# A new preconditioned AOR method for Z-matrices

Guangbin Wang[+], Ning Zhang, and Fuping Tan



*Abstract*—In this paper, we present a preconditioned AOR-type iterative method for solving the linear systems $Ax = b$, where $A$ is a Z-matrix. And give some comparison theorems to show that the rate of convergence of the preconditioned AOR-type iterative method is faster than the rate of convergence of the AOR-type iterative method.

*Keywords*—Z-matrix, AOR-type iterative method, precondition, comparison.

## I. Introduction

FOR solving linear system

$$Ax = b, \tag{1}$$

where $A$ is an $n \times n$ square matrix, and $x$ and $b$ are n-dimensional vectors, the basic iterative method is

$$Mx^{k+1} = Nx^k + b,\ k = 0, 1, \cdots, \tag{2}$$

where $A = M - N$ and $M$ is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c,\ k = 0, 1, ...,$$

where $T = M^{-1}N$, $c = M^{-1}b$.

Assuming $A$ has unit diagonal entries and let $A = I - L - U$ where $I$ is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of $A$, respectively. Then,
(I) the iteration matrix of the classical Gauss-Seidel-type method is given by

$$T = (I - L)^{-1}U \tag{3}$$

(II) the iteration matrix of the classical SOR-type method is given by

$$L_r = (I - rL)^{-1}[(1 - r)I + rU] \tag{4}$$

where $r \neq 0$ is a parameter called the relaxation parameter.
(III) the iteration matrix of the classical AOR-type method is given by

$$L_{r,w} = (I - L)^{-1}[(1 - w)I + (w - r)L + wU] \tag{5}$$

where $w$ and $r$ are real parameters and $w \neq 0$.

Transform the original system (1) into the preconditioned form

$$PAx = Pb.$$

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb,\ k = 0, 1, \dots,$$

where $PA = M_p - N_p$ and $M_p$ is nonsingular. Thus the equation above can also be written as

$$x^{k+1} = Tx^k + c,\ k = 0, 1, \dots,$$

where $T = M_p^{-1}N_p,\ c = M_p^{-1}Pb$.

In paper [1], Meijun Wu et al. presented the preconditioned AOR-type iterative method with

$$P_\alpha = I + S_\alpha = \begin{pmatrix} 1 & -\alpha_1 a_{12} & & & \\ & 1 & -\alpha_2 a_{23} & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha_{n-1}a_{n-1,n} \\ & & & & 1 \end{pmatrix} \tag{6}$$

and $\alpha_i (i = 1, 2, \cdots, n-1)$ are nonnegative real numbers, and obtained some comparison results.

In this paper, we will present the preconditioned AOR-type iterative method with

$$P_\beta = I + K_\beta = \begin{pmatrix} 1 & & & & \\ -\beta_1 a_{12} & 1 & & & \\ & -\beta_2 a_{23} & \ddots & & \\ & & \ddots & 1 & \\ & & & -\beta_{n-1}a_{n-1,n} & 1 \end{pmatrix} \tag{7}$$

and $\beta_i(\ i = 1, 2, \cdots, n-1)$ are nonnegative real numbers.

In the following, we consider three splittings for $\tilde{A}$:

$$\tilde{A} = \begin{cases} (\tilde{D} - \tilde{L}) - \tilde{U} \\ \frac{1}{r}(\tilde{D} - r\tilde{L}) - \frac{1}{r}[(1-r)\tilde{D} + r\tilde{U}] \\ \frac{\tilde{D} - r\tilde{L}}{w} - \frac{1}{w}[(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U}] \end{cases} \tag{8}$$

where $\tilde{D}$ , $-\tilde{L}$ and $-\tilde{U}$ are diagonal, strictly lower and strictly upper triangular parts of $\tilde{A}$, respectively.

In view of (8), the iteration matrices associated with $\tilde{A}$ are:

$$\tilde{T} = (\tilde{D} - \tilde{L})^{-1}\tilde{U} \tag{9}$$

$$\tilde{L}_r = (\tilde{D} - r\tilde{L})^{-1}[(1-r)\tilde{D} + r\tilde{U}] \tag{10}$$

$$\tilde{L}_{r,w} = (\tilde{D} - r\tilde{L})^{-1}[(1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U}] \tag{11}$$

In this paper, we will discuss the preconditioned iterative methods with the preconditioner $P_\beta$ for solving Z-matrices linear systems and present comparison theorems of these methods.

Guangbin Wang and Ning Zhang are with the Department of Mathematics, Qingdao University of Science and Technology, Qingdao, 266061, China.

Fuping Tan is with the Department of Mathematics, Shanghai University, Shanghai, 200444, China.

[+] Corresponding author. E-mail: wguangbin750828@sina.com. This work was supported by Natural Science Fund of Shandong Province of China (Y2008A13).

## II. Comparison results of preconditioned AOR-type methods with preconditioner $P_\beta$

We need the following definitions and results.

**Definition 2.1** (Young [3]). $A$ is a Z-matrix if $a_{ij} \le 0$, for all $i, j = 1, 2, \dots n$, $i \ne j$.

**Lemma 2.2** (Young [3]). Let $A \ge 0$ be an irreducible matrix. Then

(1) $A$ has a positive real eigenvalue equals to its spectral radius;

(2) To $\rho(A)$ there corresponds an eigenvector $x > 0$;

(3) $\rho(A)$ is a simple eigenvalue of $A$.

**Lemma 2.3** (Varga [4]). Let $A$ be a nonnegative matrix. Then

(1) If $\alpha x \le Ax$ for some nonnegative vector $x$, $x \ne 0$, then $\alpha \le \rho(A)$;

(2) If $Ax \le \beta x$ for some positive vector $x$, then $\rho(A) \le \beta$. Moreover, if $A$ is irreducible and if $0 \ne \alpha x \le Ax \le \beta x$ for some nonnegative vector $x$, then $\alpha \le \rho(A) \le \beta$ and $x$ is a positive vector.

**Lemma 2.4** ([5]). Let $A = M - N$ be an M-splitting of $A$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is a nonsingular M-matrix.

**Lemma 2.5** ([6]). Let $A$ be a Z-matrix. Then $A$ is a nonsingular M-matrix if and only if there is a positive vector $x$ such that $Ax \ge 0$.

For the linear system (1), we consider its preconditioned form

$$P_\beta A x = P_\beta b$$

with the preconditioner $P_\beta = I + K_\beta$ in this section.

We apply the AOR method to it and have the corresponding preconditioned AOR iteration matrix

$$\hat{L}_{r,w} = [D_\beta - rL_\beta]^{-1}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta], \quad (12)$$

where $D_\beta$ , $-L_\beta$ and $-U_\beta$ are diagonal, strictly lower and strictly upper triangular parts of $A_\beta = P_\beta A$, respectively.

Now we give the main results as follows.

**Theorem 2.1** Let $A = I - L - U \in R^{n\times n}$ be a nonsingular Z-matrix, $L_{r,w}$and $\hat{L}_{r,w}$ be the iteration matrices given by (5) and (12). Assume that $0 < r < w < 1$, and $0 < \beta_i < 1$, $i = 1, 2, \dots, n-1$.

(I) If $\rho(L_{r,w}) < 1$, then

$$\rho(\hat{L}_{r,w}) \le \rho(L_{r,w}) < 1$$

(II) Let $A$ be irreducible. Assume that

$$a_{i,i-1}a_{i-1,i} < 1, \ \ i = 2, \dots, n.$$

then

(1) $\rho(\hat{L}_{r,w}) > \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) > 1$;

(2) $\rho(\hat{L}_{r,w}) = \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) = 1$;

(3) $\rho(\hat{L}_{r,w}) < \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) < 1$.

**Proof.** Let

$$\begin{aligned}
M &= \tfrac{1}{w}(I - rL)\\
N &= \tfrac{1}{w}[(1-w)I + (w-r)L + wU]\\
E_\beta &= \tfrac{1}{w}(D_\beta - rL_\beta)\\
F_\beta &= \tfrac{1}{w}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta]\\
M_\beta &= \tfrac{1}{w}(I + K_\beta)(I - rL)\\
N_\beta &= \tfrac{1}{w}(I + K_\beta)[(1-w)I + (w-r)L + wU]
\end{aligned}$$

Then, we have

$$A = M - N, \quad A_\beta = E_\beta - F_\beta = M_\beta - N_\beta$$

(I) Since $A$ is a nonsingular Z-matrix and $0 < r < w < 1$, $w \ne 0$, it is clear that $M = \frac{1}{w}(I - rL)$ is a nonsingular M-matrix and the splitting

$$A = M - N = \frac{1}{w}(I - rL) - \frac{1}{w}[(1-w)I + (w-r)L + wU]$$

is an M-splitting. Since $\rho(L_{r,w}) < 1$, it follows from Lemma 2.4 that $A$ is a nonsingular M-matrix. Then by Lemma 2.5, there is a positive vector $x$ such that $Ax \ge 0$ , so $A_\beta x = (I + K_\beta)Ax \ge 0$.

By Lemma 2.5, $A_\beta$ is also a nonsingular M-matrix.

Obviously, we can get $D_\beta$ is an positive diagonal matrix, and $L_\beta$ is nonnegative. From $r > 0$ we know that $E_\beta$ is a Z-matrix. Since $rD_\beta^{-1}L_\beta \ge 0$ is a strictly lower triangular matrix so that $\rho(rD_\beta^{-1}L_\beta) = 0 < 1$, we have $(I - rD_\beta^{-1}L_\beta)^{-1} \ge 0$. Then

$$E_\beta = (I - rD_\beta^{-1}L_\beta)^{-1}D_\beta^{-1} \ge 0$$

Hence $E_\beta$ is an nonsingular M-matrix.

By $F_\beta \ge 0$ we can prove that $A_\beta = E_\beta - F_\beta$ is an M-splitting. It follows from Lemma 2.4 that

$$\rho(\hat{L}_{r,w}) = \rho(E_\beta^{-1}F_\beta) < 1.$$

Since $A_\beta = E_\beta - F_\beta$ and $A = M - N$ are both M-splitting and $M_\beta^{-1}N_\beta = M^{-1}N$, two splittings $A_\beta = E_\beta - F_\beta = M_\beta - N_\beta$ are nonnegative.

On the other hand,

$$\begin{aligned}
&M_\beta - E_\beta\\
&= \tfrac{1}{w}(I + K_\beta)(I - rL) - \tfrac{1}{w}(D_\beta - rL_\beta)\\
&= \tfrac{1}{w}(I + K_\beta - rL - rK_\beta L - D_\beta + rL_\beta)\\
&= \tfrac{1}{w}(I + K_\beta - rL - rK_\beta L - D_\beta\\
&\quad + r(D_\beta - I + L - K_\beta + K_\beta L))\\
&= \tfrac{1}{w}(I + K_\beta - rL - rK_\beta L - D_\beta\\
&\quad + rD_\beta - rI + rL - rK_\beta + rK_\beta L)\\
&= \tfrac{1}{w}(I + K_\beta - D_\beta + rD_\beta - rI - rK_\beta)\\
&= \tfrac{1}{w}[(I - r)(I - D_\beta) + (1-r)K_\beta]\\
&\ge 0
\end{aligned}$$

which implies

$$A_\beta^{-1}M_\beta - A_\beta^{-1}E_\beta = A_\beta^{-1}(M_\beta - E_\beta) \ge 0,$$

Therefore, $A_\beta^{-1}M_\beta \ge A_\beta^{-1}E_\beta \ge 0$. So we have $\rho(E_\beta^{-1}F_\beta) \le \rho(M_\beta^{-1}N_\beta)$, that is

$$\rho(\hat{L}_{r,w}) \le \rho(L_{r,w}) < 1.$$

(II) Let $A = I - L - U$ be irreducible. Since

$$\begin{aligned}
L_{r,w} &= (I - rL)^{-1}[(1-w)I + (w-r)L + wU]\\
&= (1-w)I + w(1-r)L + wU + Q
\end{aligned}$$

with $Q = (I - rL)^{-1}rL[w(1-r)L + wU] \ge 0$

We have $L_{r,w}$ is a nonnegative and irreducible matrix. According to Lemma 2.2, there exits a positive vector $x$, such that

$$L_{r,w}x = \lambda x,$$

From the expression of $L_{r,w}$ we obtain the following equality

$$[(1-w)I + (w-r)L + wU]x = \lambda(I - rL)x$$

which is equivalent to

$$[(1-w-r)I + (w-r+\lambda r)L + wU]x = 0, \qquad (13)$$

and

$$(\lambda - 1)(I - rL)xw(L + U - I)x \qquad (14)$$

Let $K_\beta U = K_1 + K_2$, where $K_1,\ K_2$ are the diagonal and lower triangular parts of $K_\beta U$, respectively. So

$$\begin{aligned} A_\beta &= D_\beta - L_\beta - U_\beta \\ &= (I - K_1) - (L - K_\beta + K_\beta L) - (U + K_2) \end{aligned}$$

where $D_\beta = I - K_1$, $L_\beta = L - K_\beta + K_\beta L$, $U_\beta = U + K_2$

By (13) and (14), we have

$$\begin{aligned} &\hat{L}_{r,w}x - \lambda x \\ &= (D_\beta - rL_\beta)^{-1}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta \\ &\quad - \lambda(D_\beta - rL_\beta)]x \\ &= (D_\beta - rL_\beta)^{-1}[(1-w-\lambda)D_\beta \\ &\quad + (w-r+\lambda r)L_\beta + wU_\beta]x \\ &= (D_\beta - rL_\beta)^{-1}[(1-w-\lambda)(I-K_1) \\ &\quad + (w-r+\lambda r)(L - K_\beta + K_\beta L) + w(U+K_2)]x \\ &= (D_\beta - rL_\beta)^{-1}\{[(1-w-\lambda)I + (w-r+\lambda r)L + wU] \\ &\quad + [-(1-w-\lambda)K_1 \\ &\quad +(w-r+\lambda r)(-K_\beta + K_\beta L) + wK_2]\}\, x \\ &= (D_\beta - rL_\beta)^{-1}[-(1-w-\lambda)K_1 \\ &\quad + (w-r+\lambda r)(-K_\beta + K_\beta L) + wK_2]x \\ &= (D_\beta - rL_\beta)^{-1}[(\lambda-1)K_1 + r(\lambda-1)(K_\beta L - K_\beta) \\ &\quad + wK_\beta(L + U - I)]x \\ &= (D_\beta - rL_\beta)^{-1}[(\lambda-1)K_1 + r(\lambda-1)(K_\beta L - K_\beta) \\ &\quad + (\lambda-1)(I - rL)K_\beta]x \\ &= (D_\beta - rL_\beta)^{-1}[(\lambda-1)K_1 - r(\lambda-1)K_\beta + (\lambda-1)K_\beta]x \\ &= (\lambda-1)(D_\beta - rL_\beta)^{-1}[K_1 + (1-r)K_\beta]x \end{aligned}$$

Here $(D_\beta - rL_\beta)^{-1} \geq 0, \quad K_1 \geq 0, \quad (1-r)K_\beta \geq 0$

(1) If $\lambda > 1$, then $\hat{L}_{r,w} \geq 0$ but not equal to 0. Therefore

$$\hat{L}_{r,w} \geq \lambda x.$$

By Lemma 2.3, we get $\rho(\hat{L}_{r,w}) > \lambda = \rho(L_{r,w})$.

(2) If $\lambda = 1$, then $\hat{L}_{r,w} = 0$but not equal to 0. Therefore

$$\hat{L}_{r,w} = \lambda x.$$

By Lemma 2.3, we get $\rho(\hat{L}_{r,w}) = \lambda = \rho(L_{r,w})$.

(3) If $\lambda < 1$, then $\hat{L}_{r,w} \leq 0$but not equal to 0. Therefore

$$\hat{L}_{r,w} \leq \lambda x.$$

By Lemma 2.3, we get $\rho(\hat{L}_{r,w}) < \lambda = \rho(L_{r,w})$.

**Corollary 2.2** Let $A = I - L - U \in R^{n\times n}$ be a nonsingular Z-matrix, $L_r$ and $\hat{L}_r$ be the iteration matrices of classical SOR-type methods and preconditioned SOR-type methods with preconditioner $P_\beta$, respectively. Assume that $0 < r < 1$, and $0 < \beta_i < 1$, $i = 1, 2, \ldots, n-1$.

(I) If $\rho(L_r) < 1$, then $\rho(\hat{L}_r) \leq \rho(L_r) < 1$;

(II) Let $A$ be irreducible. Assume that $a_{i,i-1}a_{i-1,i} < 1,\ i = 2, \ldots, n$, then

(1) $\rho(\hat{L}_{r,w}) > \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) > 1$;

(2) $\rho(\hat{L}_{r,w}) = \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) = 1$;

(3) $\rho(\hat{L}_{r,w}) < \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) < 1$.

**Corollary 2.3** Let $A = I - L - U \in R^{n\times n}$ be a nonsingular Z-matrix, $T$ and $\hat{T}$ be the iteration matrices of classical Gauss-Seidel-type methods and preconditioned Gauss-Seidel-type methods with preconditioner $P_\beta$, respectively. $0 < \beta_i < 1$, $i = 1, 2, \ldots, n-1$.

(I) If $\rho(T) < 1$, then $\rho(\hat{T}) \leq \rho(T) < 1$;

(II) Let $A$ be irreducible. Assume that $a_{i,i-1}a_{i-1,i} < 1,\ i = 2, \ldots n$, then

(1) $\rho(\hat{T}) > \rho(T)$ if $\rho(T) > 1$;

(2) $\rho(\hat{T}) = \rho(T)$ if $\rho(T) = 1$;

(3) $\rho(\hat{T}) < \rho(T)$ if $\rho(T) < 1$.

## REFERENCES

[1] Meijun Wu, Li Wang, Yongzhong Song. Preconditioned AOR iterative method for linear systems. *Appl. Num. Math.* pp.672-685,2007.
[2] Jicheng Li T.Z. Huang, Preconditioned Methods of Z-matrices, *Acta Mathematica Scientia*, vol.25A, pp.5-10, 2005.
[3] D.M. Young, *Iterative solution of large linear systems*, Academic Press, New York, 1971.
[4] R.S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1981.
[5] W.Li, W.W.Sun, Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices, *Linear Algebra Appl.* vol.317,pp.227-240, 2000.
[6] A.Berman, R.J.Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.