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Modeling Hybrid Systems with MLD Approach and Analysis of the Model Size and Complexity

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***Abstract*—** Recently, a great amount of interest has been shown in the field of modeling and controlling hybrid systems. One of the efficient and common methods in this area utilizes the mixed logical-dynamical (MLD) systems in the modeling. In this method, the system constraints are transformed into mixed-integer inequalities by defining some logic statements. In this paper, a system containing three tanks is modeled as a nonlinear switched system by using the MLD framework. Comparing the model size of the three-tank system with that of a two-tank system, it is deduced that the number of binary variables, the size of the system and its complexity tremendously increases with the number of tanks, which makes the control of the system more difficult. Therefore, methods should be found which result in fewer mixed-integer inequalities.

***Keywords*—**Hybrid systems, mixed-integer inequalities, mixed logical dynamical systems, multi-tank system.

I. INTRODUCTION

HYBRID systems are referred to those systems having different parts or different processes with different characteristics. In the area of control and modelling, hybrid systems are referred to systems which are comprised of discrete and continuous parts. Hybrid systems may also be defined as a combination of time-driven and event-driven components. In past, dynamics of these systems were studied separately. Models like automata or Petri net were used for the event-driven part, and for the time-driven part, differential or difference equations were used. In processes where the discrete and continuous parts work together and a significant interaction is observed between them, a thorough analysis of the system performance and achieving high efficiency requires that all the dynamic parts and their interactions be studied completely. In this way, the exact analysis and optimization of a system becomes possible. As a result, many researchers have concentrated on modelling and controlling hybrid systems. So far, no general method has been presented for the analysis and design of hybrid systems. Therefore, the researchers have concentrated on specific classes of such systems and have presented analysis and design methods for them. Mixed logical-dynamical systems (MLD), piecewise-affine systems (PWA), linear complementary systems (LC), max-min-plus-scaling systems (MMPS), and extended linear complementary systems (ELC) are some important classes which have been considered before [1]-[5]. The aforementioned classes are equivalent in some conditions. This equivalence is important in the sense that it allows the invented methods for a specific class to be used for other classes in some conditions [6].

In this paper, the MLD method is used to model a system containing multiple tanks. A detailed model is presented for a three-tank system and the size and complexity of the system is compared with those of a two-tank one. It is observed that by adding one tank to the system, its size and complexity increases significantly.

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II. MIXED LOGICAL DYNAMICAL MODELING

Mixed logical dynamical is one of the powerful methods of modeling in the theory of hybrid systems, which was first presented in [1]. The principles of MLD modeling are discussed in this section.

In this modeling method, a binary variable is assigned to each logic statement. If and only if the logic statement is true, the value of the binary variable is one:

$$X_i \equiv True \leftrightarrow \delta_i = 1 \quad (1)$$

Combination of the logic statements may be described with the combination of binary variables. For example the term $X_1 \vee X_2$ is equivalent to $\delta_1 + \delta_2 \geq 1$. Indeed, the mentioned logic term is true when at least one of the logic statements X_1 and X_2 is true. Equivalently, the inequality $\delta_1 + \delta_2 \geq 1$ is true when at least one of the binary variables equals one. The inequalities like $\delta_1 + \delta_2 \geq 1$, which contain a linear combination of binary variables, are called linear integer inequalities. If some continuous variables are also included, the inequality is called mixed-integer. Another basic principle of MLD modeling is the relationship between dynamic and binary variables. In fact, it can be shown that the logic statement $[y(t) \geq 0] \leftrightarrow [\delta(t) = 1]$ is true if and only if the following inequalities are true:

$$\begin{aligned} &-(M+\varepsilon)\delta(t) \leq -y(t) - \varepsilon \\ &-m\delta(t) \leq y(t) - m \end{aligned} \quad (2)$$

where M and m are the maximum and minimum of y(t) respectively, and ε is a small positive number (typically the machine precision). Sometimes terms like $\delta f(x)$ are encountered in MLD modeling method, which are in fact the product of a binary variable and a continuous function. In these cases y is defined as $y = \delta f(x)$ and it can be shown that it is equivalent to four mixed-integer inequalities:

$$\begin{array}{ll} y \le f(x) - m(1-\delta) & y \le M\delta \\ y \ge f(x) - M(1-\delta) & y \ge m\delta \end{array} \qquad (3)$$

where M and m are the maximum and minimum of $f(x)$ respectively.

In fact, with these principles, the problem constraints and the auxiliary variables (which are usually products of binary and continuous variables) are transformed into mixed integer inequalities. Therefore, the system is modeled as an MLD in the following way:

$$\begin{aligned} x[k+1] &= Ax[k] + B_1 u[k] + B_2 \delta[k] + B_3 z[k] + B_0 \\ y[k] &= Cx[k] + D_1 u[k] + D_2 \delta[k] + D_3 z[k] + D_0 \\ E_2 \delta[k] + E_3 z[k] &\le E_1 u[k] + E_4 x[k] + E_5 \end{aligned} \qquad (4)$$

where x is a state vector of the system and contains continuous and binary variables. y and u are respectively output and input vectors of the system, which consist of continuous and discrete parts. The vectors $\delta \in \{0,1\}^{r_l}$ and $z \in R^{r_c}$ are auxiliary binary and continuous variables, respectively.

III. Hybrid Modeling of a Three-Tank System with MLD Approach

In this section, the dynamics of a three-tank system and the constraints of the problem are described with MLD approach. This system is suitable to our needs, because the number of tanks may be changed easily and the resulting systems can be modeled. In this paper, the variation of system size and its complexity is studied by changing the number of tanks.

The system contains three tanks which are connected to each other. The tanks are filled by gas flows and these flows are controlled by four control valves. The control task is to fill three tanks up to predefined pressures while preventing the state trajectory to enter the forbidden regions in the state space.

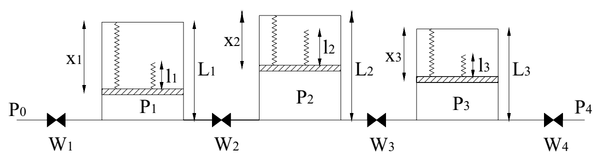

Fig. 1 Three-tank system

The flow passing through the control valves is proportional to the difference between the pressures of the two sides of the valve:

$$q_i = k_{i+1} w_{i+1} (P_i - P_{i+1}) \qquad (i = 0,1,2,3) \qquad (5)$$

Where k_i is the valve constant and w_i is the control signal of the V_i valve. The equations of gas volume and its variations in each tank are as the following:

$$V_i = A_i (L_i - x_i) \Rightarrow \dot{V}_i = -A_i \dot{x}_i \qquad (7)$$

$$\dot{V}_i = q_{i-1} - q_i \qquad (i = 1,2,3) \qquad (8)$$

where A_i is the cross-section area of the i'th tank, L_i is the height of the i'th tank and x_i is the distance between the moving plane and the ceiling of the i'th tank. The equations regarding the gas pressure and its variations in each tank are as the following:

$$P_i = \begin{cases} P_i^* + c_i (L_i - x_i) & x_i \ge l_i \\ P_i^* + c_i (L_i - x_i) + c_i' (l_i - x_i) & x_i < l_i \end{cases} \qquad (9)$$

Therefore,

$$\dot{P}_i = \begin{cases} -c_i \dot{x}_i & x_i \ge l_i \\ -(c_i + c_i') \dot{x}_i & x_i < l_i \end{cases} \qquad (i = 1,2,3) \qquad (10)$$

where P_i^* is a constant pressure, and c_i', c_i are stiffness factors of the tank springs. It is supposed that the control signals and state variables are continuous variables in the intervals $[w_{\min}, w_{\max}]$ and $[P_{\min}, P_{\max}]$. Since there are three tanks in the system and each tank has two operation regions, the nonlinear dynamics of the system can be explicitly defined in eight different regions. For example, regarding the equations (9) and (10), the dynamics of the system in region $x_1 \ge l_1$, $x_2 < l_2$ and $x_3 \ge l_3$ is as the following:

$$\dot{P} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{A_1}(k_1 p_0 w_1 - (k_1 w_1 + k_2 w_2) p_1 + k_2 p_2 w_2) \\ \frac{(c_2 + c_2')}{A_2}(k_2 p_1 w_2 - (k_2 w_2 + k_3 w_3) p_2 + k_3 p_3 w_3) \\ \frac{c_3}{A_3}(k_3 p_2 w_3 - (k_3 w_3 + k_4 w_4) p_3 + k_4 p_4 w_4) \end{bmatrix} \qquad (11)$$

By linearizing the eight nonlinear state equations around the equilibrium point (x_{eq}, u_{eq}), the nonlinear dynamics of the system can be transformed into an approximate piecewise linear model. In this way, the following affine models are derived for the eight operation regions:

$$\dot{P} = F_1 P + G_1 u + H_1 \quad \text{for } x_1 > l_1, \quad x_2 > l_2, x_3 > l_3 \qquad (12)$$

$$\dot{P} = F_2 P + G_2 u + H_2 \quad \text{for } x_1 > l_1, \quad x_2 > l_2, x_3 \le l_3 \qquad (13)$$

$$\dot{P} = F_3 P + G_3 u + H_3 \quad \text{for } x_1 > l_1, \; x_2 \le l_2, x_3 > l_3 \qquad (14)$$

$$\dot{P} = F_4 P + G_4 u + H_4 \quad \text{for } x_1 > l_1, x_2 \le l_2, x_3 \le l_3 \qquad (15)$$

$\dot{P} = F_5P + G_5u + H_5$ for $x_1 \le l_1$, $x_2 > l_2, x_3 > l_3$ (16)

$\dot{P} = F_6P + G_6u + H_6$ for $x_1 \le l_1$, $x_2 > l_2, x_3 \le l_3$ (17)

$\dot{P} = F_7P + G_7u + H_7$ for $x_1 \le l_1$, $x_2 \le l_2, x_3 > l_3$ (18)

$\dot{P} = F_8P + G_8u + H_8$ for $x_1 \le l_1$, $x_2 \le l_2, x_3 \le l_3$ (19)

where $u = [w_1 \ w_2 \ w_3 \ w_4]^T$.

By defining δ_1, δ_2 ... , δ_{10} as 10 binary variables and defining the following logic statements:

$$[\delta_1 = 1] \leftrightarrow [x_1 \le l_1] \tag{20}$$

$$[\delta_2 = 1] \leftrightarrow [x_2 \le l_2] \tag{21}$$

$$[\delta_3 = 1] \leftrightarrow [x_3 \le l_3] \tag{22}$$

$$[\delta_4 = 1] \leftrightarrow [\delta_1 = 0] \wedge [\delta_2 = 0] \wedge [\delta_3 = 1] \tag{23}$$

$$[\delta_5 = 1] \leftrightarrow [\delta_1 = 0] \wedge [\delta_2 = 1] \wedge [\delta_3 = 0] \tag{24}$$

$$[\delta_6 = 1] \leftrightarrow [\delta_1 = 0] \wedge [\delta_2 = 1] \wedge [\delta_3 = 1] \tag{25}$$

$$[\delta_7 = 1] \leftrightarrow [\delta_1 = 1] \wedge [\delta_2 = 0] \wedge [\delta_3 = 0] \tag{26}$$

$$[\delta_8 = 1] \leftrightarrow [\delta_1 = 1] \wedge [\delta_2 = 0] \wedge [\delta_3 = 1] \tag{27}$$

$$[\delta_9 = 1] \leftrightarrow [\delta_1 = 1] \wedge [\delta_2 = 1] \wedge [\delta_3 = 0] \tag{28}$$

$$[\delta_{10} = 1] \leftrightarrow [\delta_1 = 1] \wedge [\delta_2 = 1] \wedge [\delta_3 = 1] \tag{29}$$

the system can be modeled in the following way:

$$\dot{P} = F_1P + G_1u + H_1 + \sum_{i=2}^{8} \delta_{i+2}[(F_i - F_1)P + (G_i - G_1)u + (H_i - H_1)] \tag{30}$$

Using the relationships derived from the linearization, the matrices $F_i - F_1, G_i - G_1$ and $H_i - H_1$ $(i = 2,3,...,8)$ can be calculated. As an example:

$$H_6 - H_1 = \begin{bmatrix} h_{11}^6 \\ 0 \\ h_{31}^6 \end{bmatrix} \quad F_4 - F_1 = \begin{bmatrix} 0 & 0 & 0 \\ f_{21}^4 & f_{22}^4 & f_{23}^4 \\ 0 & f_{32}^4 & f_{33}^4 \end{bmatrix}$$

$$G_8 - G_1 = \begin{bmatrix} g_{11}^8 & g_{12}^8 & 0 & 0 \\ 0 & g_{22}^8 & g_{23}^8 & 0 \\ 0 & 0 & g_{33}^8 & g_{34}^8 \end{bmatrix}$$

The logic statement (20) is equivalent to the following mixed-integer inequalities:

$$\begin{cases} M_1\delta_1 \le -x_1 + l_1 + M_1 \\ (m_1 - \varepsilon)\delta_1 \le x_1 - l_1 - \varepsilon \end{cases} \tag{31}$$

where $m_1 = \min(x_1) - l_1$ and $M_1 = \max(x_1) - l_1$. Similarly each of the logic statements (21) and (22) are equivalent to two mixed-integer inequalities.

The logic statements (23-29) are also equivalent to the inequality sets (32):

$$\begin{cases} -\delta_2 + \delta_5 \le 0 \\ \delta_1 + \delta_5 \le 1 \\ \delta_3 + \delta_5 \le 1 \\ -\delta_1 + \delta_2 - \delta_3 - \delta_5 \le 0 \end{cases} \quad \begin{cases} -\delta_3 + \delta_4 \le 0 \\ \delta_1 + \delta_4 \le 1 \\ \delta_2 + \delta_4 \le 1 \\ -\delta_1 - \delta_2 + \delta_3 - \delta_4 \le 0 \end{cases}$$

$$\begin{cases} -\delta_1 + \delta_7 \le 0 \\ \delta_2 + \delta_7 \le 1 \\ \delta_3 + \delta_7 \le 1 \\ \delta_1 - \delta_2 - \delta_3 - \delta_7 \le 0 \end{cases} \quad \begin{cases} -\delta_2 + \delta_6 \le 0 \\ -\delta_3 + \delta_6 \le 0 \\ \delta_1 + \delta_6 \le 1 \\ -\delta_1 + \delta_2 + \delta_3 - \delta_6 \le 0 \end{cases}$$

$$\begin{cases} -\delta_1 + \delta_8 \le 0 \\ -\delta_3 + \delta_8 \le 0 \\ \delta_2 + \delta_8 \le 1 \\ \delta_1 - \delta_2 + \delta_3 - \delta_8 \le 1 \end{cases} \quad \begin{cases} -\delta_1 + \delta_9 \le 0 \\ -\delta_2 + \delta_9 \le 0 \\ \delta_3 + \delta_9 \le 1 \\ \delta_1 + \delta_2 - \delta_3 - \delta_9 \le 1 \end{cases}$$

$$\begin{cases} -\delta_1 + \delta_{10} \le 0 \\ -\delta_2 + \delta_{10} \le 0 \\ -\delta_3 + \delta_{10} \le 1 \\ \delta_1 + \delta_2 + \delta_3 - \delta_{10} \le 2 \end{cases} \tag{32}$$

By replacing the values of $F_i - F_1$, $G_i - G_1$ and $H_i - H_1$ for $(i = 2,3,...,8)$ in equation (30), equations are derived in the form of products of a binary variable and a continuous value. Consequently, the auxiliary variables z_i $(i = 1,2,...,12)$ may be defined as shown below:

$$z_1 = \delta_4 \times \left\{ f_{32}^2 p_2 + f_{33}^2 p_3 + g_{33}^2 w_3 + g_{34}^2 w_4 \right\} \tag{33}$$

$$z_2 = \delta_5 \times \left\{ f_{21}^3 p_1 + f_{22}^3 p_2 + f_{23}^3 p_3 + g_{22}^3 w_2 + g_{23}^3 w_3 \right\} \tag{34}$$

$$z_3 = \delta_6 \times \left\{ f_{21}^4 p_1 + f_{22}^4 p_2 + f_{23}^4 p_3 + g_{22}^4 w_2 + g_{23}^4 w_3 \right\} \tag{35}$$

$$z_4 = \delta_6 \times \left\{ f_{32}^4 p_2 + f_{33}^4 p_3 + g_{33}^4 w_3 + g_{34}^4 w_4 \right\} \tag{36}$$

$$z_5 = \delta_7 \times \left\{ f_{11}^5 p_1 + f_{12}^5 p_2 + g_{11}^5 w_1 + g_{12}^5 w_2 \right\} \tag{37}$$

$$z_6 = \delta_8 \times \left\{ f_{11}^6 p_1 + f_{12}^6 p_2 + g_{11}^6 w_1 + g_{12}^6 w_2 \right\} \tag{38}$$

$$z_7 = \delta_8 \times \left\{ f_{32}^6 p_2 + f_{33}^6 p_3 + g_{33}^6 w_3 + g_{34}^6 w_4 \right\} \tag{39}$$

$$z_8 = \delta_9 \times \left\{ f_{11}^7 p_1 + f_{12}^7 p_2 + g_{11}^7 w_1 + g_{12}^7 w_2 \right\} \tag{40}$$

$$z_9 = \delta_9 \times \left\{ f_{21}^7 p_1 + f_{22}^7 p_2 + f_{23}^7 p_3 + g_{22}^7 w_2 + g_{23}^7 w_3 \right\} \tag{41}$$

$$z_{10} = \delta_{10} \times \left\{ f_{11}^8 p_1 + f_{12}^8 p_2 + g_{11}^8 w_1 + g_{12}^8 w_2 \right\} \tag{42}$$

$$z_{11} = \delta_{10} \times \left\{ f_{21}^8 p_1 + f_{22}^8 p_2 + f_{23}^8 p_3 + g_{22}^8 w_2 + g_{23}^8 w_3 \right\} \tag{43}$$

$$z_{12} = \delta_{10} \times \left\{ f_{32}^8 p_2 + f_{33}^8 p_3 + g_{33}^8 w_3 + g_{34}^8 w_4 \right\} \tag{44}$$

Each of the equations (33-44) is equivalent to four mixed-integer inequalities. For example equation (33) is equivalent to the following inequalities:

$$\begin{cases} z_1 \le M_{z_1}\delta_4 \\ z_1 \ge m_{z_1}\delta_4 \\ z_1 \ge f_{32}^2\, p_2 + f_{33}^2\, p_3 + g_{33}^2 w_3 + g_{34}^2 w_4 - M_{z_1}(1-\delta_4) \\ z_1 \le f_{32}^2\, p_2 + f_{33}^2\, p_3 + g_{33}^2 w_3 + g_{34}^2 w_4 - m_{z_1}(1-\delta_4) \end{cases} \tag{45}$$

where

$$M_{z_1} = \max\{f_{32}^2\, p_2 + f_{33}^2\, p_3 + g_{33}^2 w_3 + g_{34}^2 w_4\}$$

$$m_{z_1} = \min\{f_{32}^2\, p_2 + f_{33}^2\, p_3 + g_{33}^2 w_3 + g_{34}^2 w_4\}$$

As mentioned earlier in this paper, the state space has forbidden regions where the state trajectory is not allowed to enter. Suppose that the forbidden regions are defined as shown below:

$$\begin{cases} p_1 \ge p_{11f} \\ p_2 \le p_{21f} \\ p_3 \le p_{31f} \end{cases}, \begin{cases} p_1 \le p_{12f} \\ p_2 \ge p_{22f} \\ p_3 \le p_{32f} \end{cases} \text{ and } \begin{cases} p_1 \le p_{13f} \\ p_2 \le p_{23f} \\ p_3 \ge p_{33f} \end{cases} \tag{46}$$

In order to express the constraints resulting from the forbidden regions in the form of mixed-integer inequalities, the binary variables $\delta_{19}, \delta_{18}, ..., \delta_{12}, \delta_{11}$ and the following logic statements are defined:

$$\begin{array}{ll} [p_1 \ge p_{11f}] \leftrightarrow [\delta_{11} = 1] & [p_2 \ge p_{21f}] \leftrightarrow [\delta_{12} = 1] \\ [p_3 \ge p_{31f}] \leftrightarrow [\delta_{13} = 1] & [p_1 \ge p_{12f}] \leftrightarrow [\delta_{14} = 1] \\ [p_2 \ge p_{22f}] \leftrightarrow [\delta_{15} = 1] & [p_3 \ge p_{32f}] \leftrightarrow [\delta_{16} = 1] \\ [p_1 \ge p_{13f}] \leftrightarrow [\delta_{17} = 1] & [p_2 \ge p_{23f}] \leftrightarrow [\delta_{18} = 1] \\ [p_3 \ge p_{33f}] \leftrightarrow [\delta_{19} = 1] & \end{array} \tag{47}$$

The state trajectory does not pass the forbidden regions, if and only if the following logic statements are true:

$$[\delta_{11} = 1] \rightarrow [\delta_{12} = 1] \vee [\delta_{13} = 1] \tag{48}$$

$$[\delta_{15} = 1] \rightarrow [\delta_{14} = 1] \vee [\delta_{16} = 1] \tag{49}$$

$$[\delta_{19} = 1] \rightarrow [\delta_{17} = 1] \vee [\delta_{18} = 1] \tag{50}$$

The above logic statements are equivalent to the following inequalities:

$$\delta_{11} - \delta_{12} - \delta_{13} \le 0 \tag{51}$$

$$\delta_{15} - \delta_{14} - \delta_{16} \le 0 \tag{52}$$

$$\delta_{19} - \delta_{17} - \delta_{18} \le 0 \tag{53}$$

Each of the logic statements in relationship (47) is equivalent to two mixed-integer inequalities. For example the statement $[p_1 \ge p_{11f}] \leftrightarrow [\delta_{11} = 1]$ is equivalent to the following inequalities:

$$\begin{cases} -m_3\delta_{11} \le p_1 - p_{11f} - m_3 \\ -(M_3 + \varepsilon)\delta_{11} \le -p_1 + p_{11f} - \varepsilon \end{cases} \tag{54}$$

where $m_3 = \min(p_1) - p_{11f}$ and $M_3 = \max(p_1) - p_{11f}$.

The higher and lower limits on input and state variables result in the following 14 inequalities:

$$w_{\min} \le w_i \le w_{\max} \qquad (i = 1,2,3,4) \tag{55}$$

$$p_{\min} \le p_i \le p_{\max} \qquad (i = 1,2,3) \tag{56}$$

Now, all the required inequalities and matrices are available to describe the three-tank system with an MLD model (Equation 4). Using the auxiliary variables, the matrices A and B_i $(i = 0,1,2,3)$ will be as the following:

$$A = F_1 \qquad B_0 = H_1 \qquad B_1 = G_1 \tag{57}$$

$$B_2 = \begin{bmatrix} 0&0&0&0&0&0&h_1^5&h_1^6&h_1^7&h_1^8&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&h_2^3&h_2^4&0&0&h_2^7&h_2^8&0&0&0&0&0&0&0&0&0 \\ 0&0&0&h_3^2&0&h_3^4&0&h_3^6&0&h_3^8&0&0&0&0&0&0&0&0&0 \end{bmatrix} \tag{58}$$

$$B_3 = \begin{bmatrix} 0&0&0&0&1&1&0&1&0&1&0&0 \\ 0&1&1&0&0&0&0&0&1&0&1&0 \\ 1&0&0&1&0&0&1&0&0&0&0&1 \end{bmatrix} \tag{59}$$

The vectors δ and z are as the following:

$$\delta = [\delta_1\ \delta_2\ \ldots\ \delta_{18}\ \delta_{19}]^T \tag{60}$$

$$z = [z_1\ z_2\ \ldots\ z_{11}\ z_{12}]^T \tag{61}$$

As mentioned before, each of the statements (20)-(22) are equivalent to two inequalities and each of the statements (23)-(29) are equivalent to four inequalities. Each of the equations (33)-(44) is also equivalent to four inequalities and each of the nine logic statements in (47) is equivalent to two inequalities. The equations (48)-(50) are also equivalent to inequalities (51)-(53). Finally, according to (55) and (56), the higher and lower limits on input and state variables result in 14 inequalities. Therefore, there are totally

$$(3\times 2) + (7\times 4) + (12\times 4) + (9\times 2) + 3 + 14 = 117$$

inequalities. In other words, the E_i matrices (in Equation 4) have 117 rows. These matrices are easily derived from the mentioned relations, but due to their large sizes, they are not shown here.

IV. Size and Complexity Comparison Between Multi-Tank Systems

In the previous section, the three tank system was described in MLD framework and it was observed that the number of mixed-integer inequalities and therefore the number of rows in E_i matrices are equal to 117. In this section, the two-tank system is modeled and compared with other multi-tank systems. Since the procedure of modeling is quite the same as that of the three-tank system, the details of modeling are not mentioned here. In two-tank system, there are four operation regions. By writing the system equations and linearizing them around the equilibrium points and introducing 5 binary variables $\delta_1, \delta_2, \delta_3, \delta_4$ and δ_5 and defining the following logic statements:

$$[\delta_1 = 1] \leftrightarrow [x_1 \le l_1] \quad (62)$$

$$[\delta_2 = 1] \leftrightarrow [x_2 \le l_2] \quad (63)$$

$$[\delta_3 = 1] \leftrightarrow [\delta_1 = 0] \wedge [\delta_2 = 1] \quad (64)$$

$$[\delta_4 = 1] \leftrightarrow [\delta_1 = 1] \wedge [\delta_2 = 0] \quad (65)$$

$$[\delta_5 = 1] \leftrightarrow [\delta_1 = 1] \wedge [\delta_2 = 1] \quad (66)$$

the system can be modelled as follows:

$$\begin{aligned}\dot{P} &= F_1 P + G_1 u + H_1 \\ &+ \sum_{i=2}^{4} \delta_{i+1}[(F_i - F_1)P + (G_i - G_1)u + (H_i - H_1)]\end{aligned} \quad (67)$$

Replacing $F_i - F_1$, $G_i - G_1$ and $H_i - H_1$ $(i = 2,3,4)$ in (67) results in definition of the following four auxiliary continuous variables:

$$z_1 = \delta_3 \times \{f_{21}^2 p_1 + f_{22}^2 p_2 + g_{22}^2 w_2 + g_{23}^2 w_3\} \quad (68)$$

$$z_2 = \delta_4 \times \{f_{11}^3 p_1 + f_{12}^3 p_2 + g_{11}^3 w_1 + g_{12}^3 w_2\} \quad (69)$$

$$z_3 = \delta_5 \times \{f_{11}^4 p_1 + f_{12}^4 p_2 + g_{11}^4 w_1 + g_{12}^4 w_2\} \quad (70)$$

$$z_4 = \delta_5 \times \{f_{21}^4 p_1 + f_{22}^4 p_2 + g_{22}^4 w_2 + g_{23}^4 w_3\} \quad (71)$$

If the forbidden regions are $p_2 \le p_{21f}, p_1 \ge p_{11f}$ and $p_2 \ge p_{22f}, p_1 \le p_{12f}$, the binary variables $\delta_9, \delta_8, \delta_7, \delta_6$ and the following logic statements are defined to express the constraints resulting from the forbidden regions:

$$[p_{11f} \le p_1] \leftrightarrow [\delta_6 = 1] \qquad [p_{21f} \le p_2] \leftrightarrow [\delta_7 = 1]$$

$$[p_1 \le p_{12f}] \leftrightarrow [\delta_8 = 1]$$

$$[p_2 \le p_{22f}] \leftrightarrow [\delta_9 = 1] \quad (72)$$

the state trajectory does not enter the forbidden regions if and only if the following logic statements are true:

$$[\delta_6 = 1] \to [\delta_7 = 1] \qquad [\delta_8 = 1] \to [\delta_9 = 1]$$

(73)

the above logic statements are equivalent to the following inequalities:

$$\delta_6 - \delta_7 \le 0 \qquad \delta_8 - \delta_9 \le 0$$

(74)

the logic statements (64)-(66) are equivalent to inequality sets (75) respectively:

$$\begin{cases}\delta_1 + \delta_3 \le 1 \\ -\delta_2 + \delta_3 \le 0 \\ -\delta_1 + \delta_2 - \delta_3 \le 0\end{cases} \qquad \begin{cases}-\delta_1 + \delta_4 \le 0 \\ \delta_2 + \delta_4 \le 1 \\ \delta_1 - \delta_2 - \delta_4 \le 0\end{cases}$$

$$\begin{cases}-\delta_1 + \delta_5 \le 0 \\ -\delta_2 + \delta_5 \le 0 \\ \delta_1 + \delta_2 - \delta_5 \le 1\end{cases} \quad (75)$$

the higher and lower limits on input and state variables result in the following 10 inequalities:

$$w_{\min} \le w_i \le w_{\max} \qquad (i = 1,2,3) \quad (76)$$

$$x_{\min} \le x_i \le x_{\max} \qquad (i = 1,2) \quad (77)$$

Therefore, the matrices A and B_i $(i = 0,1,2,3)$ in the two-tank system are derived as below:

$$A = F_1,\ B_0 = H_1,\ B_1 = G_1 \quad (78)$$

$$B_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(79)

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & h_1^3 & h_1^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_2^2 & 0 & h_2^4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(80)

In fact, each of the statements (62) and (63) is equivalent to two inequalities and the statements (64)-(66) are each equivalent to three inequalities. Each of the equations (68)-(71) is equivalent to four inequalities and each of the statements in (72) is equivalent to two inequalities. There are two inequalities in (74) and ten inequalities in (76) and (77). Therefore, to describe the two-tank system in MLD framework, there are totally

$$(2\times 2) + (3\times 3) + (4\times 4) + (4\times 2) + 2 + 10 = 49$$

inequalities. Consequently, each E_i matrix has 49 rows.

In order to have a more precise analysis, the 4, 5, 6, 7, and 8 tank systems are also modeled by MLD approach. The results are shown in the following table:

TABLE I
NUMBER OF BINARY VARIABLES, AUXILIARY CONTINUOUS VARIABLES AND MIXED-INTEGER INEQUALITIES IN TERMS OF THE NUMBER OF TANKS

Number of tanks	Number of binary variables	Number of auxiliary continuous variables	Number of mixed-integer inequalities
2	9	4	49
3	19	12	117
4	35	32	265
5	61	80	593
6	105	192	1325
7	183	448	2957
8	327	1024	6577

By plotting the number of binary variables, auxiliary continuous variables and mixed-integer inequalities in terms of the number of tanks, approximate relationships can be derived for them.

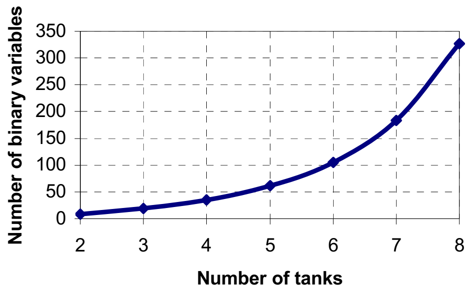

Fig.2 The number of binary variables in terms of the number of tanks

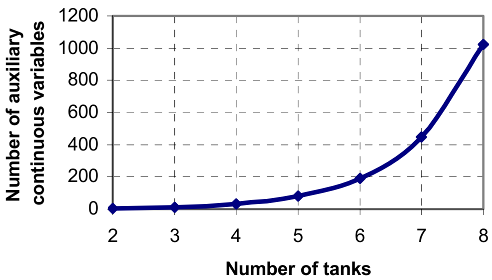

Fig. 3 The number of auxiliary continuous variables in terms of the number of tanks

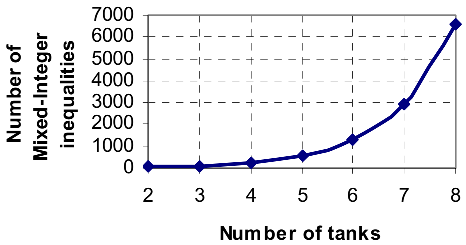

Fig. 4 The number of mixed-integer inequalities in terms of the number of tanks

If the number of tanks, the number of binary variables, the number of auxiliary continuous variables, and the number of mixed-integer inequalities are shown respectively by N, Y, Z, and W, the following approximate equations are derived:

$$Y = 3.1146 \times e^{0.586\,N} \tag{81}$$

$$Z = 0.7439 \times e^{0.9167\,N} \tag{82}$$

$$W = 10.02 \times e^{0.8131\,N} \tag{83}$$

It is observed that the number of binary variables, the number of auxiliary continuous variables and the number of mixed-integer inequalities increase exponentially with the number of tanks. Consequently the system size and its complexity grow exponentially with the number of tanks. Since the control methods presented in MLD systems like predictive control, use numerical methods to solve the control problems, the increase in the number of mixed-integer inequalities and therefore the system dimension is considered a critical limitation. For example, ILOG AMPL CPLEX software can be used for predictive control of a two-tank system in MLD framework, with control horizon of $T = 4$, while it is not possible to use this software for the predictive control of a three-tank system (since the number of inequalities rises to 117) [11],[12]. As a result, recently many researchers have concentrated on finding solutions to decrease the number of mixed-integer inequalities in MLD method.

V. Conclusion

In this paper, the details of modeling a three-tank system in MLD framework are discussed. By comparing the results with the MLD models of 2, 4, 5, 6, 7, and 8-tank systems, it is observed that the number of binary variables, auxiliary continuous variables and generated mixed-integer inequalities increase exponentially with the number of tanks. Therefore, by increasing the number of tanks, the size of the system and consequently its complexity increases exponentially. Since in the MLD systems (like predictive control and optimal control) the numerical methods are used to solve control problems, the increase in the system dimension is considered a critical limitation. Therefore, procedures should be found to decrease the number of mixed-integer inequalities in MLD framework. This subject has recently attracted many researchers and a significant amount of interest has been shown in this field.

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