On the Wreath Product of Group by Some Other Groups

Basmah H. Shafee

Abstract—In this paper, we will generate the wreath product \( M_{11,\text{wr}M_{12}} \) using only two permutations. Also, we will show the structure of some groups containing the wreath product \( M_{11,\text{wr}M_{12}} \). The structure of the groups founded is determined in terms of wreath product \( (M_{11,\text{wr}M_{12}})_{wr} C_k \). Some related cases are also included.

Also, we will show that \( S_{132K+1} \) and \( A_{132K+1} \) can be generated using the wreath product \( (M_{11,\text{wr}M_{12}})_{wr} C_k \) and a transposition in \( S_{132K+1} \) and an element of order 3 in \( A_{132K+1} \). We will also show that \( S_{132K+1} \) and \( A_{132K+1} \) can be generated using the wreath product \( M_{11,\text{wr}M_{12}} \) and an element of order \( k+1 \).

Keywords—Group presentation, group generated by n-cycle, Wreath product, Mathieu group.

I. INTRODUCTION

HAMMAS and Al-Amri [1], have shown that \( A_{2n+1} \) of degree \( 2n+1 \) can be generated using a copy of \( S_n \) and an element of order 3 in \( A_{2n+1} \). They also gave the symmetric generating set of Groups \( A_{kn+1} \) and \( S_{kn+1} \) using \( S_n \) [5].

Shafee [2] showed that the groups \( A_{kn+1} \) and \( S_{kn+1} \) can be generated using the wreath product \( A_m \text{wr} S_a \) and an element of order \( k+1 \). Also she showed how to generate \( S_{kn+1} \) and \( A_{kn+1} \) symmetrically using \( n \) elements each of order \( k+1 \).

In [3], Shafee and Al-Amri have shown that the groups \( A_{10k+1} \) and \( S_{10k+1} \) can be generated using the wreath product \( M_{11,\text{wr}M_{12}} \) and an element of order \( k+1 \).

The Mathieu group \( M_1 \) and \( M_2 \) are two groups of the well known simple groups. In [6], they are fully described. In a matter of fact, they can be faintly presented in different ways. They have presentations in [6] as follows:

\[
M_{11} = \langle X, Y, Z \mid X^{11} = Y^3 = (XZ)^4 = 1, X^2 = Z^2 = (YZ)^3 = (XY)^3 = ZXY = (YZX)^2 \rangle.
\]

\[
M_{12} = \langle X, Y, Z \mid X^4 = Y^2 = Z^2 = (XY)^3 = (XZ)^4 = (YZ) = X^2 = YZ = (YZX)^2 \rangle.
\]

II. PRELIMINARY RESULTS

DEFINITION 2.1 Let \( A \) and \( B \) be groups of permutations on non empty sets \( \Omega_1 \) and \( \Omega_2 \) respectively. The wreath product of \( A \) and \( B \) is denoted by \( A \text{wr} B \) and defined as \( A \text{wr} B = A^{\Omega_2} \times B \), i.e., the direct product of \( |\Omega_2| \) copies of \( A \) and a mapping \( \theta \).

THEOREM 2.2 [4] Let \( G \) be the group generated by the \( n \)-cycle \((1,2,\ldots,n)\) and the \( 2 \)-cycle \((n,a)\). If \( 1 < a < n \) is an integer with \( n = am \), then \( G \cong S_m \text{wr} C_a \).

THEOREM 2.3 [4] Let \( 1 \leq a < b < n \) be any integers. Let \( n \) be an odd integer and let \( G \) be the group generated by the \( n \)-cycle \((1,2,\ldots,n)\) and the \( 3 \)-cycle \((n,a,b)\). If the \( \text{hcf}(n,a,b) = 1 \), then \( G = A_n \). While if \( n \) can be an even then \( G = S_n \).

THEOREM 2.4 [4] Let \( 1 \leq a < n \) be any integer. Let \( G = \langle (1,2,\ldots,n), (n,a) \rangle \). If \( \text{hcf}(n,a) = 1 \), then \( G = S_n \).

THEOREM 2.5 [4] Let \( 1 \leq a < b < n \) be any integers. Let \( n \) be an even integer and let \( G \) be the group generated by the \((n-1)\)-cycle \((1,2,\ldots,n-1)\) and \( 3 \)-cycle \((n,a,b)\). Then \( G = A_n \).
THEOREM 3.1 The wreath product $M_1 \circ M_2$ can be generated using two permutations, the first is of order 132 and the second is of order 4.

Proof: Let $G = \langle X, Y \rangle$, where: $X=(1, 2, 3, 4, \ldots, 12)$, which is a cycle of order 252, $Y=(1, 9, 2, 6, 4, 5, 7, 8, 12, 20, 23, 31, 13, 17, 15, 16, 18, 19, 24, 28, 26, 27, 29, 30, 34, 42, 56, 64, 35, 39, 37, 38, 40, 41, 45, 53, 46, 50, 48, 49, 51, 52, 57, 61, 59, 60, 62, 63, 67, 75, 68, 72, 70, 71, 73, 74)$, which is the product of two cycles each of order 4 and twenty four transpositions. Let $\alpha_1 = (\langle X \rangle, \langle Y \rangle)$. Then $\sigma_1 = \langle X \rangle$, which is a cycle of order 7. Let $\alpha_2 = \alpha_1^{-1} \sigma_1$. It is easy to show that $\alpha_2 = (1, 2, 3, \ldots, 17)(18, 19, 20, \ldots, 22) \ldots (67, 68, 69, 132)$, which is the product of seven cycles each of order 11. Let: $\gamma_1 = (1, 5, 9, 20, 23, 31, 13, 17, 15, 16, 18, 19, 24, 28, 26, 27, 29, 30, 34, 42, 56, 64, 35, 39, 37, 38, 40, 41, 45, 53, 46, 50, 48, 49, 51, 52, 57, 61, 59, 60, 62, 63, 67, 75, 68, 72, 70, 71, 73, 74)$, $\gamma_2 = (1, 45, 12, 23)$, $\gamma_3 = (1, 44, 55, 66)$ and $\beta_4 = (1, 44, 55, 66)$. Let $\alpha_3 = \alpha_2 \beta_4 \sigma_1$. Hence $\alpha_3 = (12, 24, 48, 60)$. Let $\alpha_4 = (1, 5, 9, 20, 23, 31, 13, 17, 15, 16, 18, 19, 24, 28, 26, 27, 29, 30, 34, 42, 56, 64, 35, 39, 37, 38, 40, 41, 45, 53, 46, 50, 48, 49, 51, 52, 57, 61, 59, 60, 62, 63, 67, 75, 68, 72, 70, 71, 73, 74)$, which is the product of twenty eight transpositions. Let $K = \langle \alpha_2, \alpha_3 \rangle$. Let $\theta: K \to M_{12}$ be the mapping defined by $\theta(12i+j) = j \forall 1 \leq i \leq 10, \forall 1 \leq j \leq 12$. Since $\theta(\alpha_2) = (1, 2, 3, \ldots, 12)$ and $\theta(\alpha_3) = (1, 9, 2, 6, 4, 5, 7, 8) \ldots 121$, then $K \cong \theta(K) = M_{12}$. Let $H_0 = \langle \alpha_1, \alpha_2 \rangle$. Then $H_0 \cong M_{11}$. Moreover, $K$ conjugates $H_0$ into $H_1, H_1$ into $H_2$ and so it conjugates $H_2$ into $H_0 \cong M_{12}$, where $H_0 \cong (121 \ldots 132)$. Since $\theta(\alpha_3) = (132 \ldots 1)$, where $\theta(\alpha_3) = X = \alpha_3 \sigma_1 X$, then $G \cong M_{12}$. Hence $G = M_{12}$.
THEOREM 3.5 The wreath product \((M_{11} \wr M_{12}) \wr (S_m \wr C_k)\) can be generated using three permutations, the first is of order 132k, the second and the third are involutions, where \(k = am\) be any integer with \(1 < a < k\).

Proof: Let \(\sigma = (1, 2, \ldots, 132k), \tau = (k, 9k)(2k, 6k)(4k, 8k)(12k, 20k, 23k, 31k)(13k, 17k, 16k, 18k, 19k, 24k, 28k)(6k, 12k, 30k)(34k, 42k, 56k, 64k)(35k, 39k, 40k, 41k, 45k, 53k, 46k, 50k, 48k, 49k, 51k, 52k, 57k, 61k, 59k, 60k, 62k, 63k, 67k, 76k, 72k, 70k, 71k)\) and \(\mu = (k, a)(2k, k+a)(3k, 2k+a)\) \((132k, 131k+a)\).

Since by Theorem 3.2, \(<\sigma, \tau, \mu> \cong (M_{11} \wr M_{12}) \wr C_k\) and \((1, \ldots, k)(k+1, \ldots, 2k)\) \((131k+1, \ldots, 132k)\) \((M_{11} \wr M_{12}) \wr C_k\), then
\[
\langle (1, \ldots, k)(k+1, \ldots, 2k) \rangle \cong (S_m \wr C_k).
\]

Therefore, \(G = \langle \sigma, \tau, \mu \rangle \cong (M_{11} \wr M_{12}) \wr (S_m \wr C_k)\).

THEOREM 3.6 \(S_{132k+1}\) and \(A_{132k+1}\) can be generated using the wreath product \((M_{11} \wr M_{12}) \wr C_k\) and a transposition in \(S_{132k+1}\) for all integers \(k > 1\) and an element of order 3 in \(A_{132k+1}\) for all odd integers \(k > 1\).

Proof: Let \(\sigma = (1, 2, \ldots, 132k), \tau = (k, 9k)(2k, 6k)(4k, 8k)(12k, 20k, 23k, 31k)(13k, 17k, 16k, 18k, 19k, 24k, 28k)(6k, 12k, 30k)(34k, 42k, 56k, 64k)(35k, 39k, 40k, 41k, 45k, 53k, 46k, 50k, 48k, 49k, 51k, 52k, 57k, 61k, 59k, 60k, 62k, 63k, 67k, 76k, 72k, 70k, 71k)\) and \(\mu = (132k+1, 1)\) be four permutations, of order 132k, 2, 2 and 3 respectively. Let \(H = \langle \sigma, \tau, \mu \rangle\).

Case 1: \(G = \langle \sigma, \tau, \mu \rangle\). Let \(\alpha = \sigma \mu\), then \(\alpha = (1, 2, \ldots, 132k, 132k+1)\) which is a cycle of order 132k + 1.

By theorem 2.4 \(G < \sigma, \tau, \mu' \geq \geq \sigma, \mu \geq S_{132k+1}\).

Case 2: \(G = \langle \sigma, \tau, \mu_0 \rangle\).

By theorem 2.5 \(G < \sigma, \mu_0 \geq A_{132k+1}\). Since \(\tau\) is an even permutation, then \(G \cong A_{132k+1}\).

THEOREM 3.7 \(S_{132k+1}\) and \(A_{132k+1}\) can be generated using the wreath product \(L_c (1) \wr M_{12}\) and an element of order \(k + 1\) in \(S_{132k+1}\) and \(A_{132k+1}\) for all integers \(k \geq 1\).

References: