\beta})^{-1}[(\lambda - 1)K_1 + r(\lambda - 1)(K_3L - K_2)
+ wK_2]x
\]

\[
= (D_{\beta} - rL_{\beta})^{-1}[(\lambda - 1)K_1 + r(\lambda - 1)(K_3L - K_2)
+ (\lambda - 1)K_3L]x
\]

\[
= (\lambda - 1)(D_{\beta} - rL_{\beta})^{-1}[K_1 + (1 - r)K_3L]x
\]

Here \( (D_{\beta} - rL_{\beta})^{-1} \geq 0, \quad K_3 \geq 0, \quad (1 - r)K_3 \geq 0 \)

Thus, if \( \lambda > 1 \), then \( \hat{L}_{r,w} \geq 0 \) but not equal to 0. Therefore

\[
\hat{L}_{r,w} \geq \lambda x.
\]

By Lemma 2.3, we get \( (L_{r,w})_{\beta} = \lambda = \rho(L_{r,w}) \).

(2) If \( \lambda = 1 \), then \( \hat{L}_{r,w} \) is not equal to 0. Therefore

\[
\hat{L}_{r,w} \geq \lambda x.
\]

By Lemma 2.3, we get \( (L_{r,w})_{\beta} = \lambda = \rho(L_{r,w}) \).

(3) If \( \lambda < 1 \), then \( \hat{L}_{r,w} \leq 0 \) but not equal to 0. Therefore

\[
\hat{L}_{r,w} \leq \lambda x.
\]

By Lemma 2.3, we get \( (L_{r,w})_{\beta} < \lambda = \rho(L_{r,w}) \).

Corollary 2.2 Let \( A = I - L - U \in R^{n \times n} \) be a nonsingular Z-matrix, \( L_r \) and \( L_r \) be the iteration matrices of classical SOR-type methods and preconditioned SOR-type methods with preconditioner \( P_{\beta} \), respectively. Assume that \( 0 < r < 1 \), and \( 0 < \beta \leq 1, i = 1, 2, \ldots, n - 1 \).

(1) If \( r(L_{r,w})_{\beta} < 1 \), then \( (L_{r,w})_{\beta} \leq (L_{r,w})_{\beta} < 1 \).

(2) If \( r(L_{r,w})_{\beta} \leq r(L_{r,w})_{\beta} < 1 \).

(3) If \( r(L_{r,w})_{\beta} < r(L_{r,w})_{\beta} < 1 \).

Corollary 2.2 Let \( A = I - L - U \in R^{n \times n} \) be a nonsingular Z-matrix, \( L_r \) and \( L_r \) be the iteration matrices of classical SOR-type methods and preconditioned SOR-type methods with preconditioner \( P_{\beta} \), respectively. Assume that \( 0 < r < 1, i = 1, 2, \ldots, n - 1 \).

(1) If \( r(L_{r,w})_{\beta} > 1 \), then \( (L_{r,w})_{\beta} > (L_{r,w})_{\beta} > 1 \).

(2) If \( r(L_{r,w})_{\beta} = r(L_{r,w})_{\beta} = 1 \).

(3) If \( r(L_{r,w})_{\beta} < r(L_{r,w})_{\beta} < 1 \).

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